Integral involving the extension of the Hurwitz-Lerch Zeta function,Bessel functions, a Jacobi polynomial,a class of polynomials, a multivariable Aleph-function and multivariable I-function

$F.Y.\ AYANT^1$

1 Teacher in High School , France

ABSTRACT

In the present paper we evaluate a general integral involving the product of the extension of the Hurwitz-Lerch Zeta function, product of two ,Bessel functions, multivariable Aleph-function Jacobi polynomial, the multivariable I-function defined by Prasad [4] and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializing the parameters their in.

Keywords:Multivariable Aleph-function, general class of polynomials, generalized multiple-index Mittag-Leffler function, multivariable I-function, multivariable H-function, Bessel function

1.Introduction and preliminaries.

The function Aleph of several variables generalize the multivariable I-function recently study by C.K. Sharma and Ahmad [3] , itself is an a generalisation of G and H-functions of multiple variables. The multiple Mellin-Barnes integral occuring in this paper will be referred to as the multivariables Aleph-function throughout our present study and will be defined and represented as follows.

We define : $\aleph(z_1, \cdots, z_r) = \aleph^{0,N:M_1,N_1,\cdots,M_r,N_r}_{P_i,Q_i,\tau_i;R:P_{i(1)},Q_{i(1)},\tau_{i(1)};R^{(1)};\cdots;P_{i(r)},Q_{i(r)};\tau_{i(r)};R^{(r)}} \left(\begin{array}{c|c} \mathrm{z}_1 & \\ . & \\ . & \\ . & \\ \mathrm{z}_r & \end{array} \right.$

$$[(\mathrm{a}_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)})_{1,N}] \quad ,[\tau_i(a_{ji}; \alpha_{ji}^{(1)}, \cdots, \alpha_{ji}^{(r)})_{N+1,P_i}]:$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad ,[\tau_i(b_{ji}; \beta_{ji}^{(1)}, \cdots, \beta_{ji}^{(r)})_{M+1,Q_i}]:$$

$$\left. \begin{array}{c} [(\mathrm{c}_j^{(1)}); \gamma_j^{(1)})_{1,N_1}], [\tau_{i(1)}(c_{ji(1)}^{(1)}; \gamma_{ji(1)}^{(1)})_{N_1+1,P_i^{(1)}}]; \cdots; [(\mathrm{c}_j^{(r)}); \gamma_j^{(r)})_{1,N_r}], [\tau_{i(r)}(c_{ji(r)}^{(r)}; \gamma_{ji(r)}^{(r)})_{N_r+1,P_i^{(r)}}] \\ [(\mathrm{d}_j^{(1)}); \delta_j^{(1)})_{1,M_1}], [\tau_{i(1)}(d_{ji(1)}^{(1)}; \delta_{ji(1)}^{(1)})_{M_1+1,Q_i^{(1)}}]; \cdots; [(\mathrm{d}_j^{(r)}); \delta_j^{(r)})_{1,M_r}], [\tau_{i(r)}(d_{ji(r)}^{(r)}; \delta_{ji(r)}^{(r)})_{M_r+1,Q_i^{(r)}}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \cdots, s_r) \prod_{k=1}^{r} \theta_k(s_k) y_k^{s_k} \,\mathrm{d}s_1 \cdots \mathrm{d}s_r \qquad (1.1)$$

with $\omega = \sqrt{-1}$

$$\psi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{N} \Gamma(1 - a_j + \sum_{k=1}^{r} \alpha_j^{(k)} s_k)}{\sum_{i=1}^{R} [\tau_i \prod_{j=N+1}^{P_i} \Gamma(a_{ji} - \sum_{k=1}^{r} \alpha_{ji}^{(k)} s_k) \prod_{j=1}^{Q_i} \Gamma(1 - b_{ji} + \sum_{k=1}^{r} \beta_{ji}^{(k)} s_k)]} \qquad (1.2)$$

and $\theta_k(s_k) = \dfrac{\prod_{j=1}^{M_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} s_k) \prod_{j=1}^{N_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}} [\tau_{i^{(k)}} \prod_{j=M_k+1}^{Q_{i^{(k)}}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k) \prod_{j=N_k+1}^{P_{i^{(k)}}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k)]}$ (1.3)

Suppose , as usual , that the parameters

$b_j, j = 1, \cdots, Q; a_j, j = 1, \cdots, P;$

$c_{ji^{(k)}}^{(k)}, j = n_k + 1, \cdots, P_{i^{(k)}}; c_j^{(k)}, j = 1, \cdots, N_k;$

$d_{ji^{(k)}}^{(k)}, j = M_k + 1, \cdots, Q_{i^{(k)}}; d_j^{(k)}, j = 1, \cdots, M_k;$

with $k = 1 \cdots, r, i = 1, \cdots, R, i^{(k)} = 1, \cdots, R^{(k)}$

are complex numbers , and the $\alpha's, \beta's, \gamma's$ and $\delta's$ are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} + \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} + \tau_{i^{(k)}} \sum_{j=n_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} - \sum_{j=1}^{M_k} \delta_j^{(k)}$$

$$- \tau_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} \leqslant 0$$ (1.4)

The reals numbers τ_i are positives for $i = 1$ to R , $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop , if necessary ,ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)} s_k)$ with $j = 1$ to m_k are separated from those of $\Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to N and $\Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to N_k to the left of the contour L_k . The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|arg z_k| < \dfrac{1}{2} A_i^{(k)} \pi$, where

$$A_i^{(k)} = \sum_{j=1}^{N} \alpha_j^{(k)} - \tau_i \sum_{j=N+1}^{P_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{Q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{N_k} \gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{P_{i^{(k)}}} \gamma_{ji^{(k)}}^{(k)}$$

$$+ \sum_{j=1}^{M_k} \delta_j^{(k)} - \tau_{i^{(k)}} \sum_{j=M_k+1}^{Q_{i^{(k)}}} \delta_{ji^{(k)}}^{(k)} > 0, \quad \text{with } k = 1, \cdots, r, i = 1, \cdots, R, i^{(k)} = 1, \cdots, R^{(k)}$$ (1.5)

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \to 0$

$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \to \infty$

where, with $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, M_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, N_k$$

Serie representation of Aleph-function of several variables is given by

$$\aleph(y_1, \cdots, y_r) = \sum_{G_1, \cdots, G_r = 0}^{\infty} \sum_{g_1 = 0}^{M_1} \cdots \sum_{g_r = 0}^{M_r} \frac{(-)^{G_1 + \cdots + G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \psi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r})$$

$$\times \; \theta_1(\eta_{G_1, g_1}) \cdots \theta_r(\eta_{G_r, g_r}) y_1^{-\eta_{G_1, g_1}} \cdots y_r^{-\eta_{G_r, g_r}} \tag{1.6}$$

Where $\psi(., \cdots, .), \theta_i(.), i = 1, \cdots, r$ are given respectively in (1.2), (1.3) and

$$\eta_{G_1, g_1} = \frac{d_{g_1}^{(1)} + G_1}{\delta_{g_1}^{(1)}}, \cdots, \; \eta_{G_r, g_r} = \frac{d_{g_r}^{(r)} + G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\delta_{g_i}^{(i)}[d_j^i + p_i] \neq \delta_j^{(i)}[d_{g_i}^i + G_i]$ \tag{1.7}

for $j \neq M_i, M_i = 1, \cdots \eta_{G_i, g_i}; P_i, N_i = 0, 1, 2, \cdots, ; y_i \neq 0, i = 1, \cdots, r$ \tag{1.8}

In the document , we will note :

$$G(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r}) = \phi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r})\theta_1(\eta_{G_1, g_1}) \cdots \theta_r(\eta_{G_r, g_r}) \tag{1.9}$$

where $\phi(\eta_{G_1, g_1}, \cdots, \eta_{G_r, g_r}), \theta_1(\eta_{G_1, g_1}), \cdots, \theta_r(\eta_{G_r, g_r})$ are given respectively in (1.2) and (1.3)

We will note the Aleph-function of r variables $\aleph_{u:w}^{0, N:v} \begin{pmatrix} z_1 \\ \cdots \\ z_r \end{pmatrix}$ \tag{1.10}

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, z_2, ...z_s) = I_{p_2, q_2, p_3, q_3; \cdots; p_s, q_s: p', q'; \cdots; p^{(s)}, q^{(s)}}^{0, n_2; 0, n_3; \cdots; 0, n_r: m', n'; \cdots; m^{(s)}, n^{(s)}} \begin{pmatrix} z_1 & \vline & (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1, p_2}; \cdots; \\ . & \vline & \\ . & \vline & \\ . & \vline & \\ z_s & \vline & (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1, q_2}; \cdots; \end{pmatrix}$$

$$(a_{sj}; \alpha'_{sj}, \cdots, \alpha_{sj}^{(s)})_{1, p_s} : (a'_j, \alpha'_j)_{1, p'}; \cdots; (a_j^{(s)}, \alpha_j^{(s)})_{1, p^{(s)}}$$

$$(b_{sj}; \beta'_{sj}, \cdots, \beta_{sj}^{(s)})_{1, q_s} : (b'_j, \beta'_j)_{1, q'}; \cdots; (b_j^{(s)}, \beta_j^{(s)})_{1, q^{(s)}}$$

\tag{1.11}

$$= \frac{1}{(2\pi\omega)^s} \int_{L_1} \cdots \int_{L_s} \xi(t_1, \cdots, t_s) \prod_{i=1}^{s} \phi_i(t_i) z_i^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.12}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|arg z_k| < \frac{1}{2}\Omega_i^{(k)}\pi \, , \text{ where}$$

$$\Omega_i^{(k)} = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)} \right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} \right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)} \right) \tag{1.13}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero.Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1, \cdots, z_s) = 0(\, |z_1|^{\alpha'_1}, \cdots, |z_s|^{\alpha'_s} \,) \, , max(\, |z_1|, \cdots, |z_s| \,) \to 0$$

$$I(z_1, \cdots, z_s) = 0(\, |z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s} \,) \, , min(\, |z_1|, \cdots, |z_s| \,) \to \infty$$

where, with $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$$

We will use these following notations in this paper :

$$U = p_2, q_2; p_3, q_3; \cdots; p_{s-1}, q_{s-1}; V = 0, n_2; 0, n_3; \cdots; 0, n_{s-1} \tag{1.14}$$

$$W = (p', q'); \cdots; (p^{(s)}, q^{(s)}); X = (m', n'); \cdots; (m^{(s)}, n^{(s)}) \tag{1.15}$$

$$A = (a_{2k}, \alpha'_{2k}, \alpha''_{2k}); \cdots; (a_{(s-1)k}, \alpha'_{(s-1)k}, \alpha''_{(s-1)k}, \cdots, \alpha_{(s-1)k}^{(s-1)}) \tag{1.16}$$

$$B = (b_{2k}, \beta'_{2k}, \beta''_{2k}); \cdots; (b_{(s-1)k}, \beta'_{(s-1)k}, \beta''_{(s-1)k}, \cdots, \beta_{(s-1)k}^{(s-1)}) \tag{1.17}$$

$$\mathfrak{A} = (a_{sk}; \alpha'_{sk}, \alpha''_{sk}, \cdots, \alpha_{sk}^s) : \mathfrak{B} = (b_{sk}; \beta'_{sk}, \beta''_{sk}, \cdots, \beta_{sk}^s) \tag{1.18}$$

$$A' = (a'_k, \alpha'_k)_{1,p'}; \cdots ; (a_k^{(s)}, \alpha_k^{(s)})_{1,p^{(s)}}; B' = (b'_k, \beta'_k)_{1,q'}; \cdots ; (b_k^{(s)}, \beta_k^{(s)})_{1,q^{(s)}} \qquad (1.19)$$

The multivariable I-function write :

$$I(z_1, \cdots , z_s) = I_{U:p_s,q_s;W}^{V;0,n_s;X} \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_s \end{pmatrix} \left| \begin{array}{c} \text{A} \;;\mathfrak{A};\text{A'} \\ \\ \\ \text{B}; \; \mathfrak{B}; \; \text{B'} \end{array} \right) \qquad (1.20)$$

The generalized polynomials defined by Srivastava [4], is given in the following manner :

$$S_{N'_1,\cdots,N'_t}^{M'_1,\cdots,M'_t}[y_1, \cdots , y_t] = \sum_{K_1=0}^{[N'_1/M'_1]} \cdots \sum_{K_t=0}^{[N'_t/M'_t]} \frac{(-N'_1)_{M'_1 K_1}}{K_1!} \cdots \frac{(-N'_t)_{M'_t K_t}}{K_t!}$$

$$A[N'_1, K_1; \cdots ; N'_t, K_t] y_1^{K_1} \cdots y_t^{K_t} \qquad (1.21)$$

Where $M'_1, \cdots , M'_s$ are arbitrary positive integers and the coefficients $A[N'_1, K_1; \cdots ; N'_t, K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N'_1)_{M'_1 K_1}}{K_1!} \cdots \frac{(-N'_t)_{M'_t K_t}}{K_t!} A[N'_1, K_1; \cdots ; N'_t, K_t] \qquad (1.22)$$

3. Extension of the Hurwitz-Lerch Zeta function

The extension of the Hurwitz-Lerch Zeta function $\phi(z, S, a)$ is introduced by Srivastava et al ([6],eq.(6.2), page 503) as follows :

$$\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z;\mathfrak{s}, a) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}} \prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \times \frac{z^n}{n!} \qquad (3.1)$$

with : $p, q \in \mathbb{N}_0, \lambda_j \in \mathbb{C}(j = 1, \cdots , p), a, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^* \;\; (j = 1, \cdots , q), \rho_j, \sigma_k \in \mathbb{R}^+$ $(j = 1, \cdots , p; k = 1, \cdots , q)$

where $\Delta > -1$ when $\mathfrak{s}, z \in \mathbb{C}; \Delta = -1$ and $s \in \mathbb{C}, when |z| < \bigtriangledown^*, \Delta = -1$ and $Re(\chi) > \frac{1}{2}$ when $|z| = \bigtriangledown^*$

$$\bigtriangledown^* = \prod_{j=1}^{p} \rho_j^{\rho_j} \prod_{j=1}^{q} \sigma_j^{\sigma_j} ; \Delta = \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j ; \chi = \mathfrak{s} + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}$$

We denote these conditions, the conditions (f).

3.Required formula

see Brychkov et al ([1] , 2.22.12, 11 page 602)

$$\int_{-a}^{a}(x+a)^{\sigma}(x-a)^{\beta}J_{\mu}(b\sqrt{a-x})J_{\upsilon}(b\sqrt{a-x})P_{n}^{(\rho,\sigma)}\left(\frac{x}{a}\right)\mathrm{d}x=\frac{2^{\beta+\sigma-\frac{\mu+\upsilon}{2}}}{n!}a^{\beta+\sigma+\frac{\mu+\upsilon}{2}}\,b^{\mu+\upsilon}$$

$$\left(\rho-\beta-\frac{\mu+\upsilon}{2}+1\right)_{n}\ \frac{\Gamma\left(\beta+\frac{\mu+\upsilon}{2}\right)\Gamma(\sigma+n+1)}{\Gamma(\mu+1)\Gamma(\upsilon+1)\Gamma\left(\beta+\frac{\alpha+\beta}{2}+\sigma+n+1\right)}$$

$$\times{}_{4}F_{5}\left(\begin{array}{c}\beta+\frac{\mu+\upsilon}{2},\beta+\frac{\mu+\upsilon}{2}-\rho,\frac{\mu+\upsilon+1}{2},1+\frac{\mu+\upsilon}{2}\\ \cdots\\ \mu+1,\upsilon+1,\mu+\upsilon+1,\beta+\frac{\mu+\upsilon}{2}+\sigma+n+1,\beta+\frac{\mu+\upsilon}{2}-\rho-n\end{array};-2ab^{2}\right)\tag{3.1}$$

where $a>0, Re(\sigma)>-1, Re(2\beta+\mu+\upsilon)>0$

3. Required formula

Let $X_{e,f}=(x+a)^{e}(x-a)^{f}$ and $b_{k}=\dfrac{\prod_{j=1}^{p}(\lambda_{j})_{k\rho_{j}}}{(a+k)^{\mathfrak{s}}\prod_{j=1}^{q}(\mu_{j})_{k\sigma_{j}}}$. We have the following result :

$$\int_{0}^{1}(x+a)^{\sigma}(x-a)^{\beta}J_{\mu}(b\sqrt{a-x})J_{\upsilon}(b\sqrt{a-x})P_{n}^{\rho,\sigma}\left(\frac{x}{a}\right)\phi_{(\lambda_{1},\cdots,\lambda_{p},\mu_{1},\cdots,\mu_{q})}^{(\rho_{1},\cdots,\rho_{p},\sigma_{1},\cdots,\sigma_{q})}(zX_{e,f};\mathfrak{s},a)$$

$$S_{N_{1},\cdots,N_{t}}^{M_{1}',\cdots,M_{t}'}\left(\begin{array}{c}y_{1}X_{\gamma_{1},\mu_{1}}\\ \cdots\\ y_{t}X_{\gamma_{t},\mu_{t}}\end{array}\right)\aleph_{u:w}^{0,N:v}\left(\begin{array}{c}z_{1}X_{\alpha_{1},\beta_{1}}\\ \cdots\\ z_{r}X_{\alpha_{r},\beta_{r}}\end{array}\right)I_{U:p_{s},q_{s};W}^{V;0,n_{s};X}\left(\begin{array}{c}Z_{1}X_{\eta_{1},\epsilon_{1}}\\ \cdots\\ Z_{s}X_{\eta_{s},\epsilon_{s}}\end{array}\right)\mathrm{d}x\ =\frac{2^{\beta+\sigma-\frac{\mu+\upsilon}{2}}}{n!}a^{\beta+\sigma+\frac{\mu+\upsilon}{2}}$$

$$\frac{1}{\Gamma(\mu+1)\Gamma(\upsilon+1)}\sum_{K_{1}=0}^{[N_{1}'/M_{1}']}\cdots\sum_{K_{t}=0}^{[N_{t}'/M_{t}']}\sum_{k=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{G_{1},\cdots,G_{r}=0}^{\infty}\sum_{g_{1}=0}^{M_{1}}\cdots\sum_{g_{r}=0}^{M_{r}}\frac{(-)^{G_{1}+\cdots+G_{r}}}{\delta_{g_{1}}G_{1}!\cdots\delta_{g_{r}}G_{r}!}y_{1}^{K_{1}}\cdots y_{t}^{K_{t}}$$

$$G(\eta_{G_{1},g_{1}},\cdots\eta_{G_{r},g_{r}})a_{1}\frac{b_{k}z^{k}}{k!}\ z_{1}^{\eta_{G_{1},g_{1}}}\cdots z_{r}^{\eta_{G_{r},g_{r}}}\ \frac{\left(\frac{\mu+\upsilon+1}{2}\right)_{n'}\left(1+\frac{\mu+\upsilon}{2}\right)_{n'}(-2ab^{2})^{n'}}{(\mu+1)_{n'}(\upsilon+1)_{n'}(\mu+\upsilon+1)_{n'}n'!}$$

$$I_{U:p_{s}+5,q_{s}+4;W}^{V;0,n_{s}+5;X}\left(\begin{array}{c}(2a)^{\eta_{1}+\epsilon_{1}}Z_{1}\\ \cdots\\ \cdots\\ (2a)^{\eta_{s}+\epsilon_{s}}Z_{s}\end{array}\left|\begin{array}{c}A\ ;\ (-n-\rho+\frac{\mu+\upsilon}{2}+\beta+kf+\sum_{i=1}^{t}K_{i}\mu_{i}+\sum_{i=1}^{r}\eta_{G_{i},g_{i}}\beta_{i};-\epsilon_{1},\cdots,-\epsilon_{s}),\\ \cdots\\ B\ ;\ (-\rho+\frac{\mu+\upsilon}{2}+\beta+kf+\sum_{i=1}^{t}K_{i}\mu_{i}+\sum_{i=1}^{r}\eta_{G_{i},g_{i}}\beta_{i};-\epsilon_{1},\cdots,-\epsilon_{s})\end{array}\right.\right.$$

$$(1\text{-n'-}\tfrac{\mu+\upsilon}{2}\text{-}(\beta + kf + \textstyle\sum_{i=1}^{t} K_i\mu_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),$$

$$\cdots$$

$$(\text{-n-n'-}\tfrac{\mu+\upsilon}{2} - (\sigma + \beta + k(e+f) + \textstyle\sum_{i=1}^{t} K_i(\gamma_i+\mu) + \sum_{i=1}^{r} \eta_{G_i,g_i}(\alpha_i+\beta_i)); \eta_1+\epsilon_1, \cdots, \eta_s+\epsilon_s),$$

$$(\text{n}+\rho - \tfrac{\mu+\upsilon}{2} - (\beta + kf + \textstyle\sum_{i=1}^{t} K_i\mu + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),,$$

$$\cdots$$

$$(\text{n}+\rho - n' - \tfrac{\mu+\upsilon}{2} - (\beta + kf + \textstyle\sum_{i=1}^{t} K_i\mu + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),$$

$$(1+\rho - n' - \tfrac{\mu+\upsilon}{2} - (\beta + kf + \textstyle\sum_{i=1}^{t} K_i\mu + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),,$$

$$\cdots$$

$$(1+\rho - \tfrac{\mu+\upsilon}{2} - (\beta + kf + \textstyle\sum_{i=1}^{t} K_i\mu + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),$$

$$(\text{-n}-(\sigma + ke + \textstyle\sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); \eta_1, \cdots, \eta_s), \mathfrak{A}; A'$$

$$\cdots$$

$$\mathfrak{B} \; ; \text{B'} , \qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.1)$$

Provided that

a) $min\{\alpha, \beta, e, f, \gamma_i, \mu_i, \alpha_j, \beta_j, \eta_k, \epsilon_l\} > 0, i = 1, \cdots, t, j = 1, \cdots, r, k = 1, \cdots, s \; ; Re(a) > 0$

b) $Re\big[\sigma + ke + \displaystyle\sum_{i=1}^{r} \alpha_i \min_{1\leqslant j\leqslant M_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} \eta_i \min_{1\leqslant j\leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}\big] > -1$

c) $Re\big[2\beta + \mu + \upsilon + 2kf + 2\displaystyle\sum_{i=1}^{r} \beta_i \min_{1\leqslant j\leqslant M_i} \frac{d_j^{(i)}}{\delta_j^{(i)}} + 2\sum_{i=1}^{s} \epsilon_i \min_{1\leqslant j\leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}\big] > 0$

d) $|argz_k| < \dfrac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.5) ; $i = 1, \cdots, r$

e) $|argZ_k| < \dfrac{1}{2}\Omega_i^{(k)}\pi$, where $\Omega_i^{(k)}$ is defined by (1.11) ; $i = 1, \cdots, s$

f) The conditions (f) are satisfied

g) The series occuring on the right-hand side of (4.1) is absolutely and uniformly convergent.

Proof

First, expressing the extension of the Hurwitz-Lerch Zeta function $\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(zX_{d,e}; \mathfrak{s}, a)$ in serie with the help of equation (2.1), the Aleph-function of r variables in series with the help of equation (1.6), the general class of polynomial of several variables $S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1, \cdots, y_t]$ with the help of equation (1.19) and the Prasad's multivariable I-function of s variables in Mellin-Barnes contour integral with the help of equation (1.10), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the resulting integral with the help of equation (3.1) and expressing the generalized hypergeometric function $_4F_5$ in serie, use several times the following relations

$\Gamma(a)(a)_n = \Gamma(a+n)$ and $a = \dfrac{\Gamma(a+1)}{\Gamma(a)}$ with $Re(a) > 0$. Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

The quantities $U, V, W, X, A, B, \mathfrak{A}, \mathfrak{B}, A'$ and B' are defined by the equations (1.14) to (1;19)

5. Particular case

If $U = V = A = B = 0$, the multivariable I-function defined by Prasad degenere in multivariable H-function defined by Srivastava et al [5]. We have the following result.

$$\int_0^1 (x+a)^\sigma (x-a)^\beta J_\mu(b\sqrt{a-x}) J_\upsilon(b\sqrt{a-x}) P_n^{(\rho,\sigma)}\left(\frac{x}{a}\right) \phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(zX_{d,e}; \mathfrak{s}, a)$$

$$S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'}\left(\begin{array}{c} y_1 X_{\gamma_1,\mu_1} \\ \cdots \\ y_t X_{\gamma_t,\mu_t} \end{array}\right) \aleph_{u:w}^{0,N:v}\left(\begin{array}{c} z_1 X_{\alpha_1,\beta_1} \\ \cdots \\ z_r X_{\alpha_r,\beta_r} \end{array}\right) H_{p_s,q_s;W}^{0,n_s;X}\left(\begin{array}{c} Z_1 X_{\eta_1,\epsilon_1} \\ \cdots \\ Z_s X_{\eta_s,\epsilon_s} \end{array}\right) \mathrm{d}x = \frac{2^{\beta+\sigma-\frac{\mu+\upsilon}{2}}}{n!} a^{\beta+\sigma+\frac{\mu+\upsilon}{2}}$$

$$\frac{1}{\Gamma(\mu+1)\Gamma(\upsilon+1)} \sum_{K_1=0}^{[N_1'/M_1']} \cdots \sum_{K_t=0}^{[N_t'/M_t']} \sum_{k=0}^\infty \sum_{n'=0}^\infty \sum_{G_1,\cdots,G_r=0}^\infty \sum_{g_1=0}^{M_1} \cdots \sum_{g_r=0}^{M_r} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} y_1^{K_1} \cdots y_t^{K_t}$$

$$G(\eta_{G_1,g_1}, \cdots \eta_{G_r,g_r}) a_1 \frac{b_k z^k}{k!} z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}} \frac{\left(\frac{\mu+\upsilon+1}{2}\right)_{n'} \left(1+\frac{\mu+\upsilon}{2}\right)_{n'} (-2ab^2)^{n'}}{(\mu+1)_{n'}(\upsilon+1)_{n'}(\mu+\upsilon+1)_{n'} n'!}$$

$$H_{p_s+5,q_s+4;W}^{0,n_s+5;X}\left(\begin{array}{c} (2a)^{\eta_1+\epsilon_1} Z_1 \\ \cdots \\ \cdots \\ (2a)^{\eta_s+\epsilon_s} Z_s \end{array}\middle| \begin{array}{c} (\text{-n-}\rho + \frac{\mu+\upsilon}{2} + \beta + kf + \sum_{i=1}^t K_i\mu_i + \sum_{i=1}^r \eta_{G_i,g_i}\beta_i; -\epsilon_1, \cdots, -\epsilon_s), \\ \cdots \\ (\text{-}\rho + \frac{\mu+\upsilon}{2} + \beta + kf + \sum_{i=1}^t K_i\mu_i + \sum_{i=1}^r \eta_{G_i,g_i}\beta_i; -\epsilon_1, \cdots, -\epsilon_s) \end{array}\right.$$

$$(\text{1-n'-}\tfrac{\mu+\upsilon}{2}\text{-}(\beta + kf + \sum_{i=1}^t K_i\mu_i + \sum_{i=1}^r \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),$$
$$\cdots$$
$$(\text{-n-n'-}\tfrac{\mu+\upsilon}{2} - (\sigma + \beta + k(e+f) + \sum_{i=1}^t K_i(\gamma_i+\mu) + \sum_{i=1}^r \eta_{G_i,g_i}(\alpha_i+\beta_i)); \eta_1+\epsilon_1, \cdots, \eta_s+\epsilon_s),$$

$$(\text{n+}\rho - \tfrac{\mu+\upsilon}{2} - (\beta + kf + \sum_{i=1}^t K_i\mu + \sum_{i=1}^r \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),,$$
$$\cdots$$
$$(\text{n+}\rho - n' - \tfrac{\mu+\upsilon}{2} - (\beta + kf + \sum_{i=1}^t K_i\mu + \sum_{i=1}^r \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),$$

$$(1+\rho - n' - \frac{\mu+\upsilon}{2} - (\beta + kf + \sum_{i=1}^{t} K_i \mu + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),,$$

$$\cdot \ \cdot \ \cdot$$

$$(1+\rho - \frac{\mu+\upsilon}{2} - (\beta + kf + \sum_{i=1}^{t} K_i \mu + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),$$

$$(\text{-n} - (\sigma + ke + \sum_{i=1}^{t} K_i \gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); \eta_1, \cdots, \eta_s), \mathfrak{A}; A'$$

$$\cdot \ \cdot \ \cdot$$

$$\mathfrak{B} \ ; B' \ , \qquad \Bigg) \qquad (5.1)$$

under the same notations and conditions that (4.1) with $U = V = A = B = 0$ and B_1 is defined by (4.1)

6. Conclusion

In this paper we have evaluated a generalized integral involving the product of two Bessel functions, the Jacobi polynomial, the multivariable Aleph-function, a class of polynomials of several variables, the extension of the Hurwitz-Lerch Zeta function , and the multivariable I-function defined by Prasad. The integral established in this paper is of very general nature as it contains Multivariable Aleph-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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Personal adress : 411 Avenue Joseph Raynaud

Le parc Fleuri , Bat B

83140 , Six-Fours les plages

Tel : 06-83-12-49-68

Department : VAR

Country : FRANCE