Norm of (3, 2) Jection Operator

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ABSTRACT *:In this paper, we deal with the basic concept of norm of the (3, 2)-jection operator in a Hilbert space and present the deep property concerning with this.*

Key words : *Norm, Linear transformation, Hilbert space, operator norm of a linear operator, Bounded linear operator.*

I. INTRODUCTION& PRILIMINARIES

In each space there is defined a notion of the distance from an arbitrary element to the origin, that is, a notion of the "size" of an arbitrary element. The size of an element x is a non-negative real number denoted by $||x||$ and called norm of x, in such a manner that

(N_1) $||x|| \geq 0$, and $||x|| = 0 \Leftrightarrow x = 0$.

(N_2) $||\alpha x|| = |\alpha|\, ||x||$

(N_3) $||x + y|| \leq ||x|| + ||y||$.

We know that space (Linear space) over which a norm is defined, is called a Normed linear space.

We next mention the concept of linear transformation T from a vector space X to another vector space Y with the property that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)\ \forall \alpha, \beta \in IF$.

The main fact about such transformation is that the collection of all linear transformation mapping a vector space X into another vector space Y can be viewed as a vector space by defining addition of the linear transformation T_1 and T_2 to be that transformation which takes X into T(x) + T(y) symbolically,

We have

$(T_1 + T_2)(x) = T_1(x) + T_2(x)$

as for scalar multiplication, we have

$(\alpha T)(x) = \alpha T(x)$

We now turn our attention to operator norm and Bounded linear operator.

Let the linear mapping $T : X \rightarrow Y$ where X and Y are two normed spaces over the same field IF. Then the operator norm of T in L(X, Y) is defined by

$||T|| = \sup\{||Tx|| : ||x|| = 1\}$

and the linear mapping $T \in L(X, Y)$ is said to be a bounded if there exist an $M > 0$ such that $||Tx|| \leq M$ such that $||Tx|| \leq M$ and $||T||$ is then the infimum of all such M.

In other word, T is said to be a bounded operator if $||T||$ is finite otherwise T is called unbounded operator.

Some other concepts that we shall make extensive use of are Inner product, Hilbert space and Adjoint of an operator.

Inner product : Let X be a linear space then an inner product on X is a mapping from X × X the Cartesian product space, into the scalar field IF :

$$X \times X \rightarrow IF,\ \langle x, y\rangle \rightarrow (x, y)$$

[Here (x, y) denotes the inner product of the two vectors, whereas <x, y> represents only the ordered pair in X × X] with the following operator :

[I_1] let $x, y \in X$ then $(x, y) = \overline{(y, x)}$ where the bar denotes complex conjugation.

[I_2] if α and β are scalar and x, y, z are vectors then $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

[I_3] $(x, x) \geq$ for all $x \in X$ and equal to zero if and only if x is the zero vector.

Here, it is important to note that the linear space X with the inner product defined above, is an inner product space or pre-Hilbert space.

With the help of inner product on X, we can define a norm of x by $||x||^2 = (x, x)$ *with these ideas as a background, we are now in a position to give the basic definition of Hilbert space as follows :*

Let X be an inner product then X is said to be Hilbert space if X is complete with respect to the norm derived from the inner product. We now focus on adjoint of an operator. Let T be an operator on a Hilbert space H and there exists a unique T* on H to every T on H such that (Tx, y) = (x, T*y) for al x, y $\in$ H. Then T* is called adjoint of T.

Our work in the present paper centres around a special type of operator, called a (3, 2)-jection operator.

In linear algebra and functional analysis a projection is of fundamental importance, which is defined as a linear operator E on a vector space X such that $E^2 = E$. That is, wherever E is applied twice to any element x $\in$ X, it gives the same result as if it were applied once.

As stated above, projection is a special case of idempotent. On the basis of above definition of projection, we develop a new operator called a (3, 2)-jection operator, which is a suitable generalization of projection, defined as a linear operator on a linear space X such that $E^3 = E^2$. This definition of (3, 2)-jection operator can be carried over verbation to Hilbert space H with an additional condition that E* = E where E* stands for adjoint of E.

II. MAIN RESULT

The analogous results are listed with the following theorems :

Theorem 2.1 :

If E be a (3, 2)-jection in a Hilbert space H then

$$(x, E^2x) = (E^2x, x) = (E^2x, Ex) = \|E\|^2.$$

Proof : We have

$$
\begin{aligned}
(x, E^2x) &= (E^*x, Ex) \\
&= (Ex, Ex) && \{\because\ E^* = E\} \\
&= \|Ex\|^2 && \ldots (2.1.1) \\
\text{Again, } (E^2x, x) &= (Ex, E^*x) \\
&= (Ex, Ex) && \{\because\ E^* = E\} \\
&= \|Ex\|^2 && \ldots (2.1.2) \\
\text{Again, } (E^2x, Ex) &= (Ex, E^*Ex) \\
&= (Ex, EEx) && \{\because\ E^* = E\} \\
&= (Ex, E^2x) \\
&= (x, E^*E^2x) \\
&= (x, EE^2x) && \{\because\ E^* = E\} \\
&= (x, E^3x) \\
&= (x, EE^2x) \\
&= (x, E^2x) && \{\because\ E^3 = E^2\} \\
&= (E^*x, Ex) \\
&= (Ex, Ex) && \{\because\ E^* = E\} \\
&= \|Ex\|^2 && \ldots (2.1.3)
\end{aligned}
$$

From (2.1.1), (2.1.2) and (2.1.3), we have

$$(x, E^2x) = (E^2x, x) = (E^2x, Ex) = \|Ex\|^2$$ **Proved.**

Theorem 2.2 :

If E is a (3, 2)-jection in a Hilbert space H then $\|E^2\| = \|E\|$.

Proof : Let E be a (3, 2) –jection in a Hillbert space H.

Now,

$$
\begin{aligned}
(E^2x, E^2x) &= (Ex, E^*E^2x) \\
&= (Ex, EE^2x) && \{\because\ E^* = E\} \\
&= (Ex, E^3x) \\
&= (Ex, E^2x) && \{\because\ E^3 = E^2\} \\
&= (x, E^*E^2x) \\
&= (x, EE^2x) \\
&= (x, E^3x) \\
&= (x, E^2x) && \{\because\ E^3 = E^2\} \\
&= (E^*x, Ex) \\
&= (Ex, Ex) && \{\because\ E^* = E\} \\
&= \|E\|^2
\end{aligned}
$$

i.e, $\|E^2x\|^2 = \|Ex\|^2$

$\Rightarrow$ $\|E^2x\| = \|Ex\|$

Since $\|E^2\| = \sup \{\|E^2x\| : \|x\| \leq 1\}$

$= \sup \{\|Ex\| : \|x\| \leq 1\}$

$= \|E\|$

Hence, $\|E^2\| = \|E\|$, **Proved.**

Theorem 2.3:

If E be a (3, 2)-jection operator in R^2 then $||E|| = 0$ or 1.

Proof : Since E is a (3, 2)-jectionoperator in R^2

Then $E^* = E$

$\Rightarrow$ E is a normal operator in R^2.

$\Rightarrow$ $EE^* = E^*E$

$\Rightarrow$ $EE^* - E^*E = 0$

$\Rightarrow$ $((EE^* - E^*E)x, x) = 0 \quad \forall \ x \in R^2$

$\Rightarrow$ $(EE^*x, x) - (E^*Ex, x) = 0$

$\Rightarrow$ $(EE^*x, x) = (E^*Ex, x)$

$\Rightarrow$ $(E^*x, E^*x) = (Ex, Ex)$

$\Rightarrow$ $||E^*x||^2 = ||Ex||^2$

$\Rightarrow$ $||E^*x|| = ||Ex||$ … (2.3.1)

Now we have

$||E^2x|| = ||EEx||$

$= ||Ey||$ {putting y = Ex}

$= ||E^*y||$ {using (2.3.1)}

$= ||E^*Ex||$ … (2.3.2)

Therefore,

$||E^2|| = \sup\{||E^2x|| : ||x|| \leq 1\}$

$= \sup\{|| E^*Ex || : ||x|| \leq 1\}$ {using (2.3.2)}

$= || E^*Ex ||$

$= ||E||^2$ {since $|| E^*E || = ||E||^2$}

$\Rightarrow$ $|| E || = ||E||^2$ {from Theorem (2.2)}

$\Rightarrow$ $|| E || - ||E||^2 = 0$

$\Rightarrow$ $|| E || (1 - ||E||) = 0$

$\Rightarrow$ $|| E || = 0$ or 1

Note : $||E|| = 0 \quad \Rightarrow \quad E = 0$

And if E is a non-zero (3, 2)-jectionoperator

then $|| E || = 1$

III. CONCLUSION

Motivated by the theorem 2.3, we say that a (3, 2)-jection operator on a Hilbert space is a bounded operator.

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