Linear Ideals and Linear Grills in Topological Vector Spaces

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Abstract— *In this paper we introduce the concepts of linear grills and linear ideals in topological vector spaces. We prove that the closure operators obtained from them are both Linear Čech closure operators under certain conditions. Also we introduce two new operators based on linear grills and linear ideals.*

Keywords— *Linear Čech closure spaces, semi open sets, linear grills, linear ideals.*

I. Introduction

Closure spaces were introduced by E. Čech [1] and then studied by many authors like Jeeranunt Khampaladee [8] , Chawalit Boonpok [2], David Niel Roth [4] etc. *Čech* closure spaces is a generalisation of the concept of topological spaces. The first to introduce the concept of grill topological spaces was Choquet [3] in 1947. Ideals in topological spaces have been considered since 1930. D. S. Jankovic and T. R. Hamlett [6] defined a topology obtained as an associated structure on a topological space (X,τ) induced by an ideal on X. B. Roy and M. N. Mukherjee[13] defined a topology obtained as an associated structure on a topological space (X,τ) induced by a grill on X. Later, A. Kandil et. al.[7] proved that the topological space induced by an ideal and the topological space which is induced by a grill are equivalent. Also A. A. Nasef and A. A. Azzam [12] defined and studied new operators Φ^s and Ψ^s with grill. We Tresa M. C. and Susha D. [15] introduced the concept of Linear Čech closure spaces and studied its fundamental properties. In this paper, we study the notion of linear grills and linear ideals and also we introduced two new operators on topological vector spaces.

In Section II we quote the necessary preliminaries about Linear Čech closure spaces, grills, ideals, topologies derived from grills and ideals etc. Section III deals with the concept of linear grills and the topology derived from a linear grill. In Section IV we proved the linearity of the closure operator obtained from a linear ideal. Section V contains the proof of the equivalence of the topologies obtained from linear grills and linear ideals. In Section VI we introduced some new operators based on linear grills and linear ideals.

II. Preliminaries

***Definition 2.1*[3]** A collection $\mathcal{G}$ of nonempty subsets of a set X is called a grill if

1. $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$
2. $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Let $\mathcal{G}$ be a grill on a topological space (X,τ). Consider the operator $\Phi_{\mathcal{G}}: \wp(X) \to \wp(X)$ given by $\Phi_{\mathcal{G}}(A) = \{x \in X: U \cap A \in \mathcal{G}, \forall U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}, \forall A \in \wp(X)$. Then the map $\Psi_{\mathcal{G}}: \wp(X) \to \wp(X)$ given by $\Psi_{\mathcal{G}}(A) = A \cup \Phi_{\mathcal{G}}(A)$ is a Kuratowski closure operator and hence induces a topology $\tau_{\mathcal{G}} = \{G \subseteq X: \Psi_{\mathcal{G}}(X-G) = X-G\}$, strictly finer than τ.

Definition 2.2 Let $(X,\tau,\mathcal{G})$ be a grill topological space. A subset A of a grill topological space $(X,\tau,\mathcal{G})$ is $\tau_{\mathcal{G}}-$closed [13] (resp. $\tau_{\mathcal{G}}-$dense in itself [11], $\tau_{\mathcal{G}}-$ perfect), if $\Psi_{\mathcal{G}}(A) = A$ or equivalently if $\Phi_{\mathcal{G}}(A) \subseteq A$ (resp. $A \subseteq \Phi_{\mathcal{G}}(A)$, $A = \Phi_{\mathcal{G}}(A)$).

Definition 2.3 [9] A nonempty collection I of subsets of a nonempty set X is said to be an ideal on X if

1. $A \in I$ and $B \subseteq A \Rightarrow B \in I$
2. $A \in I$ and $B \in I \Rightarrow A \cup B \in I$.

Given a topological space (X,τ) with an ideal I on X, a set operator $(.)^*: \wp(X) \to \wp(X)$ called a local function of a subset A with respect to τ and I is defined as $A^*(I,\tau) = \{x \in X | U \cap A \notin I, \forall U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}, \forall A \in \wp(X)$. Then the map $cl^*(A) = A \cup A^*$ is a Kuratowski closure operator and hence induces a topology $\tau^*(I,\tau) = \{G \subseteq X: cl^*(X-G) = (X-G)\}$, strictly finer than τ.

Definition 2.4 Let (X,τ,I) be an ideal topological space. A subset A of an ideal topological space (X,τ,I) is τ^*-closed [6] (resp.τ^*-dense in itself [5], τ^*- perfect), if $A^* \subseteq A$ (resp.$A \subseteq A^*, A = A^*$)

Definition 2.5. [1] A function $c: \wp(X) \to \wp(X)$ is called a Čech closure operator for X if

1. $c(\emptyset) = \emptyset$
2. $A \subseteq c(A)$

3. $C(A \cup B) = c(A) \cup c(B), \forall A, B \subseteq X$. Then (X, c) is called Čech closure space simply closure space. If in addition
4. $c(c(A)) = c(A), \forall A \subseteq X$, then the space (X, c) is called a Kuratowiski (topological) space.
   If further
5. for any family of subsets of X, $\{A_i\}_{(i\epsilon I)}$, $c(\cup_{i\epsilon I} A_i) = \cup_{i\epsilon I} c(A_i)$, the space is called a total closure space.

Definition 2.6. [1] A subset$A$ of a closure space (X, c) will be closed if $c(A) = A$ and open if its complement is closed, i.e. if $c(X - A) = X - A$.

Definition 2.7. [1] If $(X, c)$is a closure space, we denote the associated topology on X by t. i.e. $t = \{A^c : c(A) = A\}$

Theorem 2.1. Let (X, c) be a closure space and cl be the closure operator of the associated topology. Then $cl \leq c$ i.e. $c(A) \subseteq cl(A)$, $\forall A \subseteq X$.

Definition 2.8. [14] A map $f: (X, c) \to (Y, c')$is said to be a $c - c'$ morphism or just a morphism if $f(c(A) \subseteq c'(f(a))$.

Result: [1]

1. A mapping f of a closure space (X, c)onto another one(Y, c') is a $c - c'$ morphism at a point $x \in X$, if and only if the inverse image, $f^{-1}(V)$ of each neighbourhood V of $f(X)$ is a neighbourhood of x.
2. If f is a $c - c'$ morphism of a space (X, c) into a space(Y, c'), then the inverse image of each open subset of Y is an open subset of X.
3. If $f: (X, c) \to (Y, c')$ is a morphism, then $f: (X, t) \to (Y, t')$ is continuous.

Definition 2.9. [14] A homeomorphism is a bijective mapping f such that both f and f^{-1} are morphisms.

Definition 2.10. [10] A subset A of a topological space (X, τ) is called semi- open set if $A \subseteq cl(\text{int } A)$, where $A \subseteq X$ and the family of all semi-open sets of (X, τ) is denoted by $SO(X, \tau)$.

Definition 2.11. [15] Let V be a vector space and cbe a closure operator on V such that

1. $c(A) + c(B) \subseteq c(A + B)$
2. $\lambda\, c(A) \subseteq c(\lambda A)$. Then c is called a linear Čech closure operator and (V, c) is called a linear Čech closure space(LČCS).

Proposition 2.1. [15] Let V be a vector space and c be a closure operator on V. Then (V, c) is a linear Čech closure space if and only if $+: (V \times V, c \times c) \to (V, c)$ and $\lambda \cdot: (V, c) \to (V, c), \forall \lambda \in K$ are morphisms.

Proposition 2.2. [15] Let (V, c) be a LČCS. Then the map $T_a: (V, c) \to (V, c)$ given by $T_a(x) = a + x$ and $M_\lambda: (V, c) \to (V, c)$ given by $M_\lambda(x) = \lambda x$ are homeomorphisms.

Proposition 2.3. The topology obtained from a LČCS is a linear topology.

Result: If $(X, c)$is T_1 and finitely generated, it is the discrete closure space.

Proposition 2.4. Every LČCS is T_1 and hence Hausdorff.

Proof: Let 0 be the identity element and x be any other element of the vector space.

Then $c(\{0\}) + c(\{x\}) \subseteq c(\{0 + x\}) = c\{x\}$.

This shows that $c\{0\} = \{0\}$.

Then $c\{x\} + c\{-x\} \subseteq c\{x + (-x)\} = c\{0\} = \{0\}$.

If $y(\neq x) \in V, y + (-x) \neq 0$.

Hence $y(\neq x) \notin c\{x\}$ and $c\{x\} = \{x\}$.

We have seen in the literature that every T_1linear topological space is Hausdorff.

III. Linear Grills

Definition 3.1. A grill $\mathcal{G}$ on a linear topological space (V, τ) is called a linear grill if

1. $A, B \in \mathcal{G} \Rightarrow A + B \in \mathcal{G}$
2. $A \in \mathcal{G} \Rightarrow \lambda A \in \mathcal{G}, \forall$ scalars λ.

Proposition 3.1. If A and B are any two sets in a topological vector space with a linear grill in it then for the corresponding function $\Phi_{\mathcal{G}}, \Phi_{\mathcal{G}}(A) + \Phi_{\mathcal{G}}(B) \subseteq \Phi_{\mathcal{G}}(A + B)$. Also $\lambda\Phi_{\mathcal{G}}(A) \subseteq \Phi_{\mathcal{G}}(\lambda A)$.

Proof: Let $x \in \Phi_{\mathcal{G}}(A)$ and $y \in \Phi_{\mathcal{G}}(B)$.

Then for every $U \in \tau(x), A \cap U \in \mathcal{G}$ and for every $V \in \tau(y), B \cap V \in \mathcal{G}$.

Since $U \in \tau(x), V \in \tau(y)\ \exists U_0, V_0 \in \tau(0)$such that $U = x + U_0$ and $V = y + V_0$

Then $U_0 + V_0 \in \tau(0)$ and

$U + V = x + U_0 + y + V_0 = x + y + U_0 + V_0$

$\Rightarrow U + V \in \tau(x + y)$.

Let $W \in \tau(x + y)$. Then $\exists\, W_0 \in \tau(0)$ such that

$W = x + y + W_0$. Since addition is continuous and $0 + 0 = 0, \exists\, U_1$and $V_1 \in \tau(0)$ such that $W_0 = U_1 + V_1$. Thus corresponding to any two neighbourhoods U and V of x and y respectively, $\exists$a neighbourhood of $x + y$ and vice versa.

Now $A \cap U \in \mathcal{G}$ and $B \cap V \in \mathcal{G}$

$\Rightarrow (A \cap U) + (B \cap V) \in \mathcal{G}$, since $\mathcal{G}$ is closed under addition and

$(A \cap U) + (B \cap V) \subseteq (A + B) \cap (U + V)$

$\Rightarrow (A+B)\cap(U+V)\in\mathcal{G}$, by the property of a grill.

Now we have to show that $\lambda\Phi(A)\subseteq\Phi(\lambda A)$.

Let $x\in\Phi(A)$. Then $\lambda x\in\lambda\Phi(A)$.

$$x\in\Phi(A)\Rightarrow\forall U\in\tau(x), A\cap U\in\mathcal{G}.$$

We have to show that $\forall V\in\tau(\lambda x), \lambda A\cap V\in\mathcal{G}$, so that $\lambda x\in\Phi(\lambda A)$. Let $V\in\tau(\lambda x)$.

$\Rightarrow V=\lambda x+V'$ for some $V'\in\tau(0)$.

Since $\lambda\cdot 0=0$ and scalar multiplication is continuous in a topological vector space, $\exists V_0\in\tau(0)$ such that $V'=\lambda V_0$.

So $V=\lambda x+\lambda V_0=\lambda(x+V_0)=\lambda W$, where $W\in\tau(x)$. Now $\forall W\in\tau(x), A\cap W\in\mathcal{G}\Rightarrow$

$\lambda A\cap V=\lambda A\cap\lambda W=\lambda(A\cap W)\in\mathcal{G}$, by the second property of $\mathcal{G}$.

Proposition 3.2. If $\mathcal{G}$ is a linear grill in a linear topological space (X,τ), consisting of $\tau_{\mathcal{G}}-$perfect sets or $\tau_{\mathcal{G}}-$ dense sets, then the closure operator $\Psi_{\mathcal{G}}(A)=A\cup\Phi_{\mathcal{G}}(A)$, where $\Phi_{\mathcal{G}}(A)=\{x\in X: U\cap A\in\mathcal{G},\ \forall U\in\tau(x)\}$ is a Linear Čech closure operator.

Proof: $A\in\mathcal{G}$ is either $\tau_{\mathcal{G}}-$perfect set or $\tau_{\mathcal{G}}-$ dense set, hence $A\cup\Phi_{\mathcal{G}}(A)=\Phi_{\mathcal{G}}(A)$.

$$\begin{aligned}\Psi_{\mathcal{G}}(A)+\Psi_{\mathcal{G}}(B)&=\left(A\cup\Phi_{\mathcal{G}}(A)\right)+(B\cup\Phi_{\mathcal{G}}(B))\\&=\Phi_{\mathcal{G}}(A)+\Phi_{\mathcal{G}}(B)\\&\subseteq\Phi_{\mathcal{G}}(A+B)\\&\subseteq(A+B)\cup\Phi_{\mathcal{G}}(A+B)\\&=\Psi_{\mathcal{G}}(A+B).\end{aligned}$$

Now
$$\begin{aligned}\lambda\Psi_{\mathcal{G}}(A)&=\lambda(A\cup\Phi_{\mathcal{G}}(A))\\&=\lambda\Phi_{\mathcal{G}}(A)\\&\subseteq\Phi_{\mathcal{G}}(\lambda A)\\&\subseteq\lambda A\cup\Phi_{\mathcal{G}}(\lambda A).\end{aligned}$$

Thus $\Psi_{\mathcal{G}}$ is a Linear Čech closure operator.

Proposition 3.3. If $\mathcal{G}$ is a grill (not necessarily linear) in a linear topological space (X,τ) consisting of $\tau_{\mathcal{G}}-$perfect sets or $\tau_{\mathcal{G}}-$ closed sets, then the closure operator, $\Psi_{\mathcal{G}}(A)=A\cup\Phi_{\mathcal{G}}(A)$, where $\Phi_{\mathcal{G}}(A)=\{x\in X: U\cap A\in\mathcal{G},\ \forall U\in\tau(x)\}$ is a Linear Čech closure operator.

Proof: $A\in\mathcal{G}$ is either $\tau_{\mathcal{G}}-$perfect set or $\tau_{\mathcal{G}}-$ closed set, hence $A\cup\Phi_{\mathcal{G}}(A)=A$.

$$\begin{aligned}\Psi_{\mathcal{G}}(A)+\Psi_{\mathcal{G}}(B)&=\left(A\cup\Phi_{\mathcal{G}}(A)\right)+(B\cup\Phi_{\mathcal{G}}(B)\\&=A+B\\&\subseteq(A+B)\cup\Phi_{\mathcal{G}}(A+B)\\&=\Psi_{\mathcal{G}}(A+B).\end{aligned}$$

Now
$$\begin{aligned}\lambda\Psi_{\mathcal{G}}(A)&=\lambda\left(A\cup\Phi_{\mathcal{G}}(A)\right)\\&=\lambda A\\&\subseteq\lambda A\cup\Phi_{\mathcal{G}}(\lambda A).\end{aligned}$$

Thus $\Psi_{\mathcal{G}}$ is a Linear Čech closure operator.

***Note*:** Let A be a fixed subset of X, then the grill $\mathcal{G}_A=\{B\subseteq X: B\cap A^c\neq\emptyset\}$ is not a linear grill, because $B\cap A^c\neq\emptyset$ and $C\cap A^c\neq\emptyset$ neednot always imply $(B+C)\cap A^c\neq\emptyset$.

IV. Linear Ideals

Definition 4.1. An ideal I on a linear topological space is a linear ideal if

1. $A+B\in I\Rightarrow A\in I$ or $B\in I$
2. $\lambda A\in I\Rightarrow A\in I$

Proposition 4.1. If A and B are any two sets in a linear topological space with a linear ideal, then for the corresponding local function $A^*+B^*\subseteq(A+B)^*$. Also $\lambda A^*\subseteq(\lambda A)^*$.

Proof:

Let $x\in A^*$ and $y\in B^*$.

Then $\forall U\in\tau(x)$, $A\cap U\notin I$

And $\forall V\in\tau(y)$, $B\cap V\notin I$.

Therefore $(A\cap U)+(B\cap V)\notin I$.

i.e. $(A+B)\cap(U+V)\notin I$.

Since $U\in\tau(x), V\in\tau(y)\Leftrightarrow U+V\in\tau(x+y)$,

we get $x+y\in(A+B)^*$.

Thus $A^*+B^*\subseteq(A+B)^*$.

Now let $x\in A^*$, then $\lambda x\in\lambda A^*$

And $\forall U\in\tau(x)$, $A\cap U\notin I$.

Let $V\in\tau(\lambda x)$. Then $V=\lambda W$ for some $W\in\tau(x)$.

$$\Rightarrow\lambda A\cap V=\lambda A\cap\lambda W=\lambda(A\cap W).$$

Since

$A\cap W\notin I, \forall W\in\tau(x), \lambda(A\cap W)=\lambda A\cap V\notin I$

$\Rightarrow\lambda x\in(\lambda A)^*$ i.e. $\lambda A^*\subseteq(\lambda A)^*$.

Proposition 4.2. If I is a linear ideal in a linear topological space (X,τ), consisting of $(.)^*-$perfect sets or $(.)^*-$ dense sets in itself, then the closure operator $cl^*(A)=A\cup A^*$, where $A^*(I,\tau)=\{x\in X: U\cap A\notin I, \forall U\in\tau(x)\}$, is a Linear Čech closure operator.

Proof:

$A\in I$ is either $(.)^*-$perfect sets or $(.)^*-$ dense in itself and hence $A\subseteq A^*$.

$$\begin{aligned}cl^*(A)+cl^*(B)&=(A\cup A^*)+(B\cup B^*)\\&=A^*+B^*\\&\subseteq(A+B)^*\\&\subseteq(A+B)\cup(A+B)^*\\&=cl^*(A+B)\end{aligned}$$

Also $\lambda cl^*(A)=\lambda(A\cup A^*)=\lambda A^*$

$$\subseteq(\lambda A)^*\subseteq\lambda A\cup(\lambda A)^*=cl^*(\lambda A)$$

Showing that cl^* is a Linear Čech closure operator.

Proposition 4.3. If I is an ideal (not necessarily linear) in a linear topological space (X,τ), consisting of $(.)^*-$perfect sets or $(.)^*-$ closed sets, then the closure operator $cl^*(A)=A\cup A^*$, where $A^*(I,\tau)=\{x\in X: U\cap A\notin I, \forall U\in\tau(x)\}$, is a Linear Čech closure operator.

Proof: $A\in I$ is either $(.)^*-$perfect sets or $(.)^*-$ closed and hence $A^*\subseteq A$.

$$\begin{aligned}cl^*(A)+cl^*(B)&=(A\cup A^*)+(B\cup B^*)\\&=A+B\\&\subseteq(A+B)\cup(A+B)^*\\&=cl^*(A+B)\end{aligned}$$

Also $\lambda cl^*(A) = \lambda(A \cup A^*) = \lambda A$
$\subseteq \lambda A \cup (\lambda A)^* = cl^*(\lambda A)$, showing that cl^* is a Linear Čech closure operator.

V. Equivalence of Topologies obtained from Linear Ideals and Linear Grills

Proposition 5.1. Let V be a vector space and let $\mathcal{G} \subseteq \wp(V)$. Then $\mathcal{G}$ is a linear grill on V if and only if $I(\mathcal{G}) = \{A \in \wp(V) | A \notin \mathcal{G}\}$ is a linear ideal on V.

Proof: A. Kandil et.al.[7] proved that $\mathcal{G}$ is a grill if and only if $I(\mathcal{G})$ is an ideal.

We have to prove the linearity conditions.

Let $\mathcal{G}$ be a linear grill. Then $A, B \in \mathcal{G} \Rightarrow A + B \in \mathcal{G}$ and $A \in \mathcal{G} \Rightarrow \lambda A \in \mathcal{G}$.

Let $A, B \notin I(\mathcal{G})$. Then $A, B \in \mathcal{G} \Rightarrow A + B \in \mathcal{G}$.

$\Rightarrow A + B \notin I(\mathcal{G})$.

Also $A \notin I(\mathcal{G}) \Rightarrow A \in \mathcal{G} \Rightarrow \lambda A \in \mathcal{G} \Rightarrow \lambda A \notin I(\mathcal{G})$.

Hence $I(\mathcal{G})$ is a linear ideal.

Now assume that $I(\mathcal{G})$ is a linear ideal.

Let $A, B \in \mathcal{G}$. Then $A, B \notin I(\mathcal{G})$.

$$\Rightarrow A + B \notin I(\mathcal{G}) \Rightarrow A + B \in \mathcal{G}.$$

Also $A \in \mathcal{G} \Rightarrow A \notin I(\mathcal{G}) \Rightarrow \lambda A \notin I(\mathcal{G}) \Rightarrow \lambda A \in \mathcal{G}$.

Hence $\mathcal{G}$ is a linear grill.

Proposition 5.2. Let V be a vector space and $I \subseteq \wp(V)$. Then I is a linear ideal on V if and only if $\mathcal{G}(I) = \{A \in \wp(V) | A \notin I\}$ is a linear grill on V.

VI. New Operators using Linear Ideals and Linear Grills

Definition 6.1. [12] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. A mapping $\Phi^s: \wp(X) \to \wp(X)$, denoted $\Phi^s_{\mathcal{G}}$ for $A \in \wp(X)$ (simply $\Phi^s(A)$), is called the operator associated with $\mathcal{G}$ and τ which is defined by $\Phi^s(A) = \{x \in X: U_x \cap A \in \mathcal{G}, \forall U_x \in SO(X, \tau)\}, \forall A \in \wp(X)$.

Definition 6.2. Let (X, τ) be a topological space and I be an ideal on X. A mapping $^{s*}: \wp(X) \to \wp(X)$, denoted A^{s*} for $A \in \wp(X)$, is called the operator associated with I and τ which is defined by $A^{s*} = \{x \in X: U_x \cap A \notin I, \forall U_x \in SO(X, \tau)\}, \forall A \in \wp(X)$.

Definition 6.3. [12] Let $(X, \tau, \mathcal{G})$ be a grill topological space. An operator $\Psi^s_{\mathcal{G}}: \wp(X) \to \wp(X)$ is defined as $\Psi^s_{\mathcal{G}}(A) = \{x \in X: \exists\, U_x \in SO(X, \tau)$ such that $U - A \notin \mathcal{G}\}$, for any $A \subseteq X$ and $\Psi^s_{\mathcal{G}}(A) = X - \Phi^s(X - A)$ or $\Psi^s(A) = A \cup \Phi^s(A)$.

Definition 6.4. Let (X, τ, I) be an ideal topological space. An operator $cl^{s*}_I: \wp(X) \to \wp(X)$ is defined as $cl^{s*}_I(A) = A \cup A^{s*}$, $\forall A \in \wp(X)$.

Theorem 6.1. [12] The operator Ψ^s satisfies Kuratowski's closure axioms.

Theorem 6.2. The operator cl^{s*}_I satisfies Kuratowski's closure axioms.

Definition 6.5. [12] A grill on a space X which carries a topology τ generates a unique topology on X depends on Ψ^s and ϕ^s operators symbolized by $\tau^s_{\mathcal{G}}$ and defined by $\tau^s_{\mathcal{G}} = \{U \subseteq X: \Psi^s(X - U) = (X - U)\}$ for $A \subseteq X$.

Definition 6.6. An ideal on a space X which carries a topology τ generates a unique topology on X depends on cl^{s*}_I symbolized by τ^s_I and defined by $\tau^s_I = \{U \subseteq X: cl^{s*}_I(X - U) = (X - U)\}$, for $A \subseteq X$.

Definition 6.7. Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then corresponding to the topology $\tau^s_{\mathcal{G}}$, a set $A \in \wp(X)$ is said to be $\tau^s_{\mathcal{G}}$-closed set,[resp. $\tau^s_{\mathcal{G}}$ −dense set in itself or $\tau^s_{\mathcal{G}}$ −perfect set] if $\phi^s_{\mathcal{G}}(A) \subseteq A$ [resp.$A \subseteq \phi^s_{\mathcal{G}}(A)$ or $A = \phi^s_{\mathcal{G}}(A)$]

Similarly let (X, τ, I) be an ideal topological space. Then corresponding to the topology τ^s_I, a set $A \in \wp(X)$ is said to be τ^s_I-closed set,[resp. τ^s_I − dense set in itself or τ^s_I − perfect set] if $A^{s*} \subseteq A$ [resp.$A \subseteq A^{s*}$ or $A = A^{s*}$].

Lemma 6.1. If A and B are semi-open sets in a Linear topological space, then $A + B$ is also a semi-open set.

Proof: Since A and B are semi-open sets, $A \subseteq cl(int(A))$ and $B \subseteq cl(int(B))$.

$$\Rightarrow A + B \subseteq cl(int(A)) + cl(int(B))$$

For a linear topological closure operator,

$cl(A) + cl(B) \subseteq cl(A + B)$.

Hence

$A + B \subseteq cl(int(A) + int(B)) \subseteq cl(int(A + B))$,

again by the property of linear topological interior operator.

Thus $A + B$ is a semi-open set.

Proposition 6.1. If $\mathcal{G}$ is a linear grill in a topological vector space (X, τ), then $\Phi^s(A) + \Phi^s(B) \subseteq \Phi^s(A + B), \forall A, B \in \wp(X)$.

Proof: Let $x \in \Phi^s(A)$ and $y \in \Phi^s(B)$.

$$\Rightarrow U_x \cap A \in \mathcal{G}, \forall U_x \in SO(X, \tau)$$

And $U_y \cap B \in \mathcal{G}, \forall U_y \in SO(X, \tau)$

$\Rightarrow (U_x \cap A) + (U_y \cap B) \in \mathcal{G}, \forall U_x, U_y \in SO(X, \tau)$

$\Rightarrow (U_x + U_y) \cap (A + B) \in \mathcal{G}$, since $(U_x \cap A) + (U_y \cap B) \subseteq (U_x + U_y) \cap (A + B)$.

Let $U_{x+y} \in SO(X, \tau)$.

Then $U_{x+y} \subseteq cl(int(U_{x+y}))$.

$int(U_{x+y})$ is an open set containing $x + y$.

By the property of topological vector spaces, $\exists$ two open sets V_x and V_y containing

x and y respectively such that $V_x + V_y \subseteq \mathrm{int}(U_{x+y}) \subseteq U_{x+y}$.

$\Rightarrow (V_x + V_y) \cap (A + B) \subseteq U_{x+y} \cap (A + B)$

Since $\mathcal{G}$ is a grill, it follows that $U_{x+y} \cap (A + B)$ belongs to $\mathcal{G}$.

Hence $x + y \in \Phi^s(A + B)$

$\Rightarrow \Phi^s(A) + \Phi^s(B) \subseteq \Phi^s(A + B), \forall A, B \in \wp(X)$.

Proposition 6.2. **(1)** If $\mathcal{G}$ is a linear grill in a topological vector space (X, τ), then Ψ^s is a linear Čech closure operator if $\mathcal{G}$ has only $\tau^s_{\mathcal{G}}$ −dense set in itself or $\tau^s_{\mathcal{G}}$ −perfect set.

(2) If $\mathcal{G}$ is a grill in a topological vector space (X, τ), then Ψ^s is a linear Čech closure operator if $\mathcal{G}$ has only $\tau^s_{\mathcal{G}}$ −closed sets or $\tau^s_{\mathcal{G}}$ −perfect sets.

Proof: (1) A. A. Nasef and A. A. Azzam [12] has proved that Ψ^s is a Kuratowski closure operator.

We want to prove the linearity conditions, $\Psi^s(A) + \Psi^s(B) \subseteq \Psi^s(A + B)$.

$\Psi^s(A) = A \cup \phi^s(A) = \phi^s(A)$, since $A \subseteq \phi^s(A)$

$$\begin{aligned}\Psi^s(A) + \Psi^s(B) &= (A \cup \phi^s(A)) + (B \cup \phi^s(B)) \\ &= \phi^s(A) + \phi^s(B) \\ &\subseteq \phi^s(A + B) \\ &\subseteq (A + B) \cup \phi^s(A + B) \\ &= \Psi^s(A + B).\end{aligned}$$

Similarily $\lambda\Psi^s(A) \subseteq \Psi^s(\lambda A)$ and hence Ψ^s is a Linear Čech closure operator.

(2) If $A \subseteq X$ is $\tau^s_{\mathcal{G}}$ −closed, $A \cup \phi^s(A) = A$ and the proof follows accordingly.

Proposition 6.3. If I is a linear ideal in a topological vector space (X, τ), then the function $A^{s*}(I, \tau) = \{x \in X | U_x \cap A \notin I, \ \forall U_x \in SO(X, \tau)\}$ satisfies $A^{s*} + B^{s*} \subseteq (A + B)^{s*}$.

Proof: Let $x \in A^{s*}$ and $y \in B^{s*}$.

$\Rightarrow U_x \cap A \notin I, \forall U_x \in SO(X, \tau)$ and

$U_y \cap B \notin I, \forall U_y \in SO(X, \tau)$

$\Rightarrow (U_x \cap A) + (U_y \cap B) \notin I, \forall U_x, U_y \in SO(X, \tau)$

$\Rightarrow (U_x + U_y) \cap (A + B) \notin I$

Let $U_{x+y} \in SO(X, \tau)$. Then $U_{x+y} \subseteq \mathrm{cl}(\mathrm{int}(U_{x+y}))$.

$\mathrm{int}(U_{x+y})$ is an open set containing $x + y$.

By the property of topological vector spaces, there exists two open sets V_x and V_y containing x and y respectively such that

$V_x + V_y \subseteq \mathrm{int}(U_{x+y}) \subseteq U_{x+y}$

$\Rightarrow (V_x + V_y) \cap (A + B) \subseteq U_{x+y} \cap (A + B)$.

Hence by the property of ideal,

if $U_{x+y} \cap (A + B) \in I$, then

$(V_x + V_y) \cap (A + B) \in I$.

So $U_{x+y} \cap (A + B) \notin I \Rightarrow x + y \in (A + B)^{s*}$.

Thus $A^{s*} + B^{s*} \subseteq (A + B)^{s*}$.

Proposition 6.4. **(1)** If I is a linear ideal in a topological vector space (X, τ), then cl^{s*} is a linear Čech closure operator if I has only τ^s_I −dense set in itself or τ^s_I −perfect sets.

(2) If I is an ideal in a topological vector space (X, τ), then cl^{s*} is a linear Čech closure operator if I has only τ^s_I −closed sets or τ^s_I −perfect sets.

Proof: Proof is analogous to that of propositions **6.2** using proposition **6.3**.

VII. Conclusions

The topology obtained from a Linear Čech closure operator is a T_1 topology, hence it is Hausdorff.

The topology derived from a grill is finer than the original topology. Hence the topology we obtained from the Linear *Čech* closure operator derived from linear grills or linear ideals possesses a significant role in the theory of topological vector spaces.

Acknowledgment

The author is indebted to the University Grants Commission as the work is under the Faculty Development Programme of UGC(XII plan).

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