New forms of compactness via grills

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Abstract: *In the present paper, we introduce and study new types of compactness with respect to a grill, namely $\rho\mathcal{G}$ – compactness and $\sigma\mathcal{G}$ – compactness. Several of its properties are investigated and effects of various kinds of functions on them are studied.*

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1. Introduction

In the present paper, we consider a topological space equipped with a grill, a brilliant notion that has been initiated by Choquet [1]. A grill $\mathcal{G}$ on a topological space X is a collection of subsets of X satisfying following conditions: (1) $\phi \notin \mathcal{G}$, (2) $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$, and (3) $A \notin \mathcal{G}$ and $B \notin \mathcal{G} \Rightarrow A \cup B \notin \mathcal{G}$. $\mathcal{G}(\{\phi\}) := P(X) - \{\phi\}$ and ϕ are trivial examples of grills. Some useful grills are (i) $\mathcal{G}_\infty$, the grill of all infinite subsets of X, (ii) $\mathcal{G}_{co}$, the grill of all uncountable subsets of X, (iii) $\mathcal{G}_p$, a particular point grill on X.

A topological space (X, τ) with a grill $\mathcal{G}$ on X will be denoted by $(X, \tau, \mathcal{G})$. Roy and Mukherjee [6, 7] defined a topology obtained as an associated structure on a topological space (X, τ) induced by a grill on X. According to them, for $A \in P(X)$, $\Phi\mathcal{G}(A, \tau)$ or $\Phi\mathcal{G}(A)$ or simply $\Phi(A)$ is the set $\{ x \in X : A \cap U \in \mathcal{G}$, for every open neighborhood U of $x\}$. For a grill space $(X, \tau, \mathcal{G})$, the $\mathcal{B} = \{U - A : U \in \tau$ and $A \notin \mathcal{G}\}$ is a base for the topology $\tau\mathcal{G}$, finer than τ.

The concept of compactness modulo an ideal was introduced by Newcomb [4]. In 2006, Gupta and Noiri [2] investigated C-compactness modulo an ideal. Recently, Pachón [5] defined two new forms of strong compactness in terms of ideals. Using his idea, we introduce $\rho\mathcal{G}$ – compactness and $\sigma\mathcal{G}$ – compactness in terms of grills.

A subset A of a space (X, τ) is said to be g– closed [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$. It is clear that every closed set is g– closed, but the converse is not true.

If $(X, \tau, \mathcal{G})$ is a grill space, (Y, β) a topological space and $f : X \to Y$ a function, then $f(\mathcal{G}) = \{ f(G) : G \in \mathcal{G}\}$ is a grill on Y.

Let (X, τ) be a topological space and $A \subseteq X$ then cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ).

2. $\rho\mathcal{G}$ – Compact Spaces

We recall that a subset A of a grill space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$ – compact [7] if for every cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of A by elements of τ, there exists finite $\Lambda_0 \subseteq \Lambda$, such that $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. The grill space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$ – compact if X is $\mathcal{G}$ – compact.
It is clear that (X, τ) is compact if and only if $(X, \tau, \mathcal{G}(\{\phi\}))$ is $\mathcal{G}(\{\phi\})$ – compact. If (X, τ) is compact then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$ – compact.

***Definition 2.1*:** For a grill space $(X, \tau, \mathcal{G})$ and $A \subseteq X$, A is said to be $\rho\mathcal{G}$ – compact if for every family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$ then there exists finite $\Lambda_0 \subseteq \Lambda$, such that $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. The grill space $(X, \tau, \mathcal{G})$ is said to be $\rho\mathcal{G}$ – compact if X is $\rho\mathcal{G}$ – compact.

It is obvious that (X, τ) is $\rho\mathcal{G}$ – compact if and only if $(X, \tau, \mathcal{G}(\{\phi\}))$ is $\rho\mathcal{G}(\{\phi\})$– compact and that if $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$ – compact. The converse is not true.

***Example 2.2*:** Let $X = [0,+\infty)$, $\tau = \{ \phi, X \} \cup \{(r, +\infty) : r \geq 0\}$ and $\mathcal{G} = \mathcal{G}_\infty$ then :

i) $(X, \tau, \mathcal{G})$ is $\mathcal{G}$ – compact, because if $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover of X, then there exists $\Lambda_0 \subseteq \Lambda$ with $V\Lambda_0 = X$ and hence $X - V\Lambda_0 \notin \mathcal{G}$.

ii) $(X, \tau, \mathcal{G})$ is not $\rho\mathcal{G}$ – compact, because $X - \bigcup_{r>0}(r,+\infty) = \{0\} \notin \mathcal{G}$, but if n is a positive integer and $0 < r_1 < r_2 < \ldots < r_n$, then $X - \bigcup_{i=1}^{n}(r_i,+\infty) = X - (r_1,+\infty) \in \mathcal{G}$.

Definition 2.3: A subset A of a grill space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}g$– closed if for every $U \in \tau$, if $A - U \notin \mathcal{G}$ then $\mathrm{cl}(A) \subseteq U$.

It is easy to check if A is $\mathcal{G}g$– closed then A is g– closed. The converse is not true.

Example 2.4:**1)** Let $X = R$, the set of real numbers, $\tau = \{\phi, R\} \cup \{(r, +\infty) : r \in R\}$, $\mathcal{G} = \{B : B \subseteq R - Q\}$ and if $A = Q$, then

(a) A is g– closed because if $U \in \tau$ and A is a subset of U then $U = R$ and $\mathrm{cl}(A) = \mathrm{cl}(Q) = R \subseteq U$.

(b) A is not $\mathcal{G}g$ – closed since $A - (0,+\infty) \notin \mathcal{G}$ and $\mathrm{cl}(A) = R \not\subset (0,+\infty)$.

2) $X = \{0,1,2\}$, $\tau = \{\phi, \{0\}, \{1\}, \{0,1\}, X\}$, $\mathcal{G} = \{\{2\}, \{0,2\}, \{1,2\}, X\}$ and $A = \{2\}$, then A is $\mathcal{G}g$ – closed because if $U \in \tau$ and $A - U \notin \mathcal{G}$ then we have $A \subseteq U$, and so $U = X$ and $\mathrm{cl}(A) \subseteq U$.

Theorem 2.5: For a grill space $(X, \tau, \mathcal{G})$ and a base $\mathcal{B}$ for τ, $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact if and only if all family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open sets in $\mathcal{B}$, if $X - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$ then there exists finite $\Lambda_0 \subseteq \Lambda$, such that $X - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$.

Proof : Necessary part is easy to prove. For sufficiency, let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of non-empty open subsets of X such that $X - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. For all $\alpha \in \Lambda$ there exists a family $\{W_{\alpha\beta} : \beta \in \Lambda_\alpha\}$ of elements in $\mathcal{B}$ such that $V_\alpha = \bigcup_{\beta\in\Lambda_\alpha} W_{\alpha\beta}$. Given that $X - \bigcup_{\alpha\in\Lambda}\bigcup_{\beta\in\Lambda_\alpha} W_{\alpha\beta} \notin \mathcal{G}$ and $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact, there exists $W_{\alpha_1\beta_1}, W_{\alpha_2\beta_2}, \ldots, W_{\alpha_n\beta_n}$ such that $X - \bigcup_{i=1}^{n} W_{\alpha_i\beta_i} \notin \mathcal{G}$. But $X - \bigcup_{i=1}^{n} V_{\alpha_i} \subseteq X - \bigcup_{i=1}^{n} W_{\alpha_i\beta_i}$ and so $X - \bigcup_{i=1}^{n} V_{\alpha_i} \notin \mathcal{G}$.

Theorem 2.6: For a grill space $(X, \tau, \mathcal{G})$, the following are equivalent:

(a) $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact.
(b) $(X, \tau_{\mathcal{G}}, \mathcal{G})$ is $\rho\mathcal{G}$ – compact.
(c) For any family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of X, if $\bigcap_{\alpha\in\Lambda} F_\alpha \notin \mathcal{G}$ then there exists finite $\Lambda_0 \subseteq \Lambda$, such that $\bigcap_{\alpha\in\Lambda_0} F_\alpha \notin \mathcal{G}$.

Proof : (a) $\Leftrightarrow$ (c) This is easy to prove.

(b) $\Rightarrow$ (a) Obvious, since $\tau_{\mathcal{G}}$ is finer than τ.

(a) $\Rightarrow$ (b) We know that the $\mathcal{B} = \{U - A : U \in \tau$ and $A \notin \mathcal{G}\}$ is a base for the topology $\tau\mathcal{G}$. Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of non-empty open subsets in $\mathcal{B}$ such that $X - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. For each $\alpha \in \Lambda$ there exists $W_\alpha \in \tau$ and $A_\alpha \notin \mathcal{G}$ such that $V_\alpha = W_\alpha - A_\alpha$. Since $X - \bigcup_{\alpha\in\Lambda} W_\alpha \notin \mathcal{G}$ then there exists finite $\Lambda_0 \subseteq \Lambda$, such that $X - \bigcup_{\alpha\in\Lambda_0} W_\alpha \notin \mathcal{G}$. Now $X - \bigcup_{\alpha\in\Lambda_0} V_\alpha \subseteq (X - \bigcup_{\alpha\in\Lambda_0} W_\alpha) \cup (\bigcup_{\alpha\in\Lambda_0} A_\alpha) \notin \mathcal{G}$ and so $X - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. Therefore $(X, \tau_{\mathcal{G}}, \mathcal{G})$ is $\rho\mathcal{G}$ – compact.

Theorem 2.7: Let $(X, \tau, \mathcal{G})$ be $\rho\mathcal{G}$ – compact and $A \subseteq X$ be $\mathcal{G}g$– closed set. Then A is $\rho\mathcal{G}$ – compact.

Proof : Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets in A such that $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Since A is $\mathcal{G}g$– closed, $\mathrm{cl}(A) \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$ then $X = (X - \mathrm{cl}(A)) \cup \bigcup_{\alpha\in\Lambda} V_\alpha$ and so $X - [(X - \mathrm{cl}(A)) \cup \bigcup_{\alpha\in\Lambda} V_\alpha] = \phi \notin \mathcal{G}$. Given that X is $\rho\mathcal{G}$ – compact, there exists finite $\Lambda_0 \subseteq \Lambda$, such that $X - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$ or

$X - [(X - \text{cl}(A)) \cup \bigcup_{\alpha\in\Lambda_0} V_\alpha] \notin \mathcal{G}$. But $X - [(X - \text{cl}(A)) \cup \bigcup_{\alpha\in\Lambda_0} V_\alpha] = \text{cl}(A) - \bigcup_{\alpha\in\Lambda_0} V_\alpha$ and since $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \subseteq \text{cl}(A) - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$, we have $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G} \Rightarrow A$ is $\rho\mathcal{G}$ – compact.

Theorem 2.8: For $\rho\mathcal{G}$ – compact subsets A and B of a grill space $(X, \tau, \mathcal{G})$, $A \cup B$ is $\rho\mathcal{G}$ – compact.

Proof: Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets such that $(A \cup B) - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Since $A - \bigcup_{\alpha\in\Lambda} V_\alpha \subseteq (A \cup B) - \bigcup_{\alpha\in\Lambda} V_\alpha$ and $B - \bigcup_{\alpha\in\Lambda} V_\alpha \subseteq (A \cup B) - \bigcup_{\alpha\in\Lambda} V_\alpha$, therefore $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$ and $B - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$, and hence there exist, both finite $\Lambda_1 \subseteq \Lambda$ and $\Lambda_2 \subseteq \Lambda$, with $A - \bigcup_{\alpha\in\Lambda_1} V_\alpha \notin \mathcal{G}$ and $B - \bigcup_{\alpha\in\Lambda_2} V_\alpha \notin \mathcal{G}$. This implies that $A - \bigcup_{\alpha\in\Lambda_1\cup\Lambda_2} V_\alpha \notin \mathcal{G}$ and $B - \bigcup_{\alpha\in\Lambda_1\cup\Lambda_2} V_\alpha \notin \mathcal{G}$, and so $(A \cup B) - \bigcup_{\alpha\in\Lambda_1\cup\Lambda_2} V_\alpha \notin \mathcal{G}$.

Theorem 2.9: Let $(X, \tau, \mathcal{G})$ be a grill space and $A \subseteq X$. Suppose that for all $U \in \tau$, if $A - U \notin \mathcal{G}$ then there exist $B \subseteq X$ such that B is $\rho\mathcal{G}$ – compact, $A \subseteq B$ and $B - U \notin \mathcal{G}$. Then A is $\rho\mathcal{G}$ – compact.

Proof: Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of X such that $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. There exists $B \subseteq X$ such that B is $\rho\mathcal{G}$ – compact, $A \subseteq B$ and $B - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. There exists finite $\Lambda_0 \subseteq \Lambda$, such that $B - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. Since $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \subseteq B - \bigcup_{\alpha\in\Lambda_0} V_\alpha$ we have $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$.

Theorem 2.10: Let $(X, \tau, \mathcal{G})$ be a grill space and $A \subseteq B \subseteq X$, $B \subseteq \text{cl}(A)$ and A be $\mathcal{G}g$– closed, then A is $\rho\mathcal{G}$ – compact if and only if B is $\rho\mathcal{G}$ – compact.

Proof: First let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of X such that $B - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Then $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$, and given that A is $\rho\mathcal{G}$ – compact there exists finite $\Lambda_0 \subseteq \Lambda$, such that $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. Since A is $\mathcal{G}g$– closed, $\text{cl}(A) \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$ and so $\text{cl}(A) - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. This implies $B - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$.

Conversely, let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of X such that $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Given that A is $\mathcal{G}g$– closed, $\text{cl}(A) - \bigcup_{\alpha\in\Lambda} V_\alpha = \phi \notin \mathcal{G}$.This implies $B - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Since B is $\rho\mathcal{G}$ – compact, there exists finite $\Lambda_0 \subseteq \Lambda$, such that $B - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. Hence $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$.

Theorem 2.11: Let $(X, \tau, \mathcal{G})$ be a grill space such that (X, τ) is Hausdorff. If A is $\rho\mathcal{G}$ – compact subset of X, then A is closed in $(X, \tau_\mathcal{G})$.

Proof : Let A be $\rho\mathcal{G}$ – compact subset of Hausdorff grill space $(X, \tau, \mathcal{G})$. Let $x \notin A$. Then for each $y \in A$, there exist neighborhoods U_y and V_y of x and y respectively such that $U_y \cap V_y = \phi$. Note that $x \notin \text{cl}(V_y)$. Now $\{V_y : y \in A\}$ is a τ-open cover of A which is $A - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin \mathcal{G}$. Now $x \notin \text{cl}(V_y)$ for each y implies $x \notin \bigcup_{y\in\Lambda_0} \text{cl}(V_y) = \text{cl}(\bigcup_{y\in\Lambda_0} V_y)$. Let $U = X - \text{cl}(\bigcup_{y\in\Lambda_0} V_y)$ and $J = A - \text{cl}(\bigcup_{y\in\Lambda_0} V_y) \subseteq A - (\bigcup_{y\in\Lambda_0} V_y) = G_1$(say), with $G_1 \notin \mathcal{G}$. Then $U - J \in \tau_\mathcal{G}(x)$ and $(U - J) \cap A = \phi \Rightarrow x \notin \Phi(A)$. Hence $\Phi(A) \subset A$, so A is closed in $(X, \tau_\mathcal{G})$.

Theorem 2.12: Let $(X, \tau, \mathcal{G})$ be a $\rho\mathcal{G}$ – compact space such that (X, τ) is Hausdorff. If F and G are disjoint $\mathcal{G}g$– closed subsets of X, then there exist disjoint open subsets U and V of X, such that $F - U \notin \mathcal{G}$ and $G - V \notin \mathcal{G}$.

Proof : The result is clear if $F = \phi$ or if $G = \phi$. Suppose that $F \neq \phi$ and $G \neq \phi$. By Theorem 2.3, we have F and G are $\rho\mathcal{G}$ – compact spaces of X. We choose $g \in G$, arbitrary but fixed.

For all $f \in F$ there exist disjoint $U_f \in \tau$ and $V_f \in \tau$ such that $f \in U_f$ and $g \in V_f$. Given that $F - \bigcup_{f\in F} U_f \notin \mathcal{G}$, there exist $F_g \subseteq F$, finite with $F - \bigcup_{f\in F_g} U_f \notin \mathcal{G}$. Let $T_g = \bigcup_{f\in F_g} U_f$ and $W_g = \bigcap_{f\in F_g} V_f$. It is clear that $T_g \cap W_g = \phi$. Now since $G - \bigcup_{g\in G} W_g = \phi \notin \mathcal{G}$ and G is $\rho\mathcal{G}$ – compact, there exist $G_0 \subseteq G$, finite such that $G - \bigcup_{g\in G_0} W_g \notin \mathcal{G}$. Let $U = \bigcap_{g\in G_0} T_g$ and $V = \bigcup_{g\in G_0} W_g$, we note that $F - U = \bigcup_{g\in G_0} (F - T_g) \notin \mathcal{G}$ and $G - V \notin \mathcal{G}$. Moreover U and V are disjoint, because if $u \in U \cap V$ then there exists $g_1 \in G_0$ with $u \in W_{g_1}$ and since $u \in T_{g_1}$, we have $T_{g_1} \cap W_{g_1} \neq \phi$, contradiction.

Theorem 2.13: If $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact, $f : (X, \tau) \to (Y, \beta)$ is a continuous function and if $\mathcal{H} = \{B \subseteq Y : f^{-1}(B) \in \mathcal{G}\}$ then :

(a) $\mathcal{H}$ is a grill on Y.
(b) $(Y, \beta, \mathcal{H})$ is $\rho\mathcal{G}$ – compact.

Proof : (a) Suppose that $A \subseteq B \subseteq Y$ and $A \in \mathcal{H}$. Since $f^{-1}(A) \subseteq f^{-1}(B) \in \mathcal{G}$ then $f^{-1}(B) \in \mathcal{G}$ and so $B \in \mathcal{H}$. Now if $A \notin \mathcal{H}$ and $B \notin \mathcal{H}$ then $f^{-1}(A) \notin \mathcal{G}$ and $f^{-1}(B) \notin \mathcal{G}$ and then $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \notin \mathcal{G}$. This implies $A \cup B \notin \mathcal{G}$.

(b) Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of Y such that $Y - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{H}$ since $X - \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}(Y - \bigcup_{\alpha\in\Lambda} V_\alpha) \notin \mathcal{G}$. $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact, hence there exists finite $\Lambda_0 \subseteq \Lambda$, with $f^{-1}(Y - \bigcup_{\alpha\in\Lambda_0} V_\alpha) = X - \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha) \in \mathcal{G}$. Thus $Y - \bigcup_{\alpha\in\Lambda_0} V_\alpha \in \mathcal{H}$.

Theorem 2.14: For a $\rho\mathcal{G}$ – compact space $(X, \tau, \mathcal{G})$ and $f : (X, \tau) \to (Y, \beta)$ a bijective continuous function, one has $(Y, \beta, f(\mathcal{G}))$ is $\rho\mathcal{G}$ – compact .

Proof : Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of Y such that $Y - \bigcup_{\alpha\in\Lambda} V_\alpha \in f(\mathcal{G})$. There exists $G \in \mathcal{G}$ with $Y - \bigcup_{\alpha\in\Lambda} V_\alpha = f(\mathcal{G})$. Then $G = f^{-1}(f(G)) = X - \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) \notin \mathcal{G}$. Given that $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact, there exists finite $\Lambda_0 \subseteq \Lambda$, with $f^{-1}(Y - \bigcup_{\alpha\in\Lambda_0} V_\alpha) = X - \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha) \notin \mathcal{G}$.

If $f : X \to Y$ is an injective function and $\mathcal{H}$ is a grill on Y, then the set $f^{-1}(\mathcal{H}) = \{ f^{-1}(J) : J \in \mathcal{H}\}$ is a grill on X.

Theorem 2.15: Let $(Y, \beta, \mathcal{H})$ be $\rho\mathcal{G}$ – compact, $f : (X, \tau) \to (Y, \beta)$ a continuous function, then $(X, \tau, f^{-1}(\mathcal{H}))$ is $\rho\, f^{-1}(\mathcal{H})$ – compact .

Proof : Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of X such that $X - \bigcup_{\alpha\in\Lambda} V_\alpha \notin f^{-1}(\mathcal{H})$. There exists $J \in \mathcal{H}$ with $X - \bigcup_{\alpha\in\Lambda} V_\alpha = f^{-1}(J)$. Then $Y - \bigcup_{\alpha\in\Lambda} V_\alpha = f(f^{-1}(J)) = J \in \mathcal{H}$, and given that $(Y, \beta, \mathcal{H})$ is $\rho\mathcal{G}$ – compact then there exists finite $\Lambda_0 \subseteq \Lambda$, with $f(X - \bigcup_{\alpha\in\Lambda_0} V_\alpha) = Y - \bigcup_{\alpha\in\Lambda_0} f(V_\alpha) \notin \mathcal{H}$, this implies that $X - \bigcup_{\alpha\in\Lambda_0} V_\alpha \notin f^{-1}(\mathcal{H})$.

3. $\sigma\mathcal{G}$ – Compact Spaces

Definition 3.1: Let $(X, \tau, \mathcal{G})$ be a grill space and $A \subseteq X$, A is said to be $\sigma\mathcal{G}$ – compact if for every family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X, if $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$ then there exists, finite $\Lambda_0 \subseteq \Lambda$ such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. The grill space $(X, \tau, \mathcal{G})$ is said to be $\sigma\mathcal{G}$ – compact if X is $\sigma\mathcal{G}$ – compact.

We note that if $(X, \tau, \mathcal{G})$ is a grill space and $(X, \tau\mathcal{G}, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact, then $(X, \tau, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact. Also $(X, \tau, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact if and only if for any family $\{F_\alpha\}_{\alpha\in\Lambda}$ of closed subsets of X, if

$\bigcap_{\alpha\in\Lambda} F_\alpha \notin \mathcal{G}$ then there exists, finite $\Lambda_0 \subseteq \Lambda$ such that $\bigcap_{\alpha\in\Lambda_0} F_\alpha = \phi$.

It is clear that (X, τ) is compact if and only if $(X, \tau, \mathcal{G}(\{\phi\}))$ is $\sigma\mathcal{G}(\{\phi\})$– compact, and that if $(X, \tau, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact then $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact, and (X, τ) is compact.

Example 3.2: (1) Let $X = Z^+$, $\tau = \{ A \subseteq X : X - A$ is finite $\} \bigcup \{\phi\}$, and $\mathcal{G} = \{ A \subseteq X : A$ is infinite$\}$. Then

(a) $(X, \tau, \mathcal{G})$ is $\rho\mathcal{G}$ – compact, because if $\{F_i\}_{i\in\Lambda}$ is a family of closed subsets of X with $\bigcap_{i\in\Lambda} F_i \notin \mathcal{G}$, then there exists $i_0 \in \Lambda$ such that $F_{i_0} \neq X$ .Thus $F_{i_0} \notin \mathcal{G}$.

(b) $(X, \tau, \mathcal{G})$ is not $\sigma\mathcal{G}$ – compact, because if $F_n = \{1, 2, \ldots, n\}$ then F_n is a closed subset of X and $\bigcap_{n=1}^{\infty} F_n = \{1\} \notin \mathcal{G}$, but if $n_1, n_2, \ldots, n_r \in Z^+$ then $\bigcap_{k=1}^{r} F_{n_k} \neq \phi$.

Theorem 3.3: Let $(X, \tau, \mathcal{G})$ be a grill space and $\mathcal{B}$ a base for τ, then $(X, \tau, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact if and only if for every family $\{V_\alpha\}_{\alpha\in\Lambda}$ of open sets in $\mathcal{B}$, if $X - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$ then there exists, finite $\Lambda_0 \subseteq \Lambda$ such that $X = \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

Proof : Necessary part is easy to proof.

Conversely, let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of non-empty open subsets of X such that $X - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. For all $\alpha \in \Lambda$ there exists a family $\{W_{\alpha\beta} : \beta \in \Lambda_\alpha\}$ of elements in $\mathcal{B}$ such that $V_\alpha = \bigcup_{\beta\in\Lambda} W_{\alpha\beta}$.

Given that $X - \bigcup_{\alpha\in\Lambda} \bigcup_{\beta\in\Lambda_\alpha} W_{\alpha\beta} \notin \mathcal{G}$ and $(X, \tau, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact, there exist $W_{\alpha_1\beta_1}, W_{\alpha_2\beta_2}, W_{\alpha_3\beta_3}, \ldots, W_{\alpha_r\beta_r}$ such that $X = \bigcup_{i=1}^{r} W_{\alpha_i\beta_i}$. But $X = \bigcup_{i=1}^{r} W_{\alpha_i\beta_i} \subseteq \bigcup_{i=1}^{r} V_{\alpha_i} \subseteq X$ and so $X = \bigcup_{i=1}^{r} V_{\alpha_i}$.

Now we investigate the behavior of subspaces of a $\sigma\mathcal{G}$ – compact space.

Theorem 3.4: Let $(X, \tau, \mathcal{G})$ be $\sigma\mathcal{G}$ – compact and $A \subseteq X$ be $\mathcal{G}g$– closed, then A is $\sigma\mathcal{G}$ – compact.

Proof **:** Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of X such that $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$ .Since A is $\mathcal{G}g$– closed, $\mathrm{cl}(A) \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha$. Then $X = (X - \mathrm{cl}(A)) \cup (\bigcup_{\alpha\in\Lambda} V_\alpha)$ and so $X - [(X - \mathrm{cl}(A)) \cup (\bigcup_{\alpha\in\Lambda} V_\alpha)] = \phi \notin \mathcal{G}$. Given that X is $\sigma\mathcal{G}$ – compact, there exists finite $\Lambda_0 \subseteq \Lambda$, such that $X = (X - \mathrm{cl}(A)) \cup (\bigcup_{\alpha\in\Lambda_0} V_\alpha)$. Then $A = A \cap [(X - \mathrm{cl}(A)) \cup (\bigcup_{\alpha\in\Lambda_0} V_\alpha)] = [A \cap \bigcup_{\alpha\in\Lambda_0} V_\alpha] \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

Theorem 3.5: For A and B $\sigma\mathcal{G}$ – compact subsets of a grill space $(X, \tau, \mathcal{G})$, $A \cup B$ is $\sigma\mathcal{G}$–compact.

Proof **:** This proof is similar to that of Theorem 2.8.

Theorem 3.6: Let $(X, \tau, \mathcal{G})$ be a grill space and $A \subseteq X$. Suppose that for all $U \in \tau$, if $A - U \notin \mathcal{G}$, then there exist $B \subseteq X$ such that B is $\sigma\mathcal{G}$ – compact, $A \subseteq B$ and $B - U \notin \mathcal{G}$. Then A is $\sigma\mathcal{G}$ – compact.

Proof **:** Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of X such that $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. There exist $B \subseteq X$ such that B is $\sigma\mathcal{G}$ – compact, $A \subseteq B$ and $B - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Hence there exist, finite $\Lambda_0 \subseteq \Lambda$, such that $B \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$ and so $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

Theorem 3.7: For a grill space $(X, \tau, \mathcal{G})$ and $A \subseteq B \subseteq X$ and $B \subseteq \mathrm{cl}(A)$:

(1) If A is g– closed and $\sigma\mathcal{G}$ – compact, then B is $\sigma\mathcal{G}$ – compact.

(2) If A is $\mathcal{G}g$– closed and B is $\sigma\mathcal{G}$ – compact, then A is $\sigma\mathcal{G}$ – compact.

Proof : (1) Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of X such that $B - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Then $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$, and given that A is $\sigma\mathcal{G}$ –

compact, so there exist, finite $\Lambda_0 \subseteq \Lambda$, such that $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Since *A* is *g*-closed, $\mathrm{cl}(A) \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$, and this implies $B \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

(2) Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of *X* such that $A - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Given that *A* is $\mathcal{G}g$-closed, $\mathrm{cl}(A) - \bigcup_{\alpha\in\Lambda} V_\alpha = \phi \notin \mathcal{G}$, and this implies $B - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Since *B* is $\sigma\mathcal{G}$ – compact, therefore there exist, finite $\Lambda_0 \subseteq \Lambda$, such that $B \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$. Hence $A \subseteq \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

Theorem 3.8: Let $(X, \tau, \mathcal{G})$ be a grill space such that (X, τ) is Hausdorff. If *A* is a $\sigma\mathcal{G}$ – compact subset of *X*, then *A* is closed in $(X, \tau_{\mathcal{G}})$.

Proof : This is an easy consequence of Theorem 2.11.

Theorem 3.9: If $(X, \tau, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact, $f : (X, \tau) \to (Y, \beta)$ is a continuous surjective function and if $\mathcal{H} = \{B \subseteq Y : f^{-1}(B) \notin \mathcal{G}\}$ then $(Y, \beta, \mathcal{H})$ is $\sigma\mathcal{H}$ – compact.

Proof : In Theorem 2.13, we prove that $\mathcal{H}$ is a grill on *Y*. Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of *Y* such that $Y - \bigcup_{\alpha\in\Lambda} V_\alpha \notin \mathcal{G}$. Since $X - \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) = f^{-1}(Y - \bigcup_{\alpha\in\Lambda} V_\alpha) \notin \mathcal{G}$ and $(X, \tau, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact, therefore there exist, finite $\Lambda_0 \subseteq \Lambda$, such that $X = \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)$. Given that *f* is surjective so we have $Y = \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

Theorem 3.10: For a $\sigma\mathcal{G}$ – compact space $(X, \tau, \mathcal{G})$ and $f : (X, \tau) \to (Y, \beta)$ a continuous bijective function, we have $(Y, \beta, f(\mathcal{G}))$ is $\sigma f(\mathcal{G})$ – compact.

Proof : Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of *Y* such that $Y - \bigcup_{\alpha\in\Lambda} V_\alpha \notin f(\mathcal{G})$. There exist $G \in \mathcal{G}$ with $Y - \bigcup_{\alpha\in\Lambda} V_\alpha \neq f(G)$. Then $G = f^{-1}(f(G))$ and $X - \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha) \notin \mathcal{G}$. Given that $(X, \tau, \mathcal{G})$ is $\sigma\mathcal{G}$ – compact, therefore there exist, finite $\Lambda_0 \subseteq \Lambda$, such that $X = \bigcup_{\alpha\in\Lambda_0} f^{-1}(V_\alpha)$. Since *f* is bijective, $Y = \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

Theorem 3.11: Let $f : (X, \tau) \to (Y, \beta)$ be a bijective and open function and $(Y, \beta, \mathcal{H})$ $\sigma\mathcal{H}$ – compact space, then $(X, \tau, f^{-1}(\mathcal{H}))$ is $\sigma f^{-1}(\mathcal{H})$ – compact.

Proof : Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of open subsets of *X* such that $X - \bigcup_{\alpha\in\Lambda} V_\alpha \notin f^{-1}(\mathcal{H})$. There exist $H \in \mathcal{H}$ with $X - \bigcup_{\alpha\in\Lambda} V_\alpha = f^{-1}(H)$. Then $Y - \bigcup_{\alpha\in\Lambda} f(V_\alpha) = f(f^{-1}(H)) = H \notin \mathcal{H}$. Given that $(Y, \beta, \mathcal{H})$ is $\sigma\mathcal{H}$ – compact, therefore there exist, finite $\Lambda_0 \subseteq \Lambda$, such that $Y = \bigcup_{\alpha\in\Lambda_0} f(V_\alpha)$. This implies that $X = \bigcup_{\alpha\in\Lambda_0} V_\alpha$.

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