Integral involving the extension of the Hurwitz-Lerch Zeta function,

Bessel functions, a class of polynomials and multivariable I-functions III

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ABSTRACT

In the present paper we evaluate a general integral involving the product of the extension of the Hurwitz-Lerch Zeta function, product of two Bessel functions and product of two the multivariable I-functions defined by Prasad [4] and general class of polynomials of several variables. The importance of the result established in this paper lies in the fact they involve the Aleph-function of several variables which is sufficiently general in nature and capable to yielding a large of results merely by specializating the parameters their in.

Keywords: general class of polynomials, the extension of the Hurwitz-Lerch Zeta function, multivariable I-function, multivariable H-function, Bessel function

1. Introduction

In the present paper we evaluate a general integral involving the product of the extension of the Hurwitz-Lerch Zeta function, product of two Bessel functions and product of two the multivariable I-functions defined by Prasad [2] and general class of polynomials of several variables. We will study the particular case of multivariable H-function defined by Srivastava et [4]

The multivariable I-function is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, z_2, ...z_r) = I_{P_2,Q_2,P_3,Q_3;\cdots;P_r,Q_r:P',Q';\cdots;P^{(r)},Q^{(r)}}^{0,N_2;0,N_3;\cdots;0,N_r:m',n';\cdots;M^{(s)},N^{(r)}} \left(\begin{array}{c} z_1 \\ . \\ . \\ . \\ z_r \end{array} \right| \begin{array}{c} (A_{2j}; \gamma'_{2j}, \gamma''_{2j})_{1,P_2}; \cdots; \\ \\ \\ (B_{2j}; \delta'_{2j}, \delta''_{2j})_{1,Q_2}; \cdots; \end{array}$$

$$\left. \begin{array}{c} (A_{rj}; \gamma'_{rj}, \cdots, \gamma^{(r)}_{rj})_{1,P_r} : (A'_j, \gamma'_j)_{1,P'}; \cdots; (A^{(r)}_j, \gamma^{(r)}_j)_{1,P^{(r)}} \\ \\ (B_{rj}; \delta'_{rj}, \cdots, \delta^{(r)}_{rj})_{1,Q_r} : (B'_j, \delta'_j)_{1,Q'}; \cdots; (B^{(r)}_j, \delta^{(r)}_j)_{1,Q^{(r)}} \end{array} \right)$$

(1.1)

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi(s_1, \cdots, s_r) \prod_{i=1}^{r} \theta_i(t_i) z_i^{t_i} dt_1 \cdots dt_r$$

(1.2)

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|argz_k| < \frac{1}{2} \Omega_i^{(k)} \pi \ , \ \text{where}$$

$$\Omega_i^{(k)} = \sum_{k=1}^{N^{(i)}} \gamma_k^{(i)} - \sum_{k=N^{(i)}+1}^{P^{(i)}} \gamma_k^{(i)} + \sum_{k=1}^{M^{(i)}} \delta_k^{(i)} - \sum_{k=M^{(i)}+1}^{Q^{(i)}} \delta_k^{(i)} + \left(\sum_{k=1}^{N_2} \alpha_{2k}^{(i)} - \sum_{k=N_2+1}^{P_2} \gamma_{2k}^{(i)} \right) + \cdots +$$

$$\left(\sum_{k=1}^{N_r} \gamma_{rk}^{(i)} - \sum_{k=N_r+1}^{P_r} \gamma_{rk}^{(i)} \right) - \left(\sum_{k=1}^{Q_2} \delta_{2k}^{(i)} + \sum_{k=1}^{Q_3} \delta_{3k}^{(i)} + \cdots + \sum_{k=1}^{Q_r} \delta_{rk}^{(i)} \right) \tag{1.3}$$

where $i = 1, \cdots, r$

The complex numbers z_i are not zero. Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$$I(z_1, \cdots, z_r) = 0(|z_1|^{\gamma_1'}, \cdots, |z_r|^{\gamma_r'}), max(|z_1|, \cdots, |z_r|) \to 0$$

$$I(z_1, \cdots, z_r) = 0(|z_1|, \cdots, |z_r|^{\beta_s'}), min(|z_1|, \cdots, |z_r|) \to \infty$$

where, with $k = 1, \cdots, r : \gamma_k' = min[Re(B_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, M_k$ and

$$\delta_k' = max[Re((A_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, N_k$$

Serie representation of multivariable I-function of several variables is given by

$$\aleph(y_1, \cdots, y_r) = \sum_{G_1, \cdots, G_r=0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_r=0}^{M_r} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!} \phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r})$$

$$\times \ \theta_1(\eta_{G_1,g_1}) \cdots \theta_r(\eta_{G_r,g_r}) y_1^{-\eta_{G_1,g_1}} \cdots y_r^{-\eta_{G_r,g_r}} \tag{1.4}$$

Where $\psi(., \cdots, .), \theta_i(.), i = 1, \cdots, r$ are defined by Prasad (see integral (1.2))

$$\eta_{G_1,g_1} = \frac{B_{g_1}^{(1)} + G_1}{\delta_{g_1}^{(1)}}, \cdots, \ \eta_{G_r,g_r} = \frac{B_{g_r}^{(r)} + G_r}{\delta_{g_r}^{(r)}}$$

which is valid under the conditions $\quad \delta_{g_i}^{(i)}[B_j^i + p_i] \neq \delta_j^{(i)}[B_{g_i}^i + G_i] \tag{1.5}$

for $j \neq M_i, M_i = 1, \cdots \eta_{G_i,g_i}; P_i, N_i = 0, 1, 2, \cdots, ; y_i \neq 0, i = 1, \cdots, r \tag{1.6}$

In the document , we will note :

$$G(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}) = \phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r})\theta_1(\eta_{G_1,g_1}) \cdots \theta_r(\eta_{G_r,g_r}) \tag{1.7}$$

where $\phi(\eta_{G_1,g_1}, \cdots, \eta_{G_r,g_r}), \theta_1(\eta_{G_1,g_1}), \cdots, \theta_r(\eta_{G_r,g_r})$ are given in (1.2)

We will use these following notations in this paper :

$$U_1 = P_2, Q_2; P_3, Q_3; \cdots; P_{r-1}, q_{r-1}; V_1 = 0, N_2; 0, N_3; \cdots; 0, N_{r-1} \tag{1.8}$$

$$W_1 = (P', Q'); \cdots ; (P^{(r)}, Q^{(r)}); X_1 = (M', N'); \cdots ; (M^{(r)}, N^{(r)}) \tag{1.9}$$

$$A_1 = (A_{2k}, \gamma'_{2k}, \gamma''_{2k}); \cdots ; (A_{(r-1)k}, \gamma'_{(r-1)k}, \gamma''_{(r-1)k}, \cdots , \gamma^{(r-1)}_{(r-1)k}) \tag{1.10}$$

$$B_1 = (B_{2k}, \delta'_{2k}, \delta''_{2k}); \cdots ; (B_{(r-1)k}, \delta'_{(r-1)k}, \delta''_{(r-1)k}, \cdots , \delta^{(r-1)}_{(r-1)k}) \tag{1.11}$$

$$\mathfrak{A}_1 = (A_{rk}; \gamma'_{rk}, \gamma''_{rk}, \cdots , \gamma^r_{rk}) : \mathfrak{B}_1 = (B_{rk}; \delta'_{rk}, \delta''_{rk}, \cdots , \delta^r_{rk}) \tag{1.12}$$

$$A'_1 = (A'_k, \gamma'_k)_{1,P'}; \cdots ; (a^{(r)}_k, \gamma^{(r)}_k)_{1,P^{(r)}}; B'_1 = (B'_k, \delta'_k)_{1,Q'}; \cdots ; (B^{(r)}_k, \delta^{(r)}_k)_{1,Q^{(r)}} \tag{1.13}$$

The multivariable I-function of r-variables write :

$$I(z_1, \cdots , z_r) = I^{V_1;0,N_r;X_1}_{U_1:P_r,Q_r;W_1} \left(\begin{array}{c} z_1 \\ . \\ . \\ . \\ z_s \end{array} \middle| \begin{array}{c} A_1; \mathfrak{A}_1; A'_1 \\ \\ \\ B_1; \mathfrak{B}_1; B'_1 \end{array} \right) \tag{1.14}$$

The multivariable I-function of s-variables is defined in term of multiple Mellin-Barnes type integral :

$$I(z_1, z_2, ...z_s) = I^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(s)},n^{(s)}}_{p_2,q_2,p_3,q_3;\cdots;p_s,q_s:p',q';\cdots;p^{(s)},q^{(s)}} \left(\begin{array}{c} z_1 \\ . \\ . \\ . \\ z_s \end{array} \middle| \begin{array}{c} (a_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2}; \cdots ; \\ \\ \\ (b_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2}; \cdots ; \end{array} \right.$$

$$\left. \begin{array}{c} (a_{sj}; \alpha'_{sj}, \cdots , \alpha^{(s)}_{sj})_{1,p_s} : (a'_j, \alpha'_j)_{1,p'}; \cdots ; (a^{(s)}_j, \alpha^{(s)}_j)_{1,p^{(s)}} \\ \\ (b_{sj}; \beta'_{sj}, \cdots , \beta^{(s)}_{sj})_{1,q_s} : (b'_j, \beta'_j)_{1,q'}; \cdots ; (b^{(s)}_j, \beta^{(s)}_j)_{1,q^{(s)}} \end{array} \right) \tag{1.15}$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L_1} \cdots \int_{L_s} \xi(t_1, \cdots , t_s) \prod_{i=1}^{s} \phi_i(t_i) z_i^{t_i} \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{1.16}$$

The defined integral of the above function, the existence and convergence conditions, see Y,N Prasad [2]. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function.

The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}\Omega_i'^{(k)}\pi$, where

$$\Omega_i'^{(k)} = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) + \cdots +$$

$$\left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)}\right) \tag{1.17}$$

where $i = 1, \cdots, s$

The complex numbers z_i are not zero. Throughout this document , we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the the asymptotic expansion in the following convenient form :

$\aleph(z_1, \cdots, z_s) = 0(\ |z_1|^{\alpha'_1}, \cdots, |z_s|^{\alpha'_s}\ )\ , max(\ |z_1|, \cdots, |z_s|\ ) \to 0$

$\aleph(z_1, \cdots, z_s) = 0(\ |z_1|^{\beta'_1}, \cdots, |z_s|^{\beta'_s}\ )\ , min(\ |z_1|, \cdots, |z_s|\ ) \to \infty$

where $k = 1, \cdots, z : \alpha'_k = min[Re(b_j^{(k)}/\beta_j^{(k)})], j = 1, \cdots, m_k$ and

$\beta'_k = max[Re((a_j^{(k)} - 1)/\alpha_j^{(k)})], j = 1, \cdots, n_k$

We will use these following notations in this paper :

$$U = p_2, q_2; p_3, q_3; \cdots; p_{s-1}, q_{s-1}; V = 0, n_2; 0, n_3; \cdots; 0, n_{s-1} \tag{1.18}$$

$$W = (p', q'); \cdots; (p^{(s)}, q^{(s)}); X = (m', n'); \cdots; (m^{(s)}, n^{(s)}) \tag{1.19}$$

$$A = (a_{2k}, \alpha'_{2k}, \alpha''_{2k}); \cdots; (a_{(s-1)k}, \alpha'_{(s-1)k}, \alpha''_{(s-1)k}, \cdots, \alpha^{(s-1)}_{(s-1)k}) \tag{1.20}$$

$$B = (b_{2k}, \beta'_{2k}, \beta''_{2k}); \cdots; (b_{(s-1)k}, \beta'_{(s-1)k}, \beta''_{(s-1)k}, \cdots, \beta^{(s-1)}_{(s-1)k}) \tag{1.21}$$

$$\mathfrak{A} = (a_{sk}; \alpha'_{sk}, \alpha''_{sk}, \cdots, \alpha^s_{sk}) : \mathfrak{B} = (b_{sk}; \beta'_{sk}, \beta''_{sk}, \cdots, \beta^s_{sk}) \tag{1.22}$$

$$A' = (a'_k, \alpha'_k)_{1,p'}; \cdots; (a_k^{(s)}, \alpha_k^{(s)})_{1,p^{(s)}}; B' = (b'_k, \beta'_k)_{1,q'}; \cdots; (b_k^{(s)}, \beta_k^{(s)})_{1,q^{(s)}} \tag{1.23}$$

The multivariable I-function write :

$$I(z_1, \cdots, z_s) = I_{U:p_s,q_s;W}^{V;0,n_s;X} \left(\begin{array}{c|c} z_1 & \\ . & A \; ; \mathfrak{A} ; A' \\ . & \\ . & \\ z_s & B; \; \mathfrak{B}; \; B' \end{array} \right) \tag{1.24}$$

The generalized polynomials defined by Srivastava [3], is given in the following manner :

$$S_{N'_1,\cdots,N'_t}^{M'_1,\cdots,M'_t}[y_1, \cdots, y_t] = \sum_{K_1=0}^{[N'_1/M'_1]} \cdots \sum_{K_t=0}^{[N'_t/M'_t]} \frac{(-N'_1)_{M'_1 K_1}}{K_1!} \cdots \frac{(-N'_t)_{M'_t K_t}}{K_t!}$$

$$A[N'_1, K_1; \cdots ; N'_t, K_t] y_1^{K_1} \cdots y_t^{K_t} \tag{1.25}$$

Where $M'_1, \cdots, M'_s$ are arbitrary positive integers and the coefficients $A[N'_1, K_1; \cdots ; N'_t, K_t]$ are arbitrary constants, real or complex. In the present paper, we use the following notation

$$a_1 = \frac{(-N'_1)_{M'_1 K_1}}{K_1!} \cdots \frac{(-N'_t)_{M'_t K_t}}{K_t!} A[N'_1, K_1; \cdots ; N'_t, K_t] \tag{1.26}$$

3. Extension of the Hurwitz-Lerch Zeta function

The extension of the Hurwitz-Lerch Zeta function $\phi(z, S, a)$ is introduced by Srivastava et al ([5],eq.(6.2), page 503) as follows :

$$\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z; \mathfrak{s}, a) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^{\mathfrak{s}} \prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \times \frac{z^n}{n!} \tag{3.1}$$

with : $p, q \in \mathbb{N}_0, \lambda_j \in \mathbb{C}(j = 1, \cdots, p), a, \mu_j \in \mathbb{C} \setminus \mathbb{Z}_0^* \; (j = 1, \cdots, q), \rho_j, \sigma_k \in \mathbb{R}^+$
$(j = 1, \cdots, p; k = 1, \cdots, q)$

where $\Delta > -1$ when $\mathfrak{s}, z \in \mathbb{C}; \Delta = -1$ and $s \in \mathbb{C}, when |z| < \bigtriangledown^*, \Delta = -1$ and $Re(\chi) > \frac{1}{2}$ when $|z| = \bigtriangledown^*$

$$\bigtriangledown^* = \prod_{j=1}^{p} \rho_j^{\rho_j} \prod_{j=1}^{q} \sigma_j^{\sigma_j}; \Delta = \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j; \chi = \mathfrak{s} + \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}$$

We denote these conditions, the conditions (f).

3. Required integral

We have the following integral, see Luke ([1], 13.3.2 35 page 302)

$$\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1} x \cos^{2\beta-1} x J_\mu(c \sin^2 x) J_\upsilon(d \cos^2 x) \mathrm{d}x = \frac{\left(\frac{c}{2}\right)^\mu \left(\frac{d}{2}\right)^\upsilon \Gamma(\alpha+\mu)\Gamma(\beta+\upsilon)}{\Gamma(\mu+1)\Gamma(\upsilon+1)\Gamma(\alpha+\beta+\mu+\upsilon)}$$

$$\sum_{k=0}^{\infty} \frac{(-)^k \left(\frac{c}{2}\right)^{2k} \left(\frac{\alpha+\mu}{2}\right)_k \left(\frac{\alpha+\mu+1}{2}\right)_k}{k!(\mu+1)_k \left(\frac{\alpha+\beta+\mu+\upsilon}{2}\right)_k \left(\frac{\alpha+\beta+\mu+\upsilon+1}{2}\right)_k} \times {}_4F_3 \left(\begin{array}{c} -k, -\mu-k, \frac{\beta+\upsilon}{2}, \frac{\beta+\upsilon+1}{2} \\ \cdots \\ \frac{1-\alpha-\mu}{2}-k, \frac{2-\alpha-\mu}{2}-k, \upsilon+1 \end{array} ; \frac{d^2}{c^2} \right) \tag{3.1}$$

with $Re(\alpha+\mu) > 0, Re(\upsilon+\beta) > 0$

4. General integral

Let $X_{a,b} = sin^{2a}x cos^{2b}x$ and $b_l = \dfrac{\prod_{j=1}^{p} (\lambda_j)_{k\rho_j}}{(a+l)^{\mathfrak{s}} \prod_{j=1}^{q} (\mu_j)_{k\sigma_j}}$. We have the following result :

$$\int_0^{\frac{\pi}{2}} \sin^{2\alpha-1} x \cos^{2\beta-1} x J_\mu(c\sin^2 x) J_\upsilon(d\cos^2 x) \phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z X_{e,f}; \mathfrak{s}, a)$$

$$S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'} \left(\begin{array}{c} y_1 X_{\gamma_1,\mu_1} \\ \cdots \\ y_t X_{\gamma_t,\mu_t} \end{array} \right) I_{U_1:P_r,Q_r;W_1}^{V_1;0,N_r;X_1} \left(\begin{array}{c} z_1 X_{\alpha_1,\beta_1} \\ \cdots \\ z_r X_{\alpha_r,\beta_r} \end{array} \right) I_{U:p_s,q_s;W}^{V;0,n_s;X} \left(\begin{array}{c} Z_1 X_{\eta_1,\epsilon_1} \\ \cdots \\ Z_s X_{\eta_s,\epsilon_s} \end{array} \right) dx = \frac{\left(\frac{c}{2}\right)^\mu \left(\frac{d}{2}\right)^\upsilon}{2\Gamma(\mu+1)\Gamma(\upsilon+1)}$$

$$\sum_{K_1=0}^{[N_1'/M_1']} \cdots \sum_{K_t=0}^{[N_t'/M_t']} \sum_{k=0}^{\infty} \sum_{n=0}^{k+1} \sum_{l=0}^{\infty} \sum_{G_1,\cdots,G_r=0}^{\infty} \sum_{g_1=0}^{M_1} \cdots \sum_{g_r=0}^{M_r} \frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1} G_1! \cdots \delta_{g_r} G_r!}$$

$$G(\eta_{G_1,g_1}, \cdots \eta_{G_r,g_r}) a_1 \frac{b_l z^l}{l!} z_1^{\eta_{G_1,g_1}} \cdots z_r^{\eta_{G_r,g_r}} y_1^{K_1} \cdots y_t^{K_t} \frac{(-)^k \left(\frac{c}{2}\right)^{2k} (-\mu-k)_n (-k)_n}{k!(2\upsilon+1)_n n!} \left(\frac{d}{c}\right)^{2m}$$

$$I_{U:p_s+10,q_s+9;W}^{V;0,n_s+10;X} \left(\begin{array}{c} Z_1 \\ \cdots \\ \cdots \\ Z_s \end{array} \middle| \begin{array}{c} A \; ; \; (1-(\mu+\alpha+le+\sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); \eta_1, \cdots, \eta_s), \\ \cdots \\ B \; , \end{array} \right.$$

$$(1-(\upsilon+\beta+lf+\sum_{i=1}^{t} K_i\mu_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \epsilon_1, \cdots, \epsilon_s),$$

$$\cdots$$

$$(1-(\mu+\upsilon+\alpha+\beta+l(e+f)+\sum_{i=1}^{t} K_i(\gamma_i+\mu) + \sum_{i=1}^{r} \eta_{G_i,g_i}(\alpha_i+\beta_i)); \eta_1+\epsilon_1, \cdots, \eta_s+\epsilon_s),$$

$$(1-\tfrac{1}{2}(\mu+\upsilon+\alpha+\beta+l(e+f)+\sum_{i=1}^{t} K_i(\gamma_i+\mu) + \sum_{i=1}^{r} \eta_{G_i,g_i}(\alpha_i+\beta_i)); \tfrac{\eta_1+\epsilon_1}{2}, \cdots, \tfrac{\eta_s+\epsilon_s}{2}),$$

$$\cdots$$

$$(1-k-\tfrac{1}{2}(\mu+\upsilon+\alpha+\beta+l(e+f)+\sum_{i=1}^{t} K_i(\gamma_i+\mu) + \sum_{i=1}^{r} \eta_{G_i,g_i}(\alpha_i+\beta_i)); \tfrac{\eta_1+\epsilon_1}{2}, \cdots, \tfrac{\eta_s+\epsilon_s}{2}),$$

$(\frac{1}{2}\text{-}\frac{1}{2}(\mu + \upsilon + \alpha + \beta + l(e+f) + \sum_{i=1}^{t} K_i(\gamma_i + \mu) + \sum_{i=1}^{r} \eta_{G_i,g_i}(\alpha_i + \beta_i)); \frac{\eta_1+\epsilon_1}{2}, \cdots, \frac{\eta_s+\epsilon_s}{2}),$

$\cdots$

$(\frac{1}{2}\text{-k-}\frac{1}{2}(\mu + \upsilon + \alpha + \beta + l(e+f) + \sum_{i=1}^{t} K_i(\gamma_i + \mu) + \sum_{i=1}^{r} \eta_{G_i,g_i}(\alpha_i + \beta_i)); \frac{\eta_1+\epsilon_1}{2}, \cdots, \frac{\eta_s+\epsilon_s}{2}),$

$(\text{1-k-}\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); \frac{\eta_1}{2}, \cdots, \frac{\eta_s}{2}),$

$\cdots$

$(\text{1-}\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); \frac{\eta_1}{2}, \cdots, \frac{\eta_s}{2}),$

$(\frac{1}{2}\text{-k-}\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); \frac{\eta_1}{2}, \cdots, \frac{\eta_s)}{2},$

$\cdots$

$(\frac{1}{2}\text{-}\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); \frac{\eta_1}{2}, \cdots, \frac{\eta_s)}{2}),$

$(\text{1-n-}\frac{1}{2}(\upsilon + \beta + le + \sum_{i=1}^{t} K_i\mu_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \frac{\epsilon_1}{2}, \cdots, \frac{\epsilon_s}{2}),$

$\cdots$

$(\text{1-}\frac{1}{2}(\upsilon + \beta + le + \sum_{i=1}^{t} K_i\mu_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \frac{\epsilon_1}{2}, \cdots, \frac{\epsilon_s}{2}),$

$(\frac{1}{2}\text{-n-}\frac{1}{2}(\upsilon + \beta + le + \sum_{i=1}^{t} K_i\mu_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \frac{\epsilon_1}{2}, \cdots, \frac{\epsilon_s}{2}),$

$\cdots$

$(\frac{1}{2}\text{-}\frac{1}{2}(\upsilon + \beta + le + \sum_{i=1}^{t} K_i\mu_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\beta_i); \frac{\epsilon_1}{2}, \cdots, \frac{\epsilon_s}{2}),$

$(\frac{1}{2}\text{+k+}\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); -\frac{\eta_1}{2}, \cdots, -\frac{\eta_s}{2}),$

$\cdots$

$(\frac{1}{2}\text{+k+n+}\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); -\frac{\eta_1}{2}, \cdots, -\frac{\eta_s}{2}),$

$(\text{k+}\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); -\frac{\eta_1}{2}, \cdots, -\frac{\eta_s}{2}), \mathfrak{A}; A'$

$\cdots$

$(\text{k+n+}\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i\gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); -\frac{\eta_1}{2}, \cdots, -\frac{\eta_s}{2}), \mathfrak{B}; B'$

(5.1)

Provided that

a) $min\{\alpha, \beta, e, f, \gamma_i, \mu_i, \alpha_j, \beta_j, \eta_k, \epsilon_l\} > 0, i = 1, \cdots, t, j = 1, \cdots, r, k = 1, \cdots, s$

b) $Re\left[\alpha + \mu + ke + \sum_{i=1}^{r} \alpha_i \min_{1 \leqslant j \leqslant M^{(i)}} \frac{B_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} \eta_i \min_{1 \leqslant j \leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > 0$

c) $Re\left[\beta + \upsilon + kf + \sum_{i=1}^{r} \beta_i \min_{1 \leqslant j \leqslant M^{(i)}} \frac{B_j^{(i)}}{\delta_j^{(i)}} + \sum_{i=1}^{s} \epsilon_i \min_{1 \leqslant j \leqslant m^{(i)}} \frac{b_j^{(i)}}{\beta_j^{(i)}}\right] > 0$

d) $|argz_k| < \frac{1}{2}\Omega_i^{(k)}\pi$, where $\Omega_i^{(k)}$ is defined by (1.3) ; $i = 1, \cdots, r$

e) $|argZ_k| < \frac{1}{2}\Omega_i'^{(k)}\pi$, where $\Omega_i'^{(k)}$ is defined by (1.17) ; $i = 1, \cdots, s$

f) The conditions (f) are satisfied

Proof

First, expressing the extension of the Hurwitz-Lerch Zeta function $\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(zX_{d,e};\mathfrak{s},a)$ in serie with the help of equation (2.1), the I-function of r variables in series with the help of equation (1.4), the general class of polynomial of several variables $S_{N_1',\cdots,N_t'}^{M_1',\cdots,M_t'}[y_1,\cdots,y_t]$ with the help of equation (1.25) and the Prasad's multivariable I-function of s variables in Mellin-Barnes contour integral with the help of equation (1.16), changing the order of integration ans summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process) and then evaluating the resulting integral with the help of equation (3.1) and expressing the generalized hypergeometric function ${}_4F_3$ in serie, use several times the following relations $\Gamma(a)(a)_n = \Gamma(a+n)$ and $a = \frac{\Gamma(a+1)}{\Gamma(a)}$ with $Re(a) > 0$. Finally interpreting the result thus obtained with the Mellin-barnes contour integral, we arrive at the desired result.

The quantities $.U_1, V_1, W_1, X_1, A_1, B_1, \mathfrak{A}_1, \mathfrak{B}_1, A_1'$ and B_1' are defined to (1.8) of (1.13) and the quantities $U, V, W, X, A, B, \mathfrak{A}, \mathfrak{B}, A'$ and B' are defined by the equations (1.18) to (1;23)

5. Particular case

If $U = V = A = B = U_1 = V_1 = A_1 = B_1 = 0$, the multivariable I-function defined by Prasad degenere in multivariable H-function defined by Srivastava et al [5]. Our integral contain two multivariable H-functions.

Denoted $G_1(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r}) = G(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})|_{A_1=B_1=0}$,we obtain the following formula

$$\int_0^{\frac{\pi}{2}}\sin^{2\alpha-1} x\cos^{2\beta-1} xJ_\mu(c\sin^2 x)J_\upsilon(dcos^2 x)\phi_{(\lambda_1,\cdots,\lambda_p,\mu_1,\cdots,\mu_q)}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(zX_{e,f};\mathfrak{s},a)$$

$$S_{N_1,\cdots,N_t}^{M_1',\cdots,M_t'}\left(\begin{array}{c} y_1X_{\gamma_1,\mu_1} \\ \cdots \\ y_tX_{\gamma_t,\mu_t} \end{array}\right)H_{P_r,Q_r;W_1}^{0,N_r;X_1}\left(\begin{array}{c} z_1X_{\alpha_1,\beta_1} \\ \cdots \\ z_rX_{\alpha_r,\beta_r} \end{array}\right)H_{p_s,q_s;W}^{0,n_s;X}\left(\begin{array}{c} Z_1X_{\eta_1,\epsilon_1} \\ \cdots \\ Z_sX_{\eta_s,\epsilon_s} \end{array}\right)\mathrm{d}x= \frac{\left(\frac{c}{2}\right)^\mu\left(\frac{d}{2}\right)^\upsilon}{2\Gamma(\mu+1)\Gamma(\upsilon+1)}$$

$$\sum_{K_1=0}^{[N_1'/M_1']}\cdots\sum_{K_t=0}^{[N_t'/M_t']}\sum_{k=0}^{\infty}\sum_{n=0}^{k+1}\sum_{l=0}^{\infty}\sum_{G_1,\cdots,G_r=0}^{\infty}\sum_{g_1=0}^{M_1}\cdots\sum_{g_r=0}^{M_r}\frac{(-)^{G_1+\cdots+G_r}}{\delta_{g_1}G_1!\cdots\delta_{g_r}G_r!}$$

$$G_1(\eta_{G_1,g_1},\cdots\eta_{G_r,g_r})a_1\frac{b_lz^l}{l!}\ z_1^{\eta_{G_1,g_1}}\cdots z_r^{\eta_{G_r,g_r}}y_1^{K_1}\cdots y_t^{K_t}\ \frac{(-)^k\left(\frac{c}{2}\right)^{2k}(-\mu-k)_n(-k)_n}{k!(2\upsilon+1)_n n!}\left(\frac{d}{c}\right)^{2m}$$

$$H^{0,n_s+10;X}_{p_s+10,q_s+9;W}\left(\begin{array}{c} Z_1 \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \\ Z_s \end{array}\right|\begin{array}{c} (1\text{-}(\mu+\alpha+le+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\eta_1,\cdots,\eta_s), \\ \cdot\cdot\cdot \\ \cdot\cdot\cdot \end{array}$$

$$(1\text{-}(\upsilon+\beta+lf+\sum_{i=1}^{t}K_i\mu_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\beta_i);\epsilon_1,\cdots,\epsilon_s),$$
$$\cdot\cdot\cdot$$
$$(1\text{-}(\mu+\upsilon+\alpha+\beta+l(e+f)+\sum_{i=1}^{t}K_i(\gamma_i+\mu)+\sum_{i=1}^{r}\eta_{G_i,g_i}(\alpha_i+\beta_i));\eta_1+\epsilon_1,\cdots,\eta_s+\epsilon_s),$$

$$(1\text{-}\tfrac{1}{2}(\mu+\upsilon+\alpha+\beta+l(e+f)+\sum_{i=1}^{t}K_i(\gamma_i+\mu)+\sum_{i=1}^{r}\eta_{G_i,g_i}(\alpha_i+\beta_i));\tfrac{\eta_1+\epsilon_1}{2},\cdots,\tfrac{\eta_s+\epsilon_s}{2}),$$
$$\cdot\cdot\cdot$$
$$(1\text{-k-}\tfrac{1}{2}(\mu+\upsilon+\alpha+\beta+l(e+f)+\sum_{i=1}^{t}K_i(\gamma_i+\mu)+\sum_{i=1}^{r}\eta_{G_i,g_i}(\alpha_i+\beta_i));\tfrac{\eta_1+\epsilon_1}{2},\cdots,\tfrac{\eta_s+\epsilon_s}{2}),$$

$$(\tfrac{1}{2}\text{-}\tfrac{1}{2}(\mu+\upsilon+\alpha+\beta+l(e+f)+\sum_{i=1}^{t}K_i(\gamma_i+\mu)+\sum_{i=1}^{r}\eta_{G_i,g_i}(\alpha_i+\beta_i));\tfrac{\eta_1+\epsilon_1}{2},\cdots,\tfrac{\eta_s+\epsilon_s}{2}),$$
$$\cdot\cdot\cdot$$
$$(\tfrac{1}{2}\text{-k-}\tfrac{1}{2}(\mu+\upsilon+\alpha+\beta+l(e+f)+\sum_{i=1}^{t}K_i(\gamma_i+\mu)+\sum_{i=1}^{r}\eta_{G_i,g_i}(\alpha_i+\beta_i));\tfrac{\eta_1+\epsilon_1}{2},\cdots,\tfrac{\eta_s+\epsilon_s}{2}),$$

$$(1\text{-k-}\tfrac{1}{2}(\mu+\alpha+le+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\tfrac{\eta_1}{2},\cdots,\tfrac{\eta_s}{2}),$$
$$\cdot\cdot\cdot$$
$$(1\text{-}\tfrac{1}{2}(\mu+\alpha+le+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\tfrac{\eta_1}{2},\cdots,\tfrac{\eta_s}{2}),$$

$$(\tfrac{1}{2}\text{-k-}\tfrac{1}{2}(\mu+\alpha+le+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\tfrac{\eta_1}{2},\cdots,\tfrac{\eta_s)}{2},$$
$$\cdot\cdot\cdot$$
$$(\tfrac{1}{2}\text{-}\tfrac{1}{2}(\mu+\alpha+le+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);\tfrac{\eta_1}{2},\cdots,\tfrac{\eta_s)}{2}),$$

$$(1\text{-n-}\tfrac{1}{2}(\upsilon+\beta+le+\sum_{i=1}^{t}K_i\mu_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\beta_i);\tfrac{\epsilon_1}{2},\cdots,\tfrac{\epsilon_s}{2}),$$
$$\cdot\cdot\cdot$$
$$(1\text{-}\tfrac{1}{2}(\upsilon+\beta+le+\sum_{i=1}^{t}K_i\mu_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\beta_i);\tfrac{\epsilon_1}{2},\cdots,\tfrac{\epsilon_s}{2}),$$

$$(\tfrac{1}{2}\text{-n-}\tfrac{1}{2}(\upsilon+\beta+le+\sum_{i=1}^{t}K_i\mu_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\beta_i);\tfrac{\epsilon_1}{2},\cdots,\tfrac{\epsilon_s}{2}),$$
$$\cdot\cdot\cdot$$
$$(\tfrac{1}{2}\text{-}\tfrac{1}{2}(\upsilon+\beta+le+\sum_{i=1}^{t}K_i\mu_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\beta_i);\tfrac{\epsilon_1}{2},\cdots,\tfrac{\epsilon_s}{2}),$$

$$(\tfrac{1}{2}\text{+k+}\tfrac{1}{2}(\mu+\alpha+le+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);-\tfrac{\eta_1}{2},\cdots,-\tfrac{\eta_s}{2}),$$
$$\cdot\cdot\cdot$$
$$(\tfrac{1}{2}\text{+k+n+}\tfrac{1}{2}(\mu+\alpha+le+\sum_{i=1}^{t}K_i\gamma_i+\sum_{i=1}^{r}\eta_{G_i,g_i}\alpha_i);-\tfrac{\eta_1}{2},\cdots,-\tfrac{\eta_s}{2}),$$

$$\left. \begin{array}{c} (\mathrm{k}+\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i \gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); -\frac{\eta_1}{2}, \cdots, -\frac{\eta_s}{2}), \mathfrak{A}; A' \\ \cdots \\ (\mathrm{k}+\mathrm{n}+\frac{1}{2}(\mu + \alpha + le + \sum_{i=1}^{t} K_i \gamma_i + \sum_{i=1}^{r} \eta_{G_i,g_i}\alpha_i); -\frac{\eta_1}{2}, \cdots, -\frac{\eta_s}{2}), \mathfrak{B}; B' \end{array} \right) \tag{5.1}$$

under the same notations and conditions that (4.1) with $U = V = A = B = U_1 = V_1 = A_1 = B_1 = 0$

6. Conclusion

In this paper we have evaluated a generalized integral involving the product of two Bessel functions, a class of polynomials of several variables, the extension of the Hurwitz-Lerch Zeta function , and product of two multivariable I-functions defined by Prasad. The integral established in this paper is of very general nature as it contains Multivariable I-function, which is a general function of several variables studied so far. Thus, the integral established in this research work would serve as a key formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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