On Bicomplex Representation Methods and Application of Quaternion Range Hermitian matrices (q-EP)

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***Abstract*:** *In this paper, a series of bicomplex representation methods of q-EP matrices is introduced. We present a new multiplication of q-EP matrices, a new determinant concept, a new inverse concept of q-EP matrix and a new similar matrix concept.*

***Keywords*:** *q-EP matrix, q-EP determinant, Inverse of q-EP matrix, similar q-EP matrix.*

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Introduction:

Throught we shall deal with nxn quaternion matrices[26]. Let A* denote the conjugate transpose of A. Any matrix $A \in H_{nXn}$ is called q-EP(27) if R(A)=R(A^*) and his called q-EP_r, if A is qs-EP and rk(A)=r, where N(A), R(A) and rk(A) denote the null space, range space and rank of A respectively. Q_E^{nxn} denote the q-EP matices. It is well known that sum, sum of parallel summable q-EP matrices, Product of q-EP and Generalized Inverses, Group Inverses And Reverse Order Law For Range Quaternion Hermitian Matrices (q-EP)[28-30].

In recent years, the algebra problems over quaternion division algebra have drawn the attention of mathematics and physics researchers [1-12]. Quaternion algebra theory is getting more and more important. In many fields of applied science, such as physics, figure and pattern recognition, spacecraft attitude control, 3-D animation, people start to make use of quaternion algebra theory to solve some actual problems. Therefore, it encourages people to do further research [13-17] on quaternion algebra theory and its applications. The main obstacle in the study of quaternion algebra is the non-commutative multiplication of quaternion. Many important conclusions over real and complex fields are different from ones over quaternion division algebra, such as determinant, the trace of matrix multiplication and solutions of quaternion equation. From the conclusions on quaternion division algebra, we find it to lack for general concepts, such as the definition of quaternion matrix determinant. There are different definitions which are given in [1,3,4,6,11,18] since Dieudonne firstly introduced the quaternion determinant in 1943. In addition, the inverse of quaternion matrix has not been well defined so far, because it depends on other algebra concepts. In the study of quaternion division algebra, people always expect to get some relations between quaternion division algebra and real algebra or complex algebra. However, some conclusions on real or complex fields are correct but not on quaternion division algebra. It makes us to consider establishing other algebra concept system over quaternion division algebra to unify the complex algebra and quaternion division algebra. Recently, Wu in [19] used real representation methods to express quaternion matrices and established some new concepts over quaternion division algebra. From these definitions, we can see that they can convert quaternion division algebra problems into real algebra problems to reduce the complexity and abstraction which exist in all kinds of definitions given in [1,3,6,10,11,20]. However, as Wu in [19] mentioned, these concept system is not suitable for complex algebra.

In this paper, based on the bicomplex form of q- EP matrix, we present some new concepts to quaternion division algebra. These new concepts can perfect the theory of Wu in [19] and unify the complex algebra and quaternion division algebra. This paper is organized as follows. In Section 1, we introduce a complex representation method of quaternion EP matrices and explore the relation between q-EP matrices and complex matrices. In Section 2, we present a series of new concepts over q-EP division algebra and study their properties. In section 3, we establish some important theorems to illustrate the applications and effectiveness of the new concept system.

1. *The Bicomplex Representation Methods of Quaternion EP matrices(q-EP) and the Relation between q-EP matrices and complex matrices:*

For any q-EP matrix $A \in Q_E^{nxn}$, A can be uniquely represented as

$$A = A_0 + A_1 j$$

$$R(A) = R(A_0 + A_1 j) \quad \text{(by [27],1.1.1)}$$

$$R(A) = R(A_0) + R(A_1)j$$

Where $A_s \in C^{nxn}$ (s=0,1), $A_1 j$ means to multiply each entries of A_1 by j from right hand side and

$rk(A_0) = rk(A_1 j)$. For above reasons we can establish a mapping relation between quaternion EP matrices and complex matrices as follows:

$$\mathbf{f : A \in Q_E^{nxn} \rightarrow (A_0, A_1)} \qquad \textbf{(1.1.2)}$$

where $A_s \in C_{nxn}$ (s=0,1).

The set of nxn quaternion matrices is written as A and the set of image of A is written as A_{img}.

Theorem 1.1

Let $f : A \in Q_E^{nxn} \rightarrow (A_0, A_1)$, ($A_s \in C_{nxn}$(s=0,1), Then the mapping f is a bijective mapping from A to A_{img}.

Proof

For any entry (A_0, A_1) in A_{img}, there exists the corresponding q-EP matrixA = $A_0+A_1 j$ in A, therefore f is a surjection from A to A_{img}. Simultaneously, since any q-EP matrix in A can be uniquely represented as the form (1.1.1), So f is an injection from A to A_{img}. Thus f is bijective mapping from a to A_{img}.The proof is complete.

Theorem 1.2

Bijection $f : A \leftrightarrow (A_0, A_1)$, $A_s \in C^{nxn}$

(s = 0,1) is an isomorphism mapping from A to A_{img}.

Proof

By the concept of isomorphism mapping, this theorem is easy to prove and we omit it here.

2. *The Bicomplex Matrix concept System over q-EP*

According to the complex representation of q-EP matrices above, a series of new definition of quaternion division algebra which are helpful to discuss the algebra problems on quaternion division algebra can be given as follows.

Definition 2.1

The matrix $\tilde{E}$ = E+Ej is said to be an nxn unit q-EP matrix if E is an nxn unit matrix, over complex field. In particular, if n=1, then $R(\tilde{E})$ = R(E)+R(E)j =R(1)+(1)j is said to be a unit

q-EP written as a_u.

Definition 2.2

Let $A = A_0+A_1 j$ and $B = B_0+B_1 j \in Q_E^{nXt}$ be given. The operator $R(A*B) = R(A_0B_0) + R(A_1B_1)j$ (where A_0B_0, A_1B_1 are both the multiplications of complex matrices) is called the * product of q-EP matrices A and B. In particular, if n=t=1, then we can drive the * product of q-EP.

Note: when $A \in C_{nxn}$ $B \in C_{nxn}$, then

A * B=AB.

Under the definition 2.1 and 2.2, we give some relative properties.For any matrix $A, B \in Q_E^{nxn}$, we have:

1) $\tilde{E}*A = A*\tilde{E} = A$, where $\tilde{E}$ is a nxn q-EP matrix:
2) A+B = B+A;
3) (A+B)*C = A*C+B*C;
4) $(A*B)^T = B^T * A^T$;
5) $T_r(A*B) = T_r(B*A)$

Similarly, we establish a new definition as follows.

Definition 2.3

Let $x \in Q_E^{nx1}$ and $a \in H$ be given. Then a*x = x*a = $a_0x_0+a_1x_1j$ is called the * product of quaternion EP and quaternion EP vector, where R(x) = $R(x_0)+R(x_1j)$, $x_0 \in C^{nx1}$, $x_1 \in C^{nx1}$, $a = a_0+a_1j$, $a_0 \in C$, $a_1 \in C$.

Definition 2.4

For any q-EP matrix $A \in Q_E^{nxn}$ $(A=A_0+A_1j)$, $\|A\| = |A_1|j$ is said to be the determinant of A, where 1.1 is the determinant of A, where 1.1 is the determinant of a complex matrix.

Note: when $A \in C^{nxn}$, then $\|A\|=A$

The definition 2.3 is reasonable. First of all, the result of a q-EP matrix determinant under definition 2.4 is a q-EP. Secondly, from 2.4 we can see that it can convert the determinant of a q-EP matrix into that of complex matrices to reduce the complexity and abstraction. Finally, the new determinant has the same fundamental properties as that over complex field. That is, if A is a nxn q-EP matrix and $i \neq j$, then we have

1) $R(\|A\|) = R(\|A^T\|)$

2) If quaternion EP matrix B is obtained from quaternion EP matrix A by interchanging two rows (or columns) of A. then $R(\|B\|) = R(-\|A\|)$

3) If quaternion EP matrix A has a zero row (or column), then $R(\|A\|)=R(\|A^T\|)= 0$

4) $R(\|K*A\|) = R(K^{*n}\|A\|)$, where $K^{*n}=\underbrace{K*K*\dots\dots*K}_{n}$, $K \in H$

5) If the j[th] row (column) of quaternion matrix A equal. A multiple of the i[th] row(column) of the matrix A, then $\|A\|=0$

6) Suppose that A, B and C are all nxn q-EP matrices If all rows of B and C both equal the corresponding to rows(columns) of A except that the i[th] row(column) of matrix A equal the sum of the i[th] of B and C, then $R(\|A\|)$ = sR $(\|B\|)+R(\|C\|)$.

7) If quaternion EP matrix B is the nxn matrix resulting from adding a multiple of the i[th] row(column) of matrix A to the j[th] row (or column) of matrix A, then $R(\|B\|)=R(\|A\|)$.

8) Let A and B be nxn q-EP matrices respectively. We have $R(\|A*B\|) = \|A\|*\|B\|$

Up to now, people still treat the inverse matrix concept of q-EP matrix A satisfies

$A^{-1}A = E$(where E is a real unit matrix), then people think that q-EP matrix A exists its inverse matrix A^{-1}. However, people pointedly ignore two questions. An issue is how to define the product of quaternion EP matrices A^{-1} and A. The other one is how to make calculation of A^{-1}. It indicates that the terminology of inverse matrix does not have a clear definition in quaternion q-EP theory.

In the following, we shall give a new definition and specific computational method for the inverse of q-EP matrix.

Definition : 2.5

Let $A = A_0+A_1j \in Q_E^{nxn}$ be given (where A_0,A_1 both are complex matrices). If the inverse matrices of A_0 and A_1^{-1} both exist, then q-EP matrix A is said to be invertible and the inverse matrix is written as $R(A^{-})$ = $R(A_0^{-1})+R(A_1^{-1})j$, where A_0^{-1}, A_1^{-1} denote the inverse of complex matrices A_0,A_1 respectively.

Note : when $A \in C^{nxn}$, then $A^{-} = A^{-1}$.

Then inverse of q-EP matrix under the new definition has the same fundamental properties as those under the traditional algebra system. It is easy to show the following facts by the new concept, namely, if a quaternion EP matrix A is invertible, then we have:

1) $R((A^{-})^{-})= R(A)$

2) $R((A^{-})^k) = R((A^k)^{-}) = R((A_0^{-1})^k)+R((A_1^{-1})^k)j$, where

$A^*=A^*A^*\ldots\ldots.*A$ is product of KA which is defined in definition 2.2

3) If $A_1,A_2,\ldots\ldots,A_m$ are all invertible q-EP matrices, then

$R((A_1*A_2*\ldots\ldots*A_m)^{-}) =$

$R(A_m^{-})*R(A_{m-1}^{-})*\ldots\ldots..*R(A_1^{-})$

Obviously, by the new definition of inverse of q-EP matrix above, people can

Determine easily whether the inverse matrix of q-EP matrices exists or not and calculate the inverse matrix if possible. Under the definition of inverse of q-EP matrix above, a new concept of similar q-EP matrices can be given as follows:

Definition 2.6

Let $A,B\in Q_E^{nxn}$, if there exists an invertible q-EP matrix P such that $A=P^{-}*B*P$, then

A and B are said to be similar q-EP matrices written as A~B.

Note: When $A,B\in C^{nxn}$, $A=P^{-}*B*P$ is equivalent to $A = P_0^{-1}BP_0$, where $P = P_0+P_1j, P_0,P_1\in C^{nxn}$.

For similar q-EP matrices, we will deduce many important properties in the next section.

2. Some applications of the Bicomplex Matrix concept System

In this section, we establish some important theorems to illustrate the applications and effectiveness of the new concept system for the research of quaternion division algebra. The eigan value is an important issue in quaternion division algebra theory. So, under the new concept of system, we will study firstly the eigen value of q-EP matrix and the relation between eigen values of similar q-EP matrices in detail.

Before showing the application, we will introduce firstly some concepts associate with eigen value.

Definition 3.1

For any matrix $A=(a_{ij}) \in Q_E^{nxn}$, if there exists non-zero quaternion vector $X\in H^{nx1}$ and a quaternion $\lambda=\lambda_0+\lambda_1 j$ (where λ_0,λ_1 are both complex numbers such that $A^*X=\lambda*X$, then λ is said to be the left eigen value of A, and X is the left eigen vector corresponding to λ.

For the sake of distinction, we call the left eigen value and the left quaternion the left q-EP eigen value and the left q-EP eigen vector respectively.

According to the new definition of q-EP matrix multiplication and A*X = A*X, we can derive that $(\lambda*E-A)*A$=0. Thus $f(\lambda)=\|\lambda*\tilde{E}-A\|$ is said to be characteristic polynomial of A (where the operator ||.|| denotes the determinant of q-EP matrix under definition 2.4).

Theorem 3.2

A nxn q-EP matrix $A = A_0 +A_1j$ (where A_0, A_1 both are complex matrices), if λ and μ are the left eigen value of A_0 and A_1 respectively, then $\lambda+aj$ and $b+\mu j$ are the left q-EP eigen value of A.

Proof

Since λ and μ are the left eigen values of A_0 and A_1 respectively, then there exist nonzero vectors. $\xi\in C^{nx1}$ and $\eta\in C^{nx1}$ such that $A_0\xi=\lambda\xi$, $A_1\eta=\mu\eta$ we have

$A*\xi =(A_0+A_1j)*(\xi+0j)=A_0\xi=\lambda\xi=(\lambda+aj)*\xi$, for all $a\in C$

So $\lambda+aj$ and $b=\mu j$ are all the left q-EP eigen value of A. The proof is complete.

Similarly, we introduce a new right q-EP eigen value concept.

Definition 3.3

For any matrix A=(a_{ij}) $\in Q_E^{nxn}$, if there exists nonzero quaternion vector $\in H^{1xn}$ and quaternion $\mu = \mu_0 + \mu_1 j$(where μ_0, μ_1 are both complex numbers) such that SY*A = μ * Y, then μ is said that to be the right q-EP eigenvalue of A and Y is the right q-EP eigenvector corresponding to μ.

For the eigen value of q-EP matrix, we have the following theorem:

Theorem 3.4

A nxn q-EP matrix A = A_0+A_1j (where A_0,A_1 are both complex matrices), if λ and μ are the right q-EP eigen value of A_0 and A_1 respectively, then $\lambda + aj$ and b+ μ j (r_1 $\forall$ a $\in$ c , $\forall$ b$\in$ c) are the right q-EP eigen values of A·

Proof

Since λ and μ are the right eigen values of A_0 and A_1 respectively, then there exists nonzero vectors $\xi \in C^{1Xn}$ and $\eta \in C^{1Xn}$ such that $\xi A_0 = \lambda \xi$, $\eta A_1 = \mu \eta$. We have

$\xi * A$ = ($\xi + 0j$)*(A_0+A_1j) = ξA_0

= $\lambda \xi$ =($\lambda + aj$)* ξ, for $a \in c$

(η j)*A = (0+η j) * (A_0+A_1j) = η A_1j = $\mu \eta j$

= s(b+ μj) * (η j), for $b \in c$

So, $\lambda + aj$ and $b + \mu j$ are the right q-EP eigenvalues of A .The proof is complete.

Theorem 3.5

If the left eigen values of A_0 are $\lambda_1, \lambda_2 .... \lambda_k$ and the left eigen values of A_1 are $\mu_1, \mu_2, ... \mu_m$ (where A_0, A_1 both are complex matrices), then the left q-EP eigen values of matrix A = A_0 + A_1j are { $\lambda_s + aj$ or $b + \mu_t j$,for all $a \in c$ $b \in c$, s=1,……,k, t=1,……,m}.

Proof

Suppose that Υ is arbitrary left q-EP eigen value of A, then $\exists$ $\psi \neq 0$

$\psi = \psi_0 + \psi_1 j \in H^{n\times 1}$, Such that A * ψ = $\Upsilon * \psi$, that is, $\begin{cases} A_0 \psi_0 = \gamma_0 \psi_0 \\ A_1 \psi_1 = \gamma_1 \psi_1 \end{cases}$,Since $\psi \neq 0$

We know that both ψ_0 and ψ_1 are not zeroes .So there are two cases as follows.

1) When, obviously, we have $\Upsilon_0 \in \{\lambda_1, \lambda_2 .... \lambda_k\}$ .So, $\Upsilon \in \{\lambda_1 + aj, i = 1, 2, ... k\}$.

2) When, obviously we have $\Upsilon \in \{\lambda_1 + aj, i = 1, 2, ... k\}$ So, $\Upsilon \in \{b + \mu_1 j, t = 1, 2, ... m\}$

To sum up 1),2) and theorem 3.2, we can draw the conclusion. The proof is complete.

Theorem 3.6

If the right eigen values of A_0 are $\lambda_1, \lambda_2, ..., \lambda_k$ and he right eigen values of A_1 are $\mu_1, \mu_2, ..., \mu_m$ (where A_0, A_1 both are complex matrices), then the right q-EP eigen values of matrix A = A_0 + A_1j are { $\lambda s + aj$ or b+ μ_t , $a \in c$, $b \in c$, s=1,……k, t = 1,…….m}.

This proof is similar to theorem 3.5 so we omit it here.

Theorem 3.7

Let A $\in Q_E^{nxn}$, then A and A^T have same q-EP left (right) eigen values

Proof

Since $A=A_0+A_1j$ where $A_0\in C^{nxn}, A_1\in C^{nxn}$), then $A^T=A_0^T+A_1^Tj$. We know A_1 and A_1^T have the same left(right) eigen values (1,2), by Theorem 3.5 and 3.6, we can draw the conclusion.

Theorem 3.8

Let $A\in Q_E^{nxn}$ and $\mu,\lambda\in H$ be given. If $\lambda(\mu)$ is the left (right) q-EP eigen value of A. Then $\lambda(\mu)$ is the right(left) q-EP eigen value of A.

Proof

Since λ is the left q-EP eigen value of A, then there exists nonzero vector ξ such that

$$A*\xi=\lambda*\xi$$

Then $(A*\xi)T=(\lambda*\xi)T$. we can have $\xi^T*A^T=\lambda*\xi^T$.

So, λ is the right q-EP eigen value of A^T by theorem 3.7, we know λ is the right q-EP eigen value of A. the same proof to μ. So, the proof is complete.

Specially, when $A\in C^{nxn}$, if $\lambda(\mu)$ is the left(right) eigen value of A, then $\lambda(\mu)$ is the right(left) eigen value of A .

Note: By the new definition of q-EP multiplication, the left q-EP eigen values of a q-EP matrix is equivalent to its right q-EP eigen value. So they are both called q-EP eigen value of the q-EP matrix.

Theorem 3.9

Let $A,B\in Q_E^{nxn}$ be given. If A~B, then A and B have the same eigen values.

Proof

Since A~B, there exists an invertible matrix $P\in Q_E^{nxn}$ such that $A = P^{-}*B*P$, that is equivalent to $A = P_0^{-1}BP0$ and $A_1 = P_1^{-1}B_1P_1$ (where $A = A_0+A_1j$, $B = B_0+B_1j$, $P = P_0+P_1j$). We know B_s and A_s (s=0,1) have the same eigen values. By theorem 3.5 and 3.6, we can draw that A and B have the same eigen values. The proof is complete.

Theorem 3.10

(the generalized Cayley-Hamilton theorem over q-EP division algebra). A q-EP matrix A must be the root of its characteristic polynomial $f(\lambda)=\|\lambda*\tilde{E}-A\|$.

Proof

According to definition 2.4, we know that

$$f(\lambda) = f(\lambda_0+\lambda_1 j) = \|\lambda*\tilde{E}-A\|$$

$$=\|(\lambda_0E+\lambda_1Ej)-(A_0+A_1j)\|$$

$$=|\lambda_0E-A_0|+|\lambda_1E-A_1|j= g(\lambda_0)+h(\lambda_1)j$$

Where $g(\lambda_0) =|\lambda_0E-A_0|$, $h(\lambda_1)=|\lambda_1E-A_1|$.

According to the Cayley-Hamilton Theorem on complex field, we know $g(A_0)=0$, $h(A_1) = 0$. So $f(A) = g(A_0) + h(A_1) = 0$. It indicates that q-EP matrix A must be the root of its characteristic polynomial $f(\lambda)$. So, the proof is complete.

Theorem 3.11

Let $A = A_0+A_1j\in Q_E^{nxn}$, $A_0, A_1\in Q_E^{nxn}$ be given. A is a diagonalizable matrix if and only if both A_0 and A_1 are diagonalizable matrices.

Proof

A is diagonalizable matrix, that is, there exists an invertible q-EP matrix P such that $A=P^{-}*\Delta*P$. It is equivalent to $A_0 = P_0^{-1}\Delta_0P_0$ and $A_1 = P_1^{-1}\Delta_1P$ (where $\Delta=\Delta_0+\Delta_1j$ is diagonal matrix). So, A is diagonalizable matrix if and only if both A_0 and A_1 are diagonalizable matrices, the proof is complete.

Corollary 3.12

Let A = $A_0+A_1j \in Q_E^{nxn}$ (where $A_0,A_1 \in C^{nxn}$) be given. If A_0 and A_1 both have n different eigen values, then A is diagonalizable matrix.

Corollary 3.13

Let A = $A_0+A_1j \in Q_E^{nxn}$ (where$A_0,A_1 \in C^{nxn}$) be given. A q-EP matrix A is diagonalizable matrix if and only if and A_1 both have n linearly independent eigenvectors.

Corollary 3.14

LetA=$A_0+A_1j \in Q_E^{nxn}$ (where$A_0,A_1 \in C^{nxn}$) be given. q-EP matrix A is diagonalizable matrix if and only if the geometric multiplicity of A_0 and A_1 both equal their algebraic multiplicity respectively.

In section 2, we have given the new definition of the inverse of q-EP matrix, but that of q-EP is not defined. In fact, a quaternion can be treats as a 1x1 matrix. So, we can define the inverse of quaternion as follows.

Definition 3.15

For any q-EP a = a_0+a_1j, if neither of a_0 and a_1 are zeroes, then a^- = $a_0^{-1}+a_1^{-1}j$ is said to be the inverse of a, where a_s^{-1}(s=0,1) is the reciprocal of a_s.

It is easy to verify the following facts. For any a,b$\in$H, we have:

1) $a_u*a = a*a_u=a$;
2) a+b = b+a;
3) (a+b)*c = a*c + b*c
4) $a^n = (a_0)^n+(a_1)^nj$
5) If a = a_0+a_1j has the inverse a^-, then $a*a^-=a_u$

In addition, we discover that there are some special phenomena about the roots at q-EP polynomial under the new definition of q-EP multiplication.

Definition 3.16

The polynomial which has the form as follows:

$a_0*x^{*0}+a_1*x^{*(n-1)}+\ldots..+a_{n-1}*x^{*1}+a_n*x^{*0}$ is said to be q-EP polynomial with complex coefficients (where a_i, i = 0,1,………n are all complex numbers, x = x_0 + x_1j,x^{*0} is the * product of i^{th} quaternion x and x^{*0} is unit q-EP.

Theorem 3.17

Let f(x) be a q-EP polynomial with complex coefficient. Then f(x) has infinite q-EP roots.

Proof

By fundamental theorem of algebra, f(x) exist at least one complex root x_0, then for any given complex number x_1, obviously, x_0+x_1j is the root of f(x).The proof is complete.

Theorem 3.18

Let f(x) be a q-EP polynomial with complex coefficient and A = $A_0+A_1j \in Q_E^{nxn}$ be a given q-EP matrix (where, both A_0 and A_1 are complex matrices). If λ is the eigen value of A_0, then

f(λ) is the eigen value of f(A).

Proof

According to the new definition of sssq-EP multiplication, we can easily obtain f(A) =f(A_0). Since λ is the eigen value of A_0, So f(λ) is the eigen value of f(A_0). The proof is complete.

Under the new concept system, we can also solve the problems of system of linear equation A*X = b, where operator '*' denotes the new multiplication of q-EP matrices.

As we known, for nay A$\in Q_E^{nxn}$,A can be represented uniquely as A = A_0+A_1j, where A_s (s=0,1) are nxn complex matrices. Let

$x = (x_{10}+x_{11}j, x_{20}+x_{21}j, \ldots, x_{n0}+x_{n1}j)^T$ and

$b = (b_{10}+b_{11}j, b_{20}+b_{21}j, \ldots, b_{n0}+b_{n1}j)T$ be nx1 q-EP vectors, then the following theorems are valid.

Theorem 3.19

Let $A = A_0+A_1j \in Q_E^{nxn}$ be given and $X=X_0+X_1j$ be nx1 q-EP vector. If $rank(A_s) = r_s$ and the fundamental system of solution to the system of homogeneous linear equation

$A_{sXs}=0$ is $\eta_{i1}, \eta_{i2}, \ldots \eta_{i(\eta-ri)}$ (s=0,1) respectively, then any solutions to the q-EP system of homogeneous linear equations A*X=0 can be expressed as follows

$X=(C_{01}\eta_{01}+C_{02}\eta_{02}+\ldots+C_{0(\eta-r_0)}\eta_{0(\eta-r_0)})+(C_{11}\eta_{11}+C_{12}\eta_{12}+\ldots C_{1(n-r_1)}\eta_{(n-r_1)})j$

where $C_{st_y} \in C$, $t_s=0,\ldots,n-r_s$, s=0,1.

Proof

By the new definition of q-EP matrix multiplication, the quaternion a system of homogeneous linear equations A*X = 0 is equivalent to the system of homogenous linear sequation $\begin{cases} A_0X_0 = b_0 \\ A_1X_1 = b_1 \end{cases}$. Since any solution to the system of homogenous linear equations $A_sX_s=0$ can be expressed as $X_s = (C_{s1}\eta_{s1}+C_{s2}\eta_{s2}+\ldots+C_{s(\eta-ro)}\eta_{s(\eta-r0)})$

(where $C_{st_s} \in C$, $t_s=1,\ldots,n-r_s$, s=0,1) and the solution of the system of homogenous linear equations A*X = 0 are $X = X_0+X_1j$, so we can draw the conclusion. So, the proof is complete.

Corollary 3.20

Let $A = A_0+A_1j$ be a given q-EP matrix (where $A_s \in c^{nxn}$, s =0,1)

If rank (A_0) = rank (A_1) =n, then the q-EP system of homogenous linear equations A*X=0 has unique solution $X = 0 = \{0,0,\ldots,0\}^T$

Corollary 3.21

Let $A = A_0+A_1j$ be a given q-EP matrix (where $A_s \in c^{nxn}$, s=0,1). If rank $(A_0) < n$ and rank(A_1) = n, then the q-EP system of homogenous linear equations A*X=0 only exists complex solution.

Theorem 3.22

Let $A = A_0+A_1j$ be a given q-EP matrix, $X = X_0+X_1j$ and $b = b_0+b_1j$ be q-EP vectors. (where $A_s \in c^{nxn}$, $X_s \in c^{nxn}$, $bs=(b_{s1},b_{s2},\ldots,b_{sn})^T$, $b_{st} \in c$, s = 0,1, t =1,2,…..n). If there is at least one $s_0 \in \{0,1\}$ such that rank $(A_{s0}) <$ rank$(A_{s0} \vdots b_{s0})$, then the q-EP system of linear equation A*X = b has no solution.

Proof

By the new definition of q-Ep matrix multiplication the q-EP system of linear equations A*X=b is equivalent to the system of linear equations $\begin{cases} A_0X_0 = b_0 \\ A_1X_1 = b_1 \end{cases}$, since

rank$(A_{s0}) <$ rank$(A_{s0} \vdots b_{s0})$, the system of linear equations $\begin{cases} A_0X_0 = b_0 \\ A_1X_1 = b_1 \end{cases}$ have no solutions A*X = b has no solution. So, the proof is complete.

Theorem 3.23

Let $A = A_0+A_1j$ be a given q-EP matrix and $X = X_0 + X_1j$ be a given q-EP vector

(where $A_s \in c^{nxn}$, $X_s \in c^{nxt}$, s=0,1). We suppose that the fundamental system of solutions to the system of linear equations $A_s X_s = 0$ is $\eta_{s1}, \eta_{s2}, \ldots \eta_{s(n-rs)}$ (s=0,1) respectively and ξ_s (s=0,1) is a special solution of the system of linear equations $A_{sxs} = b$, respectively, and rank(A_s) = rank$(A_s \vdots b_s)$ (s=0,1), then any solution to the q-EP system of linear equations A*X=b can be expressed as

$X=(\xi_0+C_{01}\eta_{01}+C_{02}\eta_{02}+\ldots+C_{0(n-r0)}\eta_{(n-r0)})+(\xi_1+C_{11}\eta_{11}+C_{12}\eta_{12}+\ldots C_{1(n-n)}\eta_{1(n-\eta)})j$

The proof is complete.

Theorem 3.24

Let $A = A_0+A_1j$ be a given q-EP matrix, $X = X_0+X_1j$ and $b=b_0+b_1j$ be q-EP vectors (where $A_s \in C^{nxn}$, $x_s \in C^{nxn}, b_s \in c^{nx1}$, (s = 0,1). If rank($a_s$) = rank($A_s \vdots b_s$) = n(s = 0,1), then the q-EP system of linear equations A*X = b exists unique solution.

Proof

By the new definition of q-EP matrix multiplication, the q-EP system of linear equations A*X=b is equivalent to the system of linear equations $\begin{cases} A_0 X_0 = b_0 \\ A_1 X_1 = b_1 \end{cases}$ and rank(A_s) = rank($A_s \vdots B_s$)=n, we know the system of linear equations $\begin{cases} A_0 X_0 = b_0 \\ A_1 X_1 = b_1 \end{cases}$ have unique solution. So the q-EP system of linear equations A*X = b exists unique solution.

Corollary 3.25

Let $A = A_0+A_1j$ be a given nxn q-EP matrix and $b = b_0+b_1j$ be a given nx1 q-EP vector. If rank(A_s) = rank($A_s \vdots B_s$) = n (s=0,1), then the solution of the system of equations A*X=b is

$X = A^{-1} * b_0$.

Corollary 3.26

Let $A \in Q_E^{nxn}$ and $b = b_0 + b_1j$ (where $b_s \in C^{nx1}$, $s=0,1, b_1 \neq 0$) be given. Then the q-EP system of linear equations A*X = b has no solution.

Corollary 3.27

Let $A \in Q_E^{nxn}$ and $b \in C^{nx1}$ be given. If rank(A) = rank($A \vdots b$), then any solution to the q-EP system of linear equations A*X = b can expressed as $X = a^{-1}b+aj$, where $a \in C_s$.

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