

# On Weyl – Concircular and Weyl – Conharmonic Sasakian Recurrent and Symmetric Spaces of Second Order

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## Abstract

Otsuki and Tashiro [5] and Walker [9] have studied and defined Ruse's spaces of recurrent curvature and curves in Kaehlerian space. Matsumoto [2] have studied Kaehlerian space with parallel or Vanishing Bochner Curvature tensor. Further, Singh [8] have defined Kaehlerian recurrent and Ricci-recurrent space of second order. Negi and Rawat [3] have studied Kaehlerian space with recurrent and symmetric Bochner Curvature tensor. Rawat and Silswal [6] studied Kaehlerian bi-recurrent and bi-symmetric spaces. Further, Rawat and Kumar [11] studied Weyl–Sasakian projective and Weyl–Sasakian conformal bi-recurrent and bi-symmetric spaces.

In the present paper, we have studied and defined Weyl–Concircular Sasakian and Weyl-Conharmonic Sasakian Recurrent and symmetric space of second order. The relation between Weyl–Concircular and Weyl–Conharmonic Curvature tensor is defined and several theorem have been established therein.

**Key words:** Sasakian spaces, Weyl-Concircular, Weyl-Conharmonic.

## Introduction

An  $n$ - dimensional Sasakian space  $S_n$  (or, normal Contact metric spaces) is a Riemannian space which admits a unit Killing Vector field  $\eta_i$  satisfying okumura [4]

$$\nabla_i \nabla_j n_k = \eta_j g_{ik} - \eta_k g_{ij} \quad \dots (1)$$

It is well known that the Sasakian space is orientable and odd dimensional. Also, we know that an  $n$  – dimensional Kaehlerian space  $K_n$  is a Riemannian space, which admits a structure tensor field  $F_i^h$  satisfying Yano [10]

$$F_j^h F_h^i = -\delta_j^i, \quad \dots (2)$$

$$F_{ij} = -F_{ji}, (F_{ij} = F_i^a g_{aj}) \quad \dots (3)$$

$$\text{and } F_{i,j}^h = 0 \quad \dots (4)$$

Where the comma (,) followed by an index denotes the operator of Covariant differentiation with respect to the metric tensor  $g_{ij}$  of the Riemannian space.

The Riemannian Curvature tensor field denoted by  $R_{ijk}^h$ , is given by

$$R_{ijk}^h = \partial_i \left\{ \begin{matrix} h \\ j k \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ i k \end{matrix} \right\} + \left\{ \begin{matrix} h \\ i \alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ j k \end{matrix} \right\} - \left\{ \begin{matrix} h \\ j \alpha \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ i k \end{matrix} \right\} \quad \dots (5)$$

where  $\partial_i = \frac{\partial}{\partial x^i}$

The Ricci - tensor and the Scalar curvature in  $S_n$  are respectively given by

$$R_{ij} = R_{ij\alpha}^\alpha \text{ and } R = R_{ij} g^{ij} \quad \dots(6)$$

If we define a tensor  $S_{ij}$  by

$$S_{ij} = F_i^\alpha R_{\alpha j} , \quad \dots (7)$$

Then, we have

$$S_{ij} = -S_{ji} , \quad \dots (8)$$

$$F_i^\alpha S_{\alpha j} = -S_{i\alpha} F_j^\alpha , \quad \dots (9)$$

$$\text{and } F_i^\alpha S_{jk,\alpha} = R_{ji,k} - R_{ki,j} \quad \dots (10)$$

It has been verified in Yano [10] that the metric tensor  $g_{ij}$  and the Ricci-tensor denoted by  $R_{ij}$  are hybrid in  $i$  and  $j$ . Therefore, we get

$$g_{ij} = g_{rs} F_i^r F_j^s , \quad \dots(11)$$

$$\text{and } R_{ij} = R_{rs} F_i^r F_j^s , \quad \dots (12)$$

Weyl-Concircular curvature and Weyl- Conharmonic curvature tensor in  $S_n$  respectively give by

$$M_{ijk}^h = R_{ijk}^h - \frac{R}{n(n-1)} (g_{ij} \delta_k^h - g_{ik} \delta_j^h) \quad \dots (13)$$

and

$$L_{ijk}^h = R_{ijk}^h - \frac{1}{(n-2)} (g_{ij} R_k^h - g_{ik} R_j^h + R_{ij} \delta_k^h - R_{ik} \delta_j^h) \quad \dots (14)$$

In View of equations (13) and (14), we have

$$L_{ijk}^h = M_{ijk}^h + \frac{R}{n(n-1)} (g_{ij} \delta_k^h - g_{ik} \delta_j^h) - \frac{1}{(n-2)} (g_{ij} R_k^h - g_{ik} R_j^h + R_{ij} \delta_k^h - R_{ik} \delta_j^h) \quad \dots (15)$$

We shall use the following :

**Definition – 1:** A Sasakian space satisfying the condition Singh [7]

$$R_{ijk,ab}^h - \lambda_{ab} R_{ijk}^h = 0, \quad \dots(16)$$

For some non-zero tensor  $\lambda_{ab}$ , will be called a Sasakian recurrent space of Second order and denoted by  $S_n^{**}$  and said to be Sasakian Ricci-recurrent space of Second order, if it satisfies

$$R_{ij,ab} - \lambda_{ab} R_{ij} = 0, \quad \dots(17)$$

Multiplying equation (17) by  $g^{ij}$ , we have

$$R_{,ab} - \lambda_{ab} R = 0, \quad \dots (18)$$

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**Definition – 2:** A Sasakian space  $S_n$  satisfying the Condition

$$M_{ijk,ab}^h - \lambda_{ab} M_{ijk}^h = 0, \quad \dots (19)$$

For some non-zero tensor  $\lambda_{ab}$ , will be called a Weyl – Sasakian Concircular recurrent space of second order and is denoted by  $M^{**}$ - space.

**Definition – 3:** A Sasakian space  $S_n$  satisfying the Condition

$$L^h_{ijk,ab} - \lambda_{ab} L^h_{ijk} = 0, \quad \dots (20)$$

For some non-zero tensor  $\lambda_{ab}$ , is said to be Weyl – Sasakian Conharmonic recurrent space of second order and is denoted by  $L^*$ - space. Now, we have the following Theorems :

**Theorem – 1 :** Every  $S_n^{**}$  - space is  $M^{**}$  - space.

**Proof :** Differentiating equation (13) Covariantly with respect to  $x^a$ , again differentiate the result thus obtained Covariantly with respect to  $x^b$ , we have

$$M^h_{ijk,ab} = R^h_{ijk,ab} - \frac{R_{,ab}}{n(n-1)} (g_{ij} \delta_k^h - g_{ik} \delta_j^h) \quad \dots (21)$$

Transvecting equation (13) by  $\lambda_{ab}$ , and subtracting the resulting equation from (21) we have

$$M^h_{ijk,ab} - \lambda_{ab} M^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} - \frac{(R_{,ab} - \lambda_{ab} R)}{n(n-1)} (g_{ij} \delta_k^h - g_{ik} \delta_j^h) \quad \dots (22)$$

If the space is  $S_n^{**}$  - space , then equation (16) and (17) are satisfied and (22) in view of (16) and (17), gives

$$M^h_{ijk,ab} - \lambda_{ab} M^h_{ijk} = 0$$

Which shows that the space is Weyl – Sasakian Concircular recurrent space of second order Or,  $M^{**}$ - space.

**Theorem – 2 :** Every  $S_n^{**}$  - space is  $L^{**}$ - space.

**Proof :** Differentiating equation (14) covariantly w.r.to  $x^a$ , again differentiate the result so obtained covariantly with respect to  $x^b$ , we have

$$L^h_{ijk,ab} = R^h_{ijk,ab} - \frac{1}{(n-2)} (g_{ij} R^h_{k,ab} - g_{ik} R^h_{j,ab} + R_{ij,ab} \delta_k^h - R_{ik,ab} \delta_j^h) \quad \dots (23)$$

Transvecting (14) by  $\lambda_{ab}$ , then subtracting the resulting equation from (23), we have

$$\begin{aligned} L^h_{ijk,ab} - \lambda_{ab} L^h_{ijk} &= R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} - \frac{1}{(n-2)} [(R^h_{k,ab} - \lambda_{ab} R^h_k) g_{ij} - (R^h_{j,ab} - \lambda_{ab} R^h_j) g_{ik} \\ &+ (R_{ij,ab} - \lambda_{ab} R_{ij}) \delta_k^h - (R_{ik,ab} - \lambda_{ab} R_{ik}) \delta_j^h] \quad \dots (24) \end{aligned}$$

If the space is  $S_n^{**}$  - space then equation (16), (17) and (18) are satisfied and equation (24), in view of equations (16), (17) and (18) gives.

$$L^h_{ijk,ab} - \lambda_{ab} L^h_{ijk} = 0$$

Which shows that the space is  $L^{**}$ - space. This Completes the proof.

**Theorem – 3 :** If in a Sasakian space  $S_n$  any two of the following properties are satisfied ;

- (i) The space is Ricci-recurrent of second order,
- (ii) The space is  $M^{**}$ - space,
- (iii) The space is  $L^{**}$ - space, Then the third is also satisfied.

**Proof :** Differentiating (15) Covariantly with respect to  $x^a$ , again differentiate the result thus obtained covariantly w.r.to  $x^b$ , we have

$$L^h_{ijk,ab} = M^h_{ijk,ab} + \frac{R_{,ab}}{n(n-1)} (g_{ij} \delta_k^h - g_{ik} \delta_j^h) - \frac{1}{(n-2)} (g_{ij} R^h_{k,ab} - g_{ik} R^h_{j,ab} + R_{ij,ab} \delta_k^h - R_{ik,ab} \delta_j^h) \quad \dots (25)$$

Transvecting equation (15) by  $\lambda_{ab}$ , then subtracting the resulting equation from (25), we have

$$L^h_{ijk,ab} - \lambda_{ab} L^h_{ijk} = M^h_{ijk,ab} - \lambda_{ab} M^h_{ijk} + \frac{(R_{,ab} - \lambda_{ab} R)}{n(n-1)} (g_{ij} \delta_k^h - g_{ik} \delta_j^h) - \frac{1}{(n-2)} [(R^h_{k,ab} - \lambda_{ab} R^h_k) g_{ij} - (R^h_{j,ab} - \lambda_{ab} R^h_j) g_{ik} + (R_{ij,ab} - \lambda_{ab} R_{ij}) \delta_k^h - (R_{ik,ab} - \lambda_{ab} R_{ik}) \delta_j^h] \dots (26)$$

Making use of (16), (17), (18), (19), (20) and (26), we obtain the proof of the theorem.

**Theorem – 4 :** The necessary and sufficient Condition for a  $M^{**}$ - space to be  $S_n^{**}$ - space, is that the space be scalar recurrent of second order.

**Proof :** Let  $M^{**}$ - space be  $S_n^{**}$ - space, so that equation (16) and (19) are satisfied and (22), in view of equation (18), reduces to

$$(R_{,ab} - \lambda_{ab} R) (g_{ij} \delta_k^h - g_{ik} \delta_j^h) = 0$$

Which after further Calculation and simplification shows that the space is scalar recurrent of second order.

Conversely, let the  $M^{**}$ - space be scalar recurrent of second order, so that equation (18) are satisfied, then equation (22), in view of equations (18) and (19), reduces to.

$$R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} = 0,$$

Which shows that the space is  $S_n^{**}$ - space. Which proofs the theorem.

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**Definition – 4 :** A Sasakian space satisfying the Condition

$$R^h_{ijk,ab} = 0, \text{ or, equivalently } R_{ijkl,ab} = 0 \dots (27)$$

is said to be Sasakian symmetric space of second order and will be called Sasakian Ricci – symmetric space of second order, if it satisfies

$$R_{ij,ab} = 0 \dots (28)$$

Multiplying equation (28) by  $g^{ij}$ , we have

$$R_{,ab} = 0 \dots (29)$$

**Remark –** From equation (27), it follows that every Sasakian symmetric space of second order is Sasakian Ricci–symmetric space of second order, but the converse is not necessarily true.

**Definition – 5 :** A Sasakian space satisfying the Condition

$$M^h_{ijk,ab} = 0, \text{ or, equivalently } M_{ijkl,ab} = 0, \dots (30)$$

Will be called a Weyl – Sasakian Concircular Symmetric space of second order.

**Definition – 6 :** A Sasakian space  $S_n$  satisfying the Condition

$$L^h_{ijk,ab} = 0, \text{ or, equivalently } L_{ijkl,ab} = 0 \dots (31)$$

Will be called a Weyl–Sasakian Conharmonic symmetric space of second order.

**Theorem – 5 :** Every Sasakian symmetric space of second order is Weyl–Sasakian Concircular symmetric space of second order.

**Proof :** If the space is Sasakian symmetric space of second order, then equations (27) and (28) are satisfied and equation (21), in view of equations (27) and ((28), reduces to

$$M_{ijk,ab}^h = 0,$$

Which shows that the space is Weyl – Sasakian Concircular symmetric space of second order.

**Theorem – 6 :** Every Sasakian symmetric space of second order is Weyl–Sasakian Conharmonic symmetric space of second order.

**Proof :** If the space is Sasakian symmetric space of second order, then equations (27) and (28) are satisfied and equation (23), in view of equations (27) and ((28), reduces to

$$L_{ijk,ab}^h = 0,$$

Which shows that the space is Weyl–Sasakian Conharmonic symmetric space of second order.

**Theorem – 7 :** If in a Sasakian space any two of the following properties are satisfied :

- (i) The Space is Ricci–symmetric of second order.
- (ii) The Space is Weyl–Sasakian Concircular symmetric space of second order.
- (iii) The Space is Weyl –Sasakian Conharmonic symmetric space of second order, then it must also satisfy the third.

**Proof :** In a Sasakian Ricci - symmetric space of second order, the relation (28) is satisfied Whereas the Weyl–Sasakian Concircular symmetric space of second order and Weyl–Sasakian Conharmonic symmetric space of second order are given respectively by the condition (30) and (31). The statement of the above theorem follows in view of equations (28), (30), (31) and (25).

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