The Existence of Concurrent Vector Fields in a Finsler Space-I

Birendra Kumar Singh[1], Prakash Chandra Yadav[2], Ashwini Dwivedi

[1,2] Department of Mathemataics, K S Saket P G college, Ayodhya, Faizabad (India)

Abstract: *In this paper a relationship between concurrent and P-concurrent vector fields in a Finsler space has been established. The present work studies the properties of concurrent vector fields [7] in Definition 1.2. Moreover, paper explains a vector field which we shall call P-concurrent vector field in a Finsler space. It generalizes the concept of concurrent vector field.*

Key words: *Finsler space, concurrent vector fields, P-concurrent vector fields.*

1. Introduction :

Let F^n be an n-dimensional Finsler space with metric function *L(x, y)*, metric tensor $g_{ij}(x,y)$, angular metric tensor h_{ij} and torsion tensor C_{ijk}. The h-and v-covariant derivatives of a vector field X_i are defined as Rund [8] :

a) $X_i|_j = \partial_j X_i - N^r_j \Delta_r X_i - X_r F^r_{ij}$,

b) $X_i \mid_j = \Delta_j X_i - X_r C^r_{ij}$,

where $N^r_j = F^r_{oj}$, ∂_j and Δ_j respectively denote partial differentiation with respect to x^j and y^j, such that an index o means contraction by *y*.

Concurrent vector fields in a Finsler space were first of all defined and studied by Tachibana [9] followed by Matsumoto [4] and others in the following form:

Definition 1.1: A vector field X^i is said to be concurrent in a Finsler space F^n if it satisfies:

(i) X^i is independent of y^i,

(ii) $X^i C_{ijk} = 0$,

(iii) $X^i\big|_j = -\delta^i_j$.

In this paper we generalize the concept of concurrent vector field and give the alternative definition as follows:

Definition 1.2: A vector field X^i is said to be p-concurrent in a Finsler space F^n if it satisfies :

(i) X^i is independent of y^i,

(ii) $X^i A_{ijk} = \alpha h_{jk}$,

(iii) $X^i\big|_j = -\delta^i_j$.

where □ is an arbitrary non-zero scalar function of *x* and *y*, $A_{ijk} = LC_{ijk}$ and $A_j = LC_j$.

The purpose of the present work is to study properties of concurrent vector fields [7] in Definition 1.2. Furthermore, we have defined and studied a vector field which we shall call P-concurrent vector field in a Finsler space. It is to be noted that P-concurrent vector field shall be different from concurrent vector field. In addition to this we have established a relationship between concurrent and P-concurrent vector fields in a Finsler space.

2. Some properties of concurrent vector fields:

We know that in a two dimensional Finsler space, A_{ijk} is expressed as [8]

(2.1) $$A_{ijk} = LC\, C_i\, C_j\, C_k \cdot$$

Let X^i be a vector in F^2, whose magnitude is *X* and which makes an angle θ with the direction of unit vector l^i, then it can be expressed as

(2.2) $$X^i = X(l^i \cos\theta + m^i \sin\theta),$$

which gives $X^i C_i = XC \sin\theta$. Now using Definition 1.2, equations (2.1) and (2.2), on simplification we get $LXC^4 \sin\theta = \alpha$. Hence we have:

Theorem 2.1: In a two dimensional Finsler space, the concurrent vector field X^i given by (2.2) in general satisfies $LXC^4 \sin\theta = \alpha \cdot$

In case $\theta = \pi/2, \alpha = LC^5$. Hence we have :

Corollary : In a two dimensional Finsler space, if the vector X^i is parallel to C^i, the parameter $\alpha = LC^5$.

In a three dimensional Finsler space A_{ijk} is expressed as [6]

(2.3) $$A_{ijk} = L[C_{(1)} m_i m_j m_k - C_{(2)}(m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) + C_{(3)}(m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + C_{(2)} n_i n_j n_k]$$

Let X^i be a vector field in F^3, which is represented by

(2.4) $$X^i = (l^i \cos\theta + m^i \cos\phi + n^i \cos\psi),$$

where cos θ, cos ϕ and cos ψ are direction cosines of this vector. Applying Definition 1.2, together with equations (2.3) and (2.4), we get

(2.5) $$L(C_{(1)} + C_{(3)}) \cos\phi = 2\alpha \cdot$$

Hence we have :

Theorem 2.2: In a three dimensional Finsler space, the concurrent vector field X^i given by (2.4), satisfies (2.5).

If we choose $X^i m_i = 1$, equation $\cos^2\theta + \cos^2\phi, \ + \cos^2\psi = 1$, leads to $\cos^2\theta + \cos^2\psi = 0$, which on simplification gives

(2.6) $$\cos(\theta+\psi)\ \cos(\theta-\psi) = -1.$$

Hence we have:

Theorem 2.3: In a three dimensional Finsler space, if the concurrent vector field X^i is in the direction of unit vector m^i, the vectors with direction consines cos (θ+ψ) and cos (θ-ψ) shall be orthogonal.

If the given Finsler space is P-reducible [6], it satisfies for $A_{ijk|o} = P_{ijk}$

(2.7) $$P_{ijk} = (n+1)^{-1}(A_{k|o}h_{ij} + A_{i|o}h_{jk} + A_{j|o}h_{ki}),$$

which by virtue of Definition 1.2, we have

(2.8) $$X^i A_{i|o} = (n-1)\,\alpha_{|o}.$$

Hence we have :

Theorem 2.4: In a P-reducible Finsler space of n-dimensions, concurrent vector field X^i satisfies equation (2.8).

Remark: In a three dimensional Finsler space, on substituting for X^i from (2.4) and, $A_{i|o} = L(C_{|o}m_i + Cn_i h_o)$, we obtain $L(C_{|o}\cos\phi + C h_o \cos\psi) = 2\alpha_{|o}.$

In a three dimensional P-reducible Finsler space of R-K type P_{ijk} is expressed as [6].

(2.9) $$P_{ijk} = L[C_{(3)|o}\{3m_i m_j m_k + (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j)\}$$
$$+ C_{(3)}h_o\{(m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) + 3n_i n_j n_k\}],$$

which for a concurrent vector field gives

(2.10) $$L\{\cos\phi\, C_{(3)|o} + \cos\psi\, C_{(3)}h_o\} = 2\alpha_{|o}.$$

Hence we have :

Theorem 2.5: A concurrent vector field in a three dimensional P-reducible Finsler space of R-K type satisfies (2.10).

It is known that in a three dimensional P-reducible Finsler space of R-K type [6], the curvature tensor P_{ijkh} is symmetric in *k* and *h* if and only if $C_{(3)}\,C_{(3)|o} = 0$. Excluding $C_{(3)} = 0$, which implies that the given space is a Riemannian space, we consider the case of $C_{(3)|o} = 0$, for which equation (2.10) gives $L\ \cos\psi\, C_{(3)}h_o = 2\alpha_{|o}$. Hence we have :

Theorem 2.6: If the hv-curvature tensor of a P-reducible Finsler space F^3 of R-K type is symmetric in last two indices, F^3 will admit a concurrent vector field if and only if the parameter is given by $L\cos\psi\, C_{(3)} h_0 = 2a_{|0}$.

It is known that a C2-like Finsler space [5] satisfies following condition

(2.11) $$C_{ijk} = C^{-2}\, C_i C_j C_k,$$

which on comparing with Definition 1.2 and covariant differentiation leads to $C_{|0}(C^2 - 1)\alpha = 0$. Hence we have :

Theorem 2.7: A C2-like Finsler space F^3 admitting a concurrent vector field satisfies $C_{|0} = 0$.

A Finsler space $F^n (n \geq 4)$ is called S3-like, if there exists a scalar S such that v-curvature tensor is written in the form

(2.12) $$L^2 S_{ijkh} = S(h_{ik}\, h_{jh} - h_{ih}\, h_{jk})$$

This for a concurrent vector field X^r can be written as

(2.13) $$L^2 S_{ijkh} = S\,\alpha^{-2} X^r X^s (A_{rik} A_{sjh} - A_{rjk} A_{sih}).$$

Which leads to

(2.14) $$\alpha^2 = (C_r C_s - C^j_{si}\, C^i_{rj})\, X^r\, X^s.$$

Hence we have:

Theorem 2.8: In a S3-like Finsler space F^n admitting a concurrent vector field X^i, coefficient □ is expressed as in (2.14).

From equation $X^i{}_{|j} = -\delta^i_j$ of Definition 1.2, we can obtain

(2.15) $$X^i{}_{|j|k} - X^i{}_{|k|j} = X^r K^i_{rjk} = 0,$$

which gives

(2.16) $$X^r{}_{|m}\, K^i_{rjk} + X^r K^i_{rjk|m} = 0.$$

From equation (2.15) and (2.16) we obtain

(2.17) $$X^r\, K^i_{rjk|m} = K^i_{mjk}, \qquad X^r X^m K^i_{rjk|m} = 0$$

Hence we have :

Theorem 2.9: If X^i is a concurrent vector field in a Finsler space F^n, the curvature tensor K^i_{rjk} satisfies equation (2.17).

If we assume that the curvature tensor K^i_{rjk} is recurrent, it satisfies [8]

(2.18) $$K^i_{rjk|m} = \lambda_m K^i_{rjk}$$

From equations (2.15), (2.16) and (2.18) we obtain $K^i_{rjk} = 0$, which should not happen because of the definition of recurrency. Hence we have :

Theorem 2.10: If X^i is a concurrent vector field in a Finsler space F^n, the curvature tensor K^i_{rjk} cannot be recurrent.

3. P-concurrent vector fields in a Finsler space:

Definition 3.1: a vector field X^i in a Finsler space F^n with a non-vanishing tensor P_{ijk} will be defined as a P-concurrent vector field if it satisfies

i) X^i is independent of y^i,

(3.1) ii) $X^i P_{ijk} = 0$

iii) $X^i|_j = -\delta^i_j$

Now we shall consider the existence of a P-concurrent vector field in a Finsler space of two and three dimensions.

Two dimensional Finsler Space F^2: Assuming X^i is a P-concurrent vector field in F^2, we take

(3.2) $$X^i P_{ijk} = 0 \cdot$$

In a two dimensional Finsler space we have (Matsumoto [5]) $P_{ijk} = P\ C_{ijk}$, therefore from (3.2) we obain either $X^i C_{ijk} = 0$ or P=0, which in turn gives $P_{ijk} = 0$. Thus we have:

Theorem 3.1: In a two dimensional non-Riemannian Finsler space, a P-concurrent vector field does not exist.

Three dimensional Finsler space F^3: In a three dimensional Finsler space any vector can be expressed as

(3.3) $$X^i = \theta l^i + \beta m^i + \gamma n^i.$$

Differentiating C_i, covariantly, by the use of Cartan's covariant derivative and $m_{i|j} = n_i h_j$ [6], we get by virtue of (3.3), $\beta C_{|0} + \gamma C h_0 = 0$. Further if we differentiate equation (3.3) and simplifying we get $\beta_{|0} = \gamma h_0$ and $\gamma_{|0} = -\beta h_0$. Hence we have :

Theorem 3.2: In a non-Riemannian Finsler space of three dimensions admitting a P-concurrent vector field X^i, the sum of the squares of the parameters □ and γ is covariantly constant.

In a three dimensional Finsler space F^3, the torsion tensor P_{ijk} is expressed by Rastogi and Kawaguchi [6] as follows :

(3.4)
$$\begin{aligned} P_{ijk} &= L[(C_{(1)|0} + 3C_{(2)}h_0)m_i m_j m_k \\ &- \{C_{(2)|0} - (C_{(1)} - 2C_{(3)})h_0\}(m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) \\ &+ (C_{(3)|0} - 3C_{(2)}h_0)(m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) \\ &+ (C_{(2)|0} + 3C_{(3)}h_0)n_i n_j n_k] \end{aligned}$$

Multiplying equation (3.4) by X^i, we get on simplification

(3.5)
$$\beta(C_{(1)|0} + C_{(3)|0}) + \gamma(C_{(1)} + C_{(3)})h_0$$

and

(3.6)
$$\gamma_{|0}(C_{(1)|0} + C_{(3)|0}) = \beta_{|0}(C_{(1)} + C_{(3)})h_0.$$

Hence we have :

Theorem 3.3: In a three dimensional Finsler space admitting a P-concurrent vector field, the covariant derivatives of the parameters □ □ □γ and coefficients $C_{(1)}, C_{(3)}$ are related by (3.6).

P-reducible Finsler space: If the given Finsler space is P-reducible it satisfies

(3.7)
$$P_{ijk} = (n+1)^{-1}(A_{k|0}h_{ij} + A_{i|0}h_{jk} + A_{j|0}h_{ki})$$

From equation (3.1) and (3.7), one can write

(3.8)
$$X^i(A_{k|0}h_{ij} + A_{i|0}h_{jk} + A_{j|0}h_{ki}) = 0.$$

which on multiplication by g^{jk} gives $X^i A_{i|0} = 0$. This in turn by virtue of [6]

(3.9)
$$C_{i|0} = (C_{(1)|0} + C_{(3)|0})m_i + C_{(1)} + C_{(3)})h_0 n_i,$$

and (3.3) will give (3.5). Hence we have :

Theorem 3.4: A P-concurrent vector field in a three dimensional P-reducible Finsler space satisfies (3.5).

P-reducible Finsler space of R-K type: From equation (2.10) for a P-concurrent vector field, X^i gives

(3.10)
$$\beta C_{(3)|0} + \gamma C_{(3)}h_0 = 0.$$

Hence we have :

Theorem 3.5: A P-concurrent vector field in a P-reducible Finsler space of R-K type satisfies (3.10).

Similar to Theorem 2.6, with the help of equations (2.10) and (3.10), we can write :

Theorem 3.6: If hv-curvature tensor of a P-reducible Finsler space F^3 is symmetric in last two indices, F^3 will admit a P-concurrent vector field if and only if either $\gamma = 0$ or $h_0 = 0$.

4. Curvature properties of P-concurrent vector fields in F^n: Using Ricci-identity [8]

(4.1) $$X^i{}_{|k|m} - X^i{}_{|m|k} = X^h R^i_{hkm} - X^i|_h R^h_{km},$$

with the help of Definition (2.1), we can obtain

(4.2) $$X^h (R^i_{hkm} - C^i_{hr} R^r_{km}) = 0.$$

Applying Cartan's covariant derivative and using the Definition 3.1 of P-concurrent vector field, we can obtain

(4.3) $$X^h (R^i_{hkm|0} - C^i_{hr} R^r_{km|0}) = y^h R^i_{hkm}.$$

Hence we have :

Theorem 4.1: A Finsler space F^n admitting a P-concurrent vector field satisfies equations (4.2) and (4.3).

Similarly for a P-concurrent vector field we obtain from the Ricci-identity [8]

(4.4) $$X^i{}_{|k}|_m - X^i|_{m|k} = X^h P^i_{hkm} - X^i|_h P^h_{km} - X^i|_h C^h_{km},$$

(4.5) $$X^h (C^i_{hm|k} - C^i_{hr} P^r_{km}) = 0,$$

Applying Cartan's covariant derivative on (4.5) and using Definition 3.1 we get

(4.6) $$X^h (C^i_{hm|k|0} - C^i_{hr} P^r_{km|0}) = 0$$

By on substituting the value of $C^i_{hm|k|0}$, we get

(4.7) $$X^h (C^r_{hm} R^i_{rk0} - C^i_{rm} R^r_{hk0} - C^i_{rh} R^r_{mk0} - C^i_{rh} P^r_{km|0} - C^i_{hm|p} R^p_{k0}) + P^i_{km} = 0$$

Hence we have :

Theorem 4.2: A Finsler space F^n admitting a P-concurrent vector field satisfies equations (4.5), (4.6) and (4.7).

If we differentiate T-tensor obtained by Matsumoto [5] and Kawaguchi [1]

(4.8) $$T_{hijk} = LC_{hij}|_k + I_h C_{ijk} + l_i C_{hjk} + l_j C_{hik} + l_k C_{hij}$$

with respect to Cartan's covariant derivative and contracting it, we get

(4.9) $$T_{hijk} = LC_{hij}|_{k|0} + l_h P_{ijk} + l_i P_{hjk} + l_j P_{hik} + l_k P_{hij}.$$

Multiplyin equation (4.9) by X^h and using Definition 3.1 we get after simplification

(4.10) $$X^h \{T_{hijk|O} - L(P_{hij}|_k - C_{hij|k}) - l_h P_{ijk}\} = O$$, and

(4.11) $$X^i X^h \{T_{hijk|O} - L(P_{hij}|_k - C_{hij|k})\} = O.$$

Hence we have:

Theorem 4.3: A Finsler space F^n admitting a P-concurrent vector field satisfies equations (4.10) and (4.11).

If we assume that a concurrent vector field is also P-concurrent vector field, by the use of Definition 3 and Definition 3.1, we get either $\alpha_{|O} = O$ or $h_{jk} = O$. Hence we have the following:

Theorem 4.4: In a non-Riemannian Finsler space F^n the necessary condition for a vector field to be both concurrent as well as P-concurrent is that the scalar $\alpha_{|O}$ vanishes.

The v-curvature tensor in a Finsler space F^n is given as [5]

(4.12) $$S^i_{hjk} = C_{(,k)}\{C^m{}_{hk} P^i_{mj}\}] = O$$

where means interchange of indices j and k and subtraction. Taking Cartan's covariant derivative of (4.12) with respect of and multiplying the resulting equation by and and using Definition 3.1, we get

(4.13) $$X^h [S^i_{hjk|O} - C_{(j,k)}\{C^m{}_{hk} P^i{}_{mj}\}] = O$$

Further taking Cartan's covariant derivative of Ricci-identity[5]

(4.14) $$X^i|_j|_k - X^i|_k|_j = X^h S^i_{hjk}$$

with respect to x^t, then with the help of Definition 3.1, we can write :

(4.15) $$X^h S^i_{hjk|O} = C_{(j,k)}\{X^h C^i_{hj}|_{k|O} + X^h|_k C^i_{hj}\}$$

Hence we have:

Theorem 4.5: The v-curvature tensor of a Finsler space F^n admitting a P-concurrent vector field satisfies equations (4.13) and (4.15).

In an n-dimensional Finsler space F^n the hv-curvature tensor is given by [5]

(4.16) $$P_{ijkh} = C_{(i,j))}\{A_{jkh|i} + A_{ikr} P^r_{jh}\}$$

and

(4.17) $$P_{ijkh} - P_{ijhk} = -S_{ijkh|O}.$$

Using equations (4.13), (4.16) and (4.17) together with the Definition 3.1 of P-concurrent vector field, we obtain

(4.18) $$X^h C_{(j,k)} \{P_{hijk|0} - C^m_{hk} P_{mij|0}\} = 0$$

Hence we have :

Theorem 4.6: The hv-curvature tensor of an n-dimensional Finsler space F^n admitting a P-concurrent vector field satisfies (4.18).

5. Some Special Cases :

C2-like Finsler spaces: If we assume that X^i is a P-concurrent vector field in a C2-like Finsler space, equation (2.11) after differentiation and by Definition 3.1, we get :

Theorem 5.1: The necessary condition or a C2-like Finsler space F^n to admit a P-concurrent vector field is that it satisfies $X^i C_{i\,|0} = 0$.

It is known that in a C2-like Finsler space the v-curvature tensor is zero, therefore by virtue of (4.17), we can observe that P_{ijkh} is symmetric in k and h. Now using the Definition 3.1 of P-concurrent vector field in equation (4.16), we can write

(5.1) $$X^i (C_{ikr} P^r_{jh} - C_{ihr} P^r_{jk}) = 0, \qquad X^i (C_{ikr} P^r_{jh\,|0} - C_{ihr} P^r_{jk|0}) = 0 \cdot$$

Hence we have :

Theorem 5.2: The necessary condition for a C2-like Finsler space F^n to admit a P-concurrent vector field is that it satisfies (5.1).

P2-like Finsler spaces: It is known that in a P2-like Finsler space F^n there exists a covariant vector field P_i such that P_{ijkh} is given by [2]

(5.2) $$P_{ijkh} = P_i C_{jkh} - P_j C_{ikh}$$

and satisfies either $P_{ijkh} = 0$ or $S_{ijkh} = 0$.

From equation (5.2) we can write $P_{ijk} = P_0 C_{ijk}$. If we assume that X^i is a P-concurrent vector field by virtue of Definition 3.1, we get

(5.3) $$X^i C_{ijk} P_0 = 0 \cdot$$

As $X^i C_{ijk} \neq 0$, we must have $P_0 = 0$, which following [2], gives $P_i = 0$. Hence we have :

Theorem 5.3: In a P2-like Finsler space F^n, admitting a P-concurrent vector field X^i, hv-curvature tensor vanishes.

If we assume that in a P2-like Finsler space X^i is a concurrent vector field, applying Definition 3 we can write $P_0 X^i A_{ijk} = \alpha_{|0} h_{jk}$, which gives

(5.4) $$P_0 X^i A_i = (n-1)\alpha_{|0}$$

Hence we have:

Theorem 5.4: In a P2-like Finsler space F^n, admitting a concurrent vector field X^i, vanishing of P_0 is the sufficient condition for the vector field to be P-concurrent.

S3-like Finsler spaces: Using definition of v-curvature tensor in a Finsler space Matsumoto [5], together with equation (2.12) and Definition 3.1, we obtain

(5.5) $$X^i X^h [(n-2)S_{|0} h_{ih} + A_{ihr} A^r_{|0}] = 0$$

Hence we have :

Theorem 5.5: A P-concurrent vector field X^i in a S3-like Finsler space F^n, satisfies equation (5.5).

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