On Conformally Flat and Weyl-Semisymmetric (ϵ) - Sasakian Manifolds

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***Abstract*—*In this paper, we study (ε)-Sasakian manifolds satisfying certain conditions on the conformal curvature tensor. Here, we consider conformally flat and Weyl-semisymmetric (ε)- Sasakian manifolds.*

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I. INTRODUCTION

In 1969, Takahashi [6] introduced almost contact manifolds equipped with associated indefinite metrics. He studied Sasakian manifolds equipped with an associated indefinite metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also called as (ε)-almost contact metric manifolds and (ε)-Sasakian manifolds, respectively [1,7]. The concept of indefinite sasakian manifolds (briefly, (ε)-Sasakian manifolds) was introduced by Bejancu and Duggal [1] and further investigations were taken up by Xufeng and Xiaoli [7]. They shown that (ε)-Sasakian manifolds are real hypersurface of indefinite Kaehlerian manifolds. The historical background of (ε)-Sasakian manifolds can be traced back to the classification of Sasakian manifolds with indefinite metrics, which play crucial role in Physics [4]. Recently, Kumar, rani and Nagaich studied some properties of (ε)-Sasakian manifolds. Indefinite Sasakian manifolds have been studied by several authors. In this paper, we we study conformally flat and Weyl-semisymmetric (ε)-Sasakian manifolds.

The paper is organized as follows: in section 2, we give a necessary information about (ε)-Sasakian manifolds. In section 3, we consider conformally flat (ε)- Sasakian manifold and show that this type of manifold is an η-Einstein manifold. We also prove that a conformally flat (ε)-Sasakian manifold is of quasi-constant curvature and hence it is a $\psi(F)_n$. In the last section, we consider Weyl-semisymmetric (ε)-Sasakian manifold and prove that this type of manifold is also a $\psi(F)_n$

II. (ε)-SASAKIAN MANIFOLDS

Let M be an n-dimensional differentiable manifold endowed with an almost contact (ϕ,ξ,η), where ϕ is a (1, 1)-tensor field, ξ is a vector field and η is a 1-form on M satisfying

$$\eta(\xi)=1, \quad \phi^2 X = -X+\eta(X)\xi \quad . \tag{2.1}$$

It follows that

$$\eta\cdot\phi=0, \quad \phi(\xi)=0, \quad rank\ \phi=2n \quad ; \tag{2.2}$$

then M is called an almost contact manifold [2]. If there exists a semi-Riemannian metric g satisfying

$$g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y), \tag{2.3}$$

$\forall X,Y\in\chi(M)$, where $\varepsilon=\pm1$, then structure (ϕ,ξ,η,g) is called an (ε)-almost contact metric structure and M is called an (ε)-almost contact metric manifold. For an (ε)-almost contact metric manifold, we have

$$\eta(X)=\varepsilon g(X,\xi) \text{ and } \varepsilon=g(\xi,\xi)\ \forall X\in\chi(M) \ , \tag{2.4}$$

hence, ξ is never a light-like vector field on M and according to the casual character of ξ we have two classes of (ε)-Sasakian manifolds. When $\varepsilon=-1$ and index of g is odd, then M is a time-like Sasakian manifold and when $\varepsilon=-1$ and index of g is even, then M is a space-like Sasakian manifold. Further M is usual Sasakian manifold for $\varepsilon=1$ and index of g is 0 and M is a Lorentz-Sasakian manifold for $\varepsilon=-1$ and index of g is 1

If $d\eta=(X,Y)=g(\phi X,Y)$, then M is said to have (ε)-contact metric structure (ϕ,ξ,η,g). If moreover, this structure is normal, that is, if

$$[\phi X,\phi Y]+\phi^2[X,Y]-\phi[X,\phi Y] -\phi[\phi X,Y]=-2d\eta(X,Y)\xi, \tag{2.5}$$

then the (ε)-contact metric structure is called an (ε)-Sasakian structure and the manifold endowed with this structure is called (ε)-Sasakian manifold.

An (ε)-almost contact metric structure (ϕ,ξ,η,g) is (ε)-Sasakian if and only if

$$(\nabla_X\phi)Y=g(X,Y)\xi-\varepsilon\eta(Y)X,\ \forall X,Y\in\chi(M), \tag{2.6}$$

where ∇ is the Levi-Civita connection with respect to g . Also we have

$$\nabla_X \xi = -\varepsilon\phi X, \ \forall X \in \chi(M). \tag{2.7}$$

In an (ε)-Sasakian manifolds, the following relations hold:

$$\eta(R(X,Y)Z) = \varepsilon(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)), \tag{2.8}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.9}$$

$$R(\xi,X)Y = \varepsilon g(X,Y)\xi - \eta(Y)X, \tag{2.10}$$

$$R(\xi,X)\xi = \varepsilon(\eta(X)\xi - X), \tag{2.11}$$

$$S(X,\xi) = \varepsilon(n-1)\eta(X), \tag{2.12}$$

for all vector fields X, Y on M

In 1972, Chen and Yano [3] introduced the notion of a manifold of quasi-constant curvature as follows:

Definition 2.1: An (ε)-Sasakian manifold will be called manifold of quasi-constant curvature if R' of type (0, 4) satisfy the condition

$$\begin{aligned} &R'(X,Y,Z,W) \\ &= a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ b[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ &+ g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)], \end{aligned} \tag{2.13}$$

where $R'(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor type (1,3); a, b are scalars functions and ρ is unit vector field defined by

$$g(X,\rho) = T(X)$$

Definition 2.2: An (ε)-Sasakian manifold will be called an η-Einstein manifold if the Ricci tensor S of type (0, 2) satisfies

$$S(X,Y) = a\,g(X,Y) + b\eta(X)\eta(Y),$$

where a and b are scalar functions.

Definition 2.3: A type of Riemannian manifold whose curvature tensor R' of type (0, 4) satisfies the condition

$$\begin{aligned} &R'(X,Y,Z,W) \\ &= F(Y,Z)F(X,W) - F(X,Z)F(Y,W), \end{aligned} \tag{2.14}$$

where B is a symmetric tensor of type (0, 2) is called a special manifold with the associated symmetric tensor B and is denoted by $\psi(F)_n$.

III. CONFORMALLY (ε)-SASAKIAN MANIFOLDS

The Weyl conformal curvature tensor C of type (1, 3) of an n-dimensional Riemannian manifold is given by

$$\begin{aligned} &C(X,Y)Z \\ &= R(X,Y)Z - \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)} \\ &[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{3.1}$$

where Q is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and r is the scalar curvature.

Let us suppose that the manifold is conformally flat. Then from the above equation, we have

$$\begin{aligned} &g(R(X,Y)Z,W) \\ &= \frac{1}{(n-2)}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] - \frac{r}{(n-1)(n-2)} \\ &[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned} \tag{3.2}$$

Putting $W = \xi$ and using the equations (2.4) and (2.12) the above equation gives

$$\begin{aligned} &\varepsilon\eta(R(X,Y)Z) \\ &= \frac{1}{(n-2)}[\varepsilon S(Y,Z)\eta(X) - \varepsilon S(X,Z)\eta(Y) \\ &+ (n-1)g(Y,Z)\eta(X) - (n-1)g(X,Z)\eta(Y) \\ &- \frac{r}{(n-1)(n-2)}[\varepsilon g(Y,Z)\eta(X) - \varepsilon g(X,Z)\eta(Y)]. \end{aligned} \tag{3.3}$$

In view of the equation (2.8) and $\varepsilon^2 = 1$, the above equation yields

$$\begin{aligned} S(Y,Z)\eta(X) = S(X,Z)\eta(Y) + \left(\frac{r}{n-1} - \varepsilon\right) \\ (g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \end{aligned} \tag{3.4}$$

For $X = \xi$, we get

$$S(Y,Z) = \left(\frac{r}{n-1} - \varepsilon\right)g(Y,Z) + \left(n - \frac{r}{n-1}\right)\eta(Y)\eta(Z). \tag{3.5}$$

Hence we can state the following:

Theorem 3.1: An n-dimensional $(n > 2)$ conformally flat (ε) −Sasakian manifold is an $\eta-$ Einstein manifold.

Using the equation (3.5) in (3.3), we get

$$g(R(X,Y)Z,W)$$
$$=\frac{1}{(n-2)}\left[\left(\frac{2r}{n-1}-2\varepsilon\right)g(Y,Z)g(X,W)-\left(\frac{2r}{n-1}-2\varepsilon\right)g(X,Z)g(Y,W)\right]$$
$$+\left(n-\frac{r}{n-1}\varepsilon\right)[\eta(Y)\eta(Z)g(X,W)-\eta(X)\eta(Z)g(Y,W)$$
$$+\eta(X)\eta(W)g(Y,Z)-\eta(Y)\eta(W)g(X,Z)]$$
$$-\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)].$$

(3.6) The above relation can be written as
$$g(R(X,Y)Z,W)$$
$$=\frac{r-2(n-1)\varepsilon}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]$$
$$-\left(n-\frac{r}{n-1}\varepsilon\right)[\eta(X)\eta(Z)g(Y,W)+\eta(Y)\eta(W)g(X,Z)$$
$$-\eta(X)\eta(W)g(Y,Z)-\eta(Y)\eta(Z)g(X,W)].$$
(3.7)

In view of definition 2.1 and the above relation we have the following:

Theorem 3.2: An n-dimensional $(n>2)$ conformally flat (ε)-Sasakian manifold is of quasi-constant curvature.

It is also proved that a $\psi(F)_n$ contains a manifold of quasi-constant curvature as a subclass:

The substitution
$$F(X,Y)=qg(X,Y)+q'\eta(X)\eta(Y)$$
(3.8)

with
$$q=\sqrt{\frac{r-2(n-1)\eta}{(n-1)(n-2)}}$$
(3.9)

and
$$q'=\left(n-\frac{r}{n-1}\varepsilon\right)\sqrt{\frac{(n-1)(n-2)}{r-2(n-1)\eta}}$$
(3.10)

leads to
$$R(X,Y,Z,W)=F(Y,Z)F(X,W)-F(X,Z)F(Y,W).$$
(3.11)

Therefore the manifold of quasi-constant curvature is a $\psi(F)_n$.

From the above condition and Theorem 3.2, we have the following:

Corollary 3.1: A conformally flat (ε)-Sasakian manifold is a $\psi(F)_n$.

IV. WEYL-SEMISYMMETRIC (ε)-SASAKIAN MANIFOLDS

An (ε)-Sasakian manifolds is said to be Weyl-semisymmetric if
$$R\cdot C=0$$
From the equation 3.1, we get
$$g(C(X,Y)Z,\xi)=g(R(X,Y)Z,\xi)-\frac{1}{n-2}[g(Y,Z)S(X,\xi)$$
$$-g(X,Z)S(Y,\xi)+S(Y,Z)g(X,\xi)$$
$$-S(X,Z)S(Y,\xi)]+\frac{r}{(n-1)(n-2)}.$$
$$[g(Y,Z)g(X,\xi)-g(X,Z)g(Y,\xi)].$$
(4.1)

From the above equation, we have
$$\eta(C(X,Y)Z)=\frac{1}{(n-2)}[\left(\frac{r}{n-1}-\varepsilon\right)(g(Y,Z)\eta(X)-g(X,Z)\eta(Y))$$
$$-S(Y,Z)\eta(X)+S(X,Z)\eta(Y)]$$
(4.2)

Putting $Z=\xi$ in the above equation and using (2.9) and (2.12), we have
$$\eta(C(X,Y)\xi)=0$$
(4.3)

Again putting $X=\xi$ in the equation (4.1) and using (2.10) and (2.12), we get
$$\eta(C(\xi,Y)Z)=\frac{1}{(n-2)}[\left(\frac{r}{n-1}-\varepsilon\right)(g(Y,Z)-\varepsilon\eta(Y)\eta(Z))$$
$$-S(Y,Z)+(n-1)\eta(Y)\eta(Z)].$$
(4.4)

If the manifold is Weyl-semisymmetric, then we have
$$g[R(\xi,Y)C(U,V)W,\xi]-g[C(R(\xi,Y)U,V)W,\xi]$$
$$-g[C(U,R(\xi,Y)V,W),\xi]-g[C(U,V)R(\xi,Y)W,\xi]=0.$$
(4.5)

From the equation (2.10), we have
$$g(R(\xi,X)Y,\xi)=g(X,Y)-\varepsilon\eta(Y)\eta(X)$$
(4.6)

Using the equation (4.6) in (4.5), we get
$$g(Y,C(U,V)W)-\varepsilon\eta(C(U,V)W)\eta(Y)$$
$$-g[C(\varepsilon g(Y,U)\xi-\eta(U)Y,V)W,\xi]$$
$$-g[C(U,\varepsilon g(Y,V)\xi-\eta(V)Y)W,\xi]$$
$$-g[C(U,V)(\varepsilon g(Y,W)\xi-\eta(W)Y),\xi]=0.$$
(4.7)

From the above equation, we have
$$C'(U,V,W,Y)-\eta(Y)\eta(C(U,V)W)$$
$$+\varepsilon\eta(U)\eta(C(Y,V)W)+\varepsilon\eta(V)\eta(C(U,Y)W)$$
$$+\varepsilon\eta(W)\eta(C(U,V)Y)-g(Y,U)\eta(C(\xi,V)W)$$
$$-g(Y,V)\eta(C(U,\xi)W)-g(Y,W)\eta(C(U,V)\xi)=0,$$
(4.8)

where $C'(U,V,W,Y)=g(C(U,V)W,Y)$.

Putting $Y=U$, we get
$$C'(U,V,W,U)-\eta(U)\eta(C(U,V)W)$$
$$+\varepsilon\eta(W)\eta(C(U,V)W)+\varepsilon\eta(V)\eta(C(U,V)W)$$
$$+\varepsilon\eta(W)\eta(C(U,V)U)-g(U,U)\eta(C(\xi,V)W)$$
$$-g(U,V)\eta(C(U,\xi)W)-g(U,W)\eta(C(U,V)\xi)=0.$$
(4.9)

Again putting $U=e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i where $i = 1, 2, 3......n,$ we get

$$\sum_{i=1}^{n} C'(e_i, V, W, e_i) = 0,$$

and using (4.3) in (4.9), we have

$$\eta(C(\xi, V)W) = 0.$$

Using the equations (4.3) and (4.9) in (4.8), we get

$$C'(U,V,W,Y) - \eta(Y)\eta(C(U,V)W) + \varepsilon\eta(U)\eta(C(Y,V)W) + \varepsilon\eta(V)\eta(C(U,Y)W) + \varepsilon\eta(W)\eta(C(U,V)Y) = 0.$$

(4.10)

Using the equations (4.2) and (4.10), we get

$$C'(U,V,W,Y) + \frac{\eta(W)}{n-2}[\left(\frac{\varepsilon r}{n-1} - 1\right) g(Y,U)\eta(U) - g(U,Y)\eta(V) - \varepsilon(S(Y,V)\eta(U) - S(Y,U)\eta(V))] + \frac{(\varepsilon-1)}{n-2}[\left(\frac{\varepsilon r}{n-1} - 1\right)(g(U,W)\eta(V)\eta(Y) - g(V,W)\eta(U)\eta(Y) - S(U,W)\eta(Y)\eta(V) + S(V,W)\eta(U)\eta(Y))] = 0.$$

(4.11)

From the equation (4.9), we have from (4.4) that

$$S(Y,Z) = \left(\frac{r}{n-1} - \varepsilon\right) g(Y,Z) - \left(\frac{r\varepsilon}{n-1} - n\right)\eta(Y)\eta(Z).$$

(4.12)

Using the equation (4.12) in (4.11)

$$C'(U,V,W,Y) = 0.$$

(4.13)

From the above equation we see that $R \cdot C = 0$ implies that $C = 0$. Hence using the condition with the help of Theorem 3.2 we have the following:

Theorem 4.3: An *n*-dimensional $(n > 2)$ Weyl-semisymmetric (ε) −Sasakian manifold is of quasi-constant curvature.

Theorem 4.3 and equation (4.13) leads the following:

Corollary 4.3: An *n*-dimensional $(n > 2)$ Weyl-semisymmetric (ε) −Sasakian manifold is a $\psi(F)_n$.

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