ON W_7- CURVATURE TENSOR OF GENERALIZED SASAKIAN-SPACE-FORMS

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ABSTRACT. The object of the present paper is to study generalized Sasakian-space-forms satisfying certain curvature conditions on W_7- curvature tensor. In this paper, we study W_7- semisymmetric, $\xi - W_7-$ flat, generalized Sasakian-space-forms satisfying $G.S = 0, W_7-$ flat. Also satisfying $G.P = 0$, $G.\widetilde{C} = 0$, $G.R = 0$.

1. INTRODUCTION

In 2011, M.M. Tripathi and P. Gupta [8] introduced and explored $\tau-$ curvature tensor in semi -Riemannian manifolds. They gave properties and some identities of $\tau-$ curvature tensor. They defined W_7- curvature tensor of type $(0,4)$ for $(2n+1)-$dimensional Riemannian manifold, as

$$W_7(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{2n}\{S(Y,Z)g(X,U) - g(Y,Z)S(X,U)\} \tag{1.1}$$

where R and S denote the Riemannian curvature tensor of type $(0,4)$ defined by $`R(X,Y,Z,U) = g(R(X,Y)Z,U)$ and the Ricci tensor of type $(0,2)$ respectively. The curvature tensor defined by (1.1) is known as W_7- curvature tensor. A manifold whose W_7- curvature tensor vanishes at every point of the manifold is called W_7- flat manifold. They also defined $\tau-$conservative semi $-$ Riemannian manifolds and gave necessary and sufficient condition for semi $-$ Riemannian manifolds to be $\tau-$ conservative.

A. Sarkar and U.C. De [1] studied some curvature properties of generalized Sasakian-space-forms. C. Özgür and M.M. Tripathi [2] have given results on about P-Sasakian manifolds satisfying certain conditions on concircular curvature tensor. In [3] C. Özgür studied $\phi-$conformally flat LP- Sasakian manifolds. J.L. Cabrerizo and et al have given results in [7] about the structure of a class of K-contact manifolds.

In differential geometry, the curvature of a Riemannian manifold (M,g) plays a fundamental role as well known, the sectional curvature of a manifold determine the curvature tensor $R-$completely. A Riemannian manifold with constant sectional curvature c is called a real-space form and its curvature tensor is given by the equation

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$$

for any vector fields X, Y, Z on M. Models for these spaces are the Euclidean space $(c = 0)$, the sphere $(c > 0)$ and the Hyperbolic space $(c < 0)$.

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A similar situation can be found in the study of complex manifolds from a Riemannian point of view. If (M, J, g) is a Kaehler manifold with constant holomorphic sectional curvature $K(X \wedge JX) = c$, then it is said to be a complex space form and it is well known that its curvature tensor satisfies the equation

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(X,JZ)JY - g(Y,JZ)JX \\ &\quad +2g(X,JY)JZ\} \end{aligned}$$

for any vector fields X, Y, Z on M. These models are C^n, CP^n and CH^n depending on $c = 0, c > 0$ and $c < 0$ respectively.

On the other hand, Sasakian-space-forms play a similar role in contact metric geometry. For such a manifold, the curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= (\frac{c+3}{4})\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad +(\frac{c-1}{4})\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z \\ &\quad +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned}$$

for any vector fields X, Y, Z on M. These spaces can also be modeled depending on cases $c > -3, c = -3$ and $c < -3$.It is known that any three-dimensional $(\alpha, \beta)-$trans Sasakian manifold with α, β depending on ξ is a generalized Sasakian-space-forms [9]. Alegre et al. give results in [11] about B.Y. Chen's inequality on submanifolds of generalized complex space-forms and generalized Sasakian-space-forms. Al. Ghefari et al. analyse the CR submanifolds of generalized Sasakian-space-forms [12, 13].Sreenivasa. G.T. Venkatesha and Bagewadi C.S. [14] have some results on $(LCS)_{2n+1}-$Manifolds. S. K. Yadav, P.K. Dwivedi and D. Suthar [15] studied $(LCS)_{2n+1}-$ Manifolds satisfying certain conditions on the concircular curvature tensor. De and Sarakar [16] have studied generalized Sasakian-space-forms regarding projective curvature tensor. Motivated by the above studies, in the present paper, we study flatness and symmetric property of generalized Sasakian-space-forms regarding $W_7 - curvature\ tensor$. The present paper is organized as follows:-

In this paper, we study the $W_7 - curvature\ tensor$ of generalized Sasakian-space-forms with certain conditions. In section 2, some preliminary results are recalled. In section 3, we study W_7 semisymmetric generalized Sasakian-space-forms. Section 4 deals with $\xi - W_7$ flat generalized Sasakian-space-forms. Generalized Sasakian-space-forms satisfying $G.S = 0$ are studied in section 5. In section 6, $W_7 - flat$ generalized Sasakian-space-forms are studied. Section 7 is devoted to study of generalized Sasakian-space-forms satisfying $G.P = 0$. In section 8 contains generalized Sasakian-space-forms satisfying $G.\widetilde{C} = 0$. The last section contains generalized Sasakian-space-forms satisfying $G.R = 0$.

2. Preliminary

An odd $-$ dimensional differentiable manifold M^{2n+1} of differentiability class C^{r+1}, there exists a vector valued real linear function Φ, a 1-form η, associated vector field ξ and the Riemannian metric g satisfying

$$\Phi^2(X) = -X + \eta(X)\xi, \Phi(\xi) = 0 \tag{2.1}$$

$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \eta(\Phi X) = 0 \tag{2.2}$$

$$g(\Phi X, \Phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for arbitary vector fields X and Y, then (M^{2n+1}, g) is said to be an almost contact metric manifold [5], and the structure (Φ, ξ, η, g) is called an almost contact metric structure to M^{2n+1}. In view of (2.1), (2.2) and (2.3), we have

$$g(\Phi X, Y) = -g(X, \Phi Y), g(\Phi X, X) = 0 \tag{2.4}$$

$$\nabla_X \eta(Y) = g(\nabla_X \xi, Y) \tag{2.5}$$

Again we know [10] that in a $(2n+1)-$ dimensional generalized Sasakian-space-forms, we have

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} \\ &+ f_2\{g(X,\Phi Z)\Phi Y - g(Y,\Phi Z)\Phi X + 2g(X,\Phi Y)\Phi Z\} \\ &+ f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \end{aligned} \tag{2.6}$$

for any vector field X, Y, Z on M^{2n+1}, where R denotes the curvature tensor of M^{2n+1} and f_1, f_2, f_3 are smooth functions on the manifold.

The Ricci tensor S and the scalar curvature r of the manifold of dimension $(2n+1)$ are respectively, given by

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y) \tag{2.7}$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi \tag{2.8}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3 \tag{2.9}$$

Also for a generalized Sasakian-space-forms, we have

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} \tag{2.10}$$

$$R(\xi, X)Y = -R(X,\xi)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\} \tag{2.11}$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{2.12}$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X) \tag{2.13}$$

$$Q\xi = 2n(f_1 - f_3)\xi \tag{2.14}$$

where Q is the Ricci Operator, i.e.

$$g(QX,Y) = S(X,Y) \tag{2.15}$$

For a $(2n+1)-$ dimensional $(n>1)$ Almost Contact Metric, the W_7- curvature tensor G is given by

$$G(X,Y)Z = R(X,Y)Z - \frac{1}{2n}\{S(Y,Z)X - g(Y,Z)QX\} \tag{2.16}$$

the W_7- curvature tensor G in a generalized Sasakia-space-forms satisfies

$$G(X,Y)\xi = (f_1-f_3)[\eta(Y)X-\eta(X)Y]-\frac{1}{2n}(3f_2+(2n-1)f_3)[\eta(X)\eta(Y)\xi-\eta(Y)X] \tag{2.17}$$

$$G(X,\xi)\xi = \frac{1}{2n}(2nf_1+3f_2-f_3)\{X-\eta(X)\xi\} \tag{2.18}$$

$$G(\xi,Y)\xi = (f_1-f_3)\{\eta(Y)\xi - Y\} \tag{2.19}$$

$$G(\xi,X)Y = (f_1-f_3)\{2g(X,Y)\xi-\eta(Y)X\} - \frac{1}{2n}S(X,Y)\xi \tag{2.20}$$

$$\begin{aligned} \eta(G(X,Y)Z) \quad &= \quad (f_1-f_3)\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\} \\ &\quad -\frac{1}{2n}(3f_2+(2n-1)f_3)\{g(Y,Z)\eta(X)-\eta(X)\eta(Y)\eta(Z)\} \end{aligned} \tag{2.21}$$

Given an $(2n+1)-$ dimensional Riemannian manifold (M,g), the Concircular curvature tensor $\widetilde{C}$ is given by

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\{g(Y,Z)X-g(X,Z)Y\} \tag{2.22}$$

$$\widetilde{C}(\xi,X)Y = [f_1-f_3-\frac{r}{2n(2n+1)}]\{g(X,Y)\xi-\eta(Y)X\} \tag{2.23}$$

and

$$\eta(\widetilde{C}(X,Y)Z) = [f_1-f_3-\frac{r}{2n(2n+1)}]\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\} \tag{2.24}$$

and Projective curvature tensor is given by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X-S(X,Z)Y] \tag{2.25}$$

and related term

$$\eta(P(X,Y)\xi) = 0 \tag{2.26}$$

$$\eta(P(X,\xi)Z) = \frac{1}{2n}S(X,Z)-(f_1-f_3)g(X,Z) \tag{2.27}$$

$$\eta(P(\xi,Y)Z) = (f_1-f_3)g(Y,Z)-\frac{1}{2n}S(Y,Z) \tag{2.28}$$

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for any vector field X, Y, Z on M.

3. W_7- Semisymmetric Generalized Sasakian-Space-Forms

Definition 1. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-forms is said to be W_7- semisymmetric if it satisfies $R.G=0$, where R is the Riemannian curvature tensor and G is the W_7- curvature tensor of the space form.*

Theorem 1. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is W_7- semisymmetric if and only if $f_1=f_3$.*

Proof. Let us suppose that the generalized Sasakian-space-forms $M^{2n+1}(f_1, f_2, f_3)$ is W_7- semisymmetric, then we have

$$R(\xi, U).G(X, Y)\xi = 0 \tag{3.1}$$

The above equation can be written as

$$R(\xi, U)G(X, Y)\xi - G(R(\xi, U)X, Y)\xi - G(X, R(\xi, U)Y)\xi - G(X, Y)R(\xi, U)\xi = 0 \tag{3.2}$$

In view of (2.2), (2.10) & (2.11) the above equation reduces to

$$\begin{aligned}(f_1 - f_3)\{&g(U, G(X, Y)\xi)\xi - \eta(G(X, Y)\xi)U - g(U, X)G(\xi, Y)\xi \\ &+\eta(X)G(U, Y)\xi - g(U, Y)G(X, \xi)\xi + \eta(Y)G(X, U)\xi \\ &-\eta(U)G(X, Y)\xi + G(X, Y)U\} \\ = \;& 0\end{aligned} \tag{3.3}$$

In view of (2.16), (2.17) & (2.18) and taking the inner product of above equation with ξ, we get

$$(f_1 - f_3)\{g(U, G(X, Y)\xi + g(G(X, Y)U, \xi)\} = 0 \tag{3.4}$$

$$(f_1 - f_3)\{g(U, G(X, Y)\xi) + \eta(G(X, Y)U)\} = 0 \tag{3.5}$$

This implies either $f_1 = f_3$ or

$$g(U, G(X, Y)\xi) + \eta(G(X, Y)U) = 0$$

In the light of equation (2.17) and (2.21), the above equation gives

$$\eta(Y)g(X, U) - \eta(X)g(U, Y) = 0 \tag{3.6}$$

which is not possible in generalized Sasakian-space-form. Conversely, if $f_1 = f_3$, then from (2.11), $R(\xi, U) = 0$. Then obviously $R.G = 0$ is satisfied. This completes the proof. □

4. $\xi - W_7-$ Flat Generalized Sasakian-Space-Forms

Definition 2. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is said to be W_7- flat* [6] *if $G(X, Y)\xi = 0$ for all $X, Y \in TM$.*

Theorem 2. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is $\xi - W_7 - flat$ if and only if it is $\eta-$ Einstein Manifold.*

Proof. Let us consider that a generalized Sasakian-space-forms is $\xi - W_7-$ flat, i.e. $G(X,Y)\xi = 0$. Then from (2.16), we have

$$R(X,Y)\xi = \frac{1}{2n}\{S(Y,\xi)X - g(Y,\xi)QX\} \tag{4.1}$$

$$R(X,Y)\xi = \frac{1}{2n}\{S(Y,\xi)X - \eta(Y)QX\} \tag{4.2}$$

By using (2.10) & (2.12) above equation becomes

$$(f_1 - f_3)\{\eta(Y)X - \eta(X)Y\} = \frac{1}{2n}\{2n(f_1 - f_3)\eta(Y)X - \eta(Y)QX\} \tag{4.3}$$

On solving, we get

$$\eta(Y)QX = 2n(f_1 - f_3)\eta(X)Y \tag{4.4}$$

putting $Y = \xi$, we obtain

$$QX = 2n(f_1 - f_3)\eta(X)\xi \tag{4.5}$$

Now, taking the inner product of the above equation with U, we get

$$S(X,U) = 2n(f_1 - f_3)\eta(X)\eta(U) \tag{4.6}$$

which implies generalised Sasakian-space-forms is an $\eta-$ Einstein Manifold. Conversely, suppose that (4.6) is satisfied. Then from (4.1) & (4.4), we get

$$G(X,Y)\xi = 0$$

This completes the proof. $\square$

5. Generalized Sasakian-Space-Forms Satisfying $G.S = 0$

Theorem 3. *A generalized Sasakian-space-form $M^{2n+1}(f_1, f_2, f_3)$ satisfies the condition $G(\xi, X).S = 0$ if and only if either $M^{2n+1}(f_1, f_2, f_3)$ has $f_1 = f_3$ or an Einstein Manifold.*

Proof. The condition $G(\xi, X).S = 0$ implies that

$$S(G(\xi,X)Y,Z) + S(Y,G(\xi,X)Z) = 0$$

for any vector fields X, Y, Z on M. Substituting (2.18) in above equation, we obtain

$$\begin{aligned} & (f_1 - f_3)\{2n(f_1 - f_3)g(X,Y)\eta(Z) - \eta(Y)S(X,Z) \\ & +2n(f_1 - f_3)g(X,Y)\eta(Z)\} - \frac{1}{2n}2n(f_1 - f_3)S(X,Y)\eta(Z) \\ & +(f_1 - f_3)\{2n(f_1 - f_3)g(X,Z)\eta(Y) - S(Y,X)\eta(Z) \\ & +2n(f_1 - f_3)g(X,Z)\eta(Y)\} - \frac{1}{2n}2n(f_1 - f_3)S(X,Z)\eta(Y) \\ = & \ 0 \end{aligned} \tag{5.1}$$

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For $Z=\xi$, the last equation is equivalent to

$$\begin{aligned}(5.2)\qquad & (f_1-f_3)\{2n(f_1-f_3)g(X,Y)-2n(f_1-f_3)\eta(Y)\eta(Z)\\ & +2n(f_1-f_3)g(X,Y)\}-(f_1-f_3)S(X,Y)\\ & +(f_1-f_3)\{2n(f_1-f_3)\eta(X)\eta(Y)-S(Y,X)\\ & +2n(f_1-f_3)\eta(X)\eta(Y)\}-2n(f_1-f_3)(f_1-f_3)\eta(X)\eta(Y)\\ = & \; 0\end{aligned}$$

Using (2.12), we obtain

$$S(X,Y)=2n(f_1-f_3)g(X,Y) \tag{5.3}$$

$$S(X,Y)=\lambda g(X,Y)$$

which implies, it is an Einstein Manifold where $\lambda=2n(f_1-f_3)$. □

6. W_7- flat Generalized Sasakian-space-forms

Theorem 4. *A $(2n+1)-$ dimensional $(n>1)$ generalized Sasakian-space-form is W_7- flat if and only if $f_1=\frac{3f_2}{(1-2n)}=f_3$.*

Proof. For a $(2n+1)-$ dimensional W_7- flat generalized Sasakian-space-forms, we have from (2.16)

$$R(X,Y)Z=\frac{1}{2n}\{S(Y,Z)X-g(Y,Z)QX\} \tag{6.1}$$

In view of (2.7) & (2.8), the above equation takes the form

$$R(X,Y)Z=\frac{1}{2n}\{-(3f_2+(2n-1)f_3)(\eta(Y)\eta(Z)X+g(Y,Z)\eta(X)\xi\} \tag{6.2}$$

By virtue of (2.6), the above equation reduces to

$$\begin{aligned}(6.3)\qquad & f_1\{g(Y,Z)X-g(X,Z)Y\}\\ & +f_2\{g(X,\Phi Z)\Phi Y-g(Y,\Phi Z)\Phi X+2g(X,\Phi Y)\Phi Z\}\\ & +f_3\{\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X+g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\}\\ = & \; \frac{1}{2n}\{-(3f_2+(2n-1)f_3)(\eta(Y)\eta(Z)X+g(Y,Z)\eta(X)\xi\}\end{aligned}$$

Now, replacing Z by ΦZ in the above equation, we obtain

$$\begin{aligned}(6.4)\qquad & f_1\{g(Y,\Phi Z)X-g(X,\Phi Z)Y\}\\ & +f_2\{g(X,\Phi^2 Z)\Phi Y-g(Y,\Phi^2 Z)\Phi X+2g(X,\Phi Y)\Phi^2 Z\}\\ & +f_3\{g(X,\Phi Z)\eta(Y)\xi-g(Y,\Phi Z)\eta(X)\xi\}\\ = & \; \frac{1}{2n}\{-(3f_2+(2n-1)f_3)g(Y,\Phi Z)\eta(X)\xi\}\end{aligned}$$

Taking inner product of above equation with ξ, we get

$$\begin{aligned} &f_1\{g(Y,\Phi Z)\eta(X)-g(X,\Phi Z)\eta(Y)\} \\ &+f_3\{g(X,\Phi Z)\eta(Y)-g(Y,\Phi Z)\eta(X)\} \\ = \;& \frac{1}{2n}\{-(3f_2+(2n-1)f_3)g(Y,\Phi Z)\eta(X)\} \end{aligned} \tag{6.5}$$

In view of (2.1) & (2.2), we obtain

$$(2nf_1+3f_2-f_3)g(Y,\Phi Z)\eta(X)-2n(f_1-f_3)g(X,\Phi Z)\eta(Y)=0 \tag{6.6}$$

Putting $X=\xi$ in above equation, we get

$$(2nf_1+3f_2-f_3)g(Y,\Phi Z)=0 \tag{6.7}$$

Since $g(Y,\Phi Z)\neq 0$ in general, we obtain

$$2nf_1+3f_2-f_3=0 \tag{6.8}$$

Again replacing X by ΦX in equation (6.3), we get

$$\begin{aligned} &f_1\{g(Y,Z)\Phi X-g(\Phi X,Z)Y\} \\ &+f_2\{g(\Phi X,\Phi Z)\Phi Y-g(Y,\Phi Z)\Phi^2X+2g(\Phi X,\Phi Y)\Phi Z\} \\ &+f_3\{\eta(\Phi X)\eta(Z)Y-\eta(Y)\eta(Z)\Phi X+g(\Phi X,Z)\eta(Y)\xi-g(Y,Z)\eta(\Phi X)\xi\} \\ = \;& \frac{1}{2n}\{-(3f_2+(2n-1)f_3)[\eta(Y)\eta(Z)\Phi X+g(Y,Z)\eta(\Phi X)\xi]\} \end{aligned} \tag{6.9}$$

Taking inner product with ξ

$$(f_1-f_3)g(\Phi X,Z)\eta(Y)=0 \tag{6.10}$$

putting $Y=\xi$, we get

$$(f_1-f_3)g(\Phi X,Z)=0 \tag{6.11}$$

Since $g(\Phi X,Z)\neq 0$ in general, we obtain

$$f_3=f_1 \tag{6.12}$$

From equation (6.8) and (6.12), we get

$$f_1=\frac{3f_2}{1-2n}=f_3 \tag{6.13}$$

Conversely, suppose that $f_1=\frac{3f_2}{1-2n}=f_3$ satisfies in generalized Sasakian-spaces-forms and then we have

$$S(X,Y)=0 \tag{6.14}$$

$$QX=0 \tag{6.15}$$

Also, in view of (2.16), we have

$$G(X,Y,Z,U)={}'R(X,Y,Z,U) \tag{6.16}$$

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where $G(X,Y,Z,U) = g(G(X,Y)Z,U)$ and ${}'R(X,Y,Z,U) = g(R(X,Y)Z,U)$. Putting $Y = Z = e_i$ in above equation and taking summation over $i, 1 \leq i \leq 2n+1$, we get

$$\sum_{i=1}^{2n+1} G(X,e_i,e_i,U) = \sum_{i=1}^{2n+1\prime} R(X,e_i,e_i,U) = S(X,U) \tag{6.17}$$

In view of (2.6) & (6.16), we have

$$\begin{aligned} G(X,Y,Z,U) &= f_1\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \\ &\quad +f_2\{g(X,\Phi Z)g(\Phi Y,U) - g(Y,\Phi Z)g(\Phi X,U) + 2g(X,\Phi Y)g(\Phi Z,U)\} \\ &\quad +f_3\{\eta(X)\eta(Z)g(Y,U) - \eta(Y)\eta(Z)g(X,U) \\ &\quad +g(X,Z)\eta(Y)\eta(U) - g(Y,Z)\eta(X)\eta(U)\} \end{aligned} \tag{6.18}$$

Now, putting $Y = Z = e_i$ in above equation and taking summation over $i, 1 \leq i \leq 2n+1$, we get

$$\sum_{i=1}^{2n+1} G(X,e_i,e_i,U) = 2nf_1g(X,U)+3f_2g(\Phi X,\Phi U)-f_3\{(2n-1)\eta(X)\eta(U)+g(X,U)\} \tag{6.19}$$

In view of (6.14) & (6.17), we have

$$2nf_1g(X,U) + 3f_2g(\Phi X,\Phi U) - f_3\{(2n-1)\eta(X)\eta(U) + g(X,U)\} = 0 \tag{6.20}$$

Putting $X = U = e_i$ in above equation and taking summation over $i, 1 \leq i \leq 2n+1$, we get $f_1 = 0$. Then in view of (6.12), $f_2 = f_3 = 0$. Therefore, we obtain from (2.6)

$$R(X,Y)Z = 0 \tag{6.21}$$

Hence in view of (6.14) , (6.15) & (6.21), we have $G(X,Y)Z = 0$.This completes the proof. $\square$

7. Generalized Sasakian-space-forms satisfying $G.P = 0$

Theorem 5. *A generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ satisfies the condition*

$$G(\xi,X).P = 0$$

if and only if $M^{2n+1}(f_1,f_2,f_3)$ has either the sectional curvature $(f_1 - f_3)$ or the function f_1, f_2 and f_3 are linearly dependent such that $(2nf_1-3f_2+(1-4n)f_3) = 0$.

Proof. The condition $G(\xi,X).P = 0$ implies that

$$(G(\xi,X)P)(Y,Z,U) = G(\xi,X)P(Y,Z)U-P(G(\xi,X)Y,Z)U-P(Y,G(\xi,X)Z)U-P(Y,Z)G(\xi,X)U = 0 \tag{7.1}$$

for any vector fields X, Y, Z on M.

In view of (2.7), we obtain from (2.25)

$$\eta(P(X,Y)Z) = 0 \tag{7.2}$$

Since,

(7.3)
$$G(\xi,X)P(Y,Z)U=(f_1-f_3)\{2g(X,P(Y,Z)U)\xi-\eta(P(Y,Z)U)X\}-\frac{1}{2n}S(X,P(Y,Z)U)\xi$$

(7.4)
$$P(G(\xi,X)Y,Z)U=(f_1-f_3)\{2g(X,Y)P(\xi,Z)U-\eta(Y)P(X,Z)U)-\frac{1}{2n}S(X,Y)P(\xi.Z)U$$

Finally, we conclude that

(7.5)
$$P(Y,Z)G(\xi,X)U=(f_1-f_3)\{2g(X,U)P(Y,Z)\xi-\eta(U)P(Y,Z)X)-\frac{1}{2n}S(X,U)P(Y,Z)\xi$$

So, substituting (7.3), (7.4) and (7.5) in (7.1), we get

$$\begin{aligned}(7.6)\qquad & (f_1-f_3)\{2g(X,P(Y,Z)U)\xi-\eta(P(Y,Z)U)X-2g(X,Y)P(\xi,Z)U\\ & +\eta(Y)P(X,Z)U-2g(X,Z)P(Y,\xi)U+\eta(Z)P(Y,X)U\\ & -2g(X,U)P(Y,Z)\xi+\eta(U)P(Y,Z)X\}-\frac{1}{2n}\{S(X,P(Y,Z)U)\xi\\ & -S(X,Y)P(\xi,Z)U-S(X,Z)P(Y,\xi)U-S(X,U)P(Y,Z)\xi\}\\ = & \ 0\end{aligned}$$

Taking inner product with $'\xi'$

$$\begin{aligned}& 2(f_1-f_3)\{g(X,R(Y,Z)U)-(f_1-f_3)(g(X,Y)g(Z,U)-g(X,Z)g(Y,U))\}\\ & -\frac{1}{2n}\{S(X,R(Y,Z)U)-(f_1-f_3)(S(X,Y)g(Z,U)-S(X,Z)g(Y,U))\}\\ = & \ 0\end{aligned}$$

Simplifying above equation, we get

(7.7)
$$(2nf_1-3f_2+(1-4n)f_3)\{g(X,R(Y,Z)U)-(f_1-f_3)(g(X,Y)g(Z,U)-g(X,Z)g(Y,U)\}=0$$

which say us $M^{2n+1}(f_1,f_2,f_3)$ has the sectional curvature (f_1-f_3) or the fucntions f_1, f_2 and f_3 are lineraly dependent such that $(2nf_1-3f_2+(1-4n)f_3)=0$. □

8. Generalized Sasakian-space-forms satisfying $G.\widetilde{C}=0$

Theorem 6. *A generalized Sasakian-space-form $M^{2n+1}(f_1,f_2,f_3)$ satisfies the condition*

$$G(\xi,X).\widetilde{C}=0$$

if and only if either the scalar curvature τ of $M^{2n+1}(f_1,f_2,f_3)$ is $\tau=(f_1-f_3)2n(2n+1)$ or the functions f_2 and f_3 are linearly dependent such that $3f_2+(2n-1)f_3=0$.

Proof. The condition $G(\xi,X).\widetilde{C}=0$ implies that

$$\begin{aligned}(8.1)\quad (G(\xi,X)\widetilde{C})(Y,Z,U) &= G(\xi,X)\widetilde{C}(Y,Z)U-\widetilde{C}(G(\xi,X)Y,Z)U\\ & -\widetilde{C}(Y,G(\xi,X)Z)U-\widetilde{C}(Y,Z)G(\xi,X)U=0\end{aligned}$$

for any vector fields X,Y,Z and U on M. From (2.22) and (2.23), we can easily to see that

(8.2)
$$G(\xi,X)\widetilde{C}(Y,Z)U=(f_1-f_3)\{2g(X,\widetilde{C}(Y,Z)U)\xi-\eta(\widetilde{C}(Y,Z)U)X\}-\frac{1}{2n}S(X,\widetilde{C}(Y,Z)U)\xi$$

(8.3)
$$\widetilde{C}(G(\xi,X)Y,Z)U=(f_1-f_3)\{2g(X,Y)\widetilde{C}(\xi,Z)U)-\eta(Y)\widetilde{C}(X,Z)U\}-\frac{1}{2n}S(X,Y)\widetilde{C}(\xi,Z)U$$

(8.4)
$$\widetilde{C}(Y,G(\xi,X)Z)U=(f_1-f_3)\{2g(X,Z)\widetilde{C}(Y,\xi)U-\eta(Z)\widetilde{C}(Y,X)U\}-\frac{1}{2n}S(X,Z)\widetilde{C}(Y,\xi)U$$

and

(8.5)
$$\widetilde{C}(Y,Z)G(\xi,X)U=(f_1-f_3)\{2g(X,U)\widetilde{C}(Y,Z)\xi-\eta(U)\widetilde{C}(Y,Z)X\}-\frac{1}{2n}S(X,U)\widetilde{C}(Y,Z)\xi$$

Thus, substituting (8.2), (8.3), (8.4) and (8.5) in (8.1) and after from necessary abbreviations, (8.1) takes from

$$(2nf_1-3f_2-(4n-1)f_3)\{g(X,R(Y,Z)U)-(f_1-f_3)(g(Z,U)g(X,Y)-g(X,Z)g(Y,U))\}$$
$$+(f_1-f_3-\frac{\tau}{2n(2n+1)})(3f_2+(2n-1)f_3)\{g(X,Z)\eta(Y)\eta(U)-g(X,Y)\eta(Z)\eta(U)\}=0$$

Now putting $U=\xi$ in the above equation, we get

$$(f_1-f_3-\frac{\tau}{2n(2n+1)})(3f_2+(2n-1)f_3)\{g(X,Z)\eta(Y)-g(X,Y)\eta(Z)\}=0$$

This equation tells us that either $M^{2n+1}(f_1,f_2,f_3)$ has either the scalar curvature $\tau=(f_1-f_3)2n(2n+1)$ or the functions f_2 and f_3 are linearly dependent such that $3f_2+(2n-1)f_3=0$. □

9. Generalized Sasakian-space-forms satisfying $G.R=0$

Theorem 7. *A $(2n+1)-$dimensioanl $(n>1)$ generalized Sasakian-space-form satisfing G.R=0 is an $\eta-$Einstein Manifold.*

Proof. The condition $G(\xi,X).R=0$ yields to

(9.1)
$$G(\xi,X)R(Y,Z)U-R(G(\xi,X)Y,Z)U-R(Y,G(\xi,X)Z)U-R(Y,Z)G(\xi,X)U=0$$

for any vector fields X,Y,Z,U on M. In view of (2.20), we obtain

$$\begin{aligned} G(\xi,X)R(Y,Z)U &= (f_1-f_3)\{2g(X,R(Y,Z)U)\xi-\eta(R(Y,Z)U)X\} \\ &\quad -\frac{1}{2n}S(X,R(Y,Z)U)\xi \end{aligned} \tag{9.2}$$

On the other hand, by direct calculations, we have

$$\begin{aligned} R(G(\xi,X)Y,Z)U &= (f_1-f_3)\{2g(X,Y)R(\xi,Z)U)-\eta(Y)R(X,Z)U\} \\ &\quad -\frac{1}{2n}S(X,Y)R(\xi,Z)U \end{aligned} \tag{9.3}$$

and

$$\begin{aligned} R(Y,G(\xi,X)Z)U &= (f_1-f_3)\{2g(X,Z)R(Y,\xi)U-\eta(Z)R(Y,X)U\} \\ &\quad -\frac{1}{2n}S(X,Z)R(Y,\xi)U \end{aligned} \tag{9.4}$$

and

$$R(Y,Z)G(\xi,X)U = (f_1-f_3)\{2g(X,U)R(Y,Z)\xi-\eta(U)R(Y,Z)X\} \tag{9.5}$$
$$-\frac{1}{2n}S(X,U)R(Y,Z)\xi$$

Substituting $(9.2),(9.3),(9.4)$ & (9.5) in (9.1), we get

$$(f_1-f_3)\{2g(X,R(Y,Z)U)\xi-(f_1-f_3)(g(Z,U)\eta(Y)X-g(Y,U)\eta(Z)X)$$
$$-2g(X,Y)(f_1-f_3)(g(Z,U)\xi-\eta(U)Z)+\eta(Y)R(X,Z)U-2g(X,Z)(f_1-f_3)(\eta(U)Y-g(Y,U)\xi)$$
$$+\eta(Z)R(Y,X)U-2g(X,U)(f_1-f_3)(\eta(Z)Y-\eta(Y)Z)+\eta(U)R(Y,Z)X$$
$$+\frac{1}{2n}\{-S(X,R(Y,Z)U)\xi+S(X,Y)(f_1-f_3)(g(Z,U)\xi-\eta(U)Z)$$
$$+S(X,Z)(f_1-f_3)(\eta(U)Y-g(Y,U)\xi)+S(X,U)(f_1-f_3)(\eta(Z)Y-\eta(Y)Z\}=0$$

Taking inner product with ξ, above equation implies that

$$(f_1-f_3)\{2g(X,R(Y,Z)U)-2(f_1-f_3)g(X,Y)g(Z,U)+(f_1-f_3)g(X,Y)\eta(Z)\eta(U)$$
$$-(f_1-f_3)g(X,Z)\eta(Y)\eta(U)+2(f_1-f_3)g(X,Z)g(Y,U)\}$$
$$+\frac{1}{2n}\{-S(X,R(Y,Z)U)+(f_1-f_3)S(X,Y)(g(Z,U)$$
$$-\eta(Z)\eta(U))+(f_1-f_3)S(X,Z)(\eta(U)\eta(Y)-g(Y,U)\}=0$$

In consequence of (2.6), (2.10), (2.11) and (2.12) the above equation takes the form

$$(f_1-f_3)\{2f_3g(X,Y)g(Z,U)-2f_3g(X,Z)g(Y,U)+2f_2g(X,\Phi Z)g(Y,\Phi U)$$
$$-2f_2g(Z,\Phi U)g(X,\Phi Y)+4f_2g(X,\Phi U)g(Y,\Phi Z)+(3f_3-f_1)g(X,Z)\eta(Y)\eta(U)$$
$$+(f_1-3f_3)g(X,Y)\eta(Z)\eta(U)+2f_3g(Y,U)\eta(X)\eta(Z)-2f_3g(Z,U)\eta(X)\eta(Y)\}$$
$$+\frac{1}{2n}\{-f_3S(X,Y)g(Z,U)+f_3S(X,Z)g(Y,U)-f_2S(X,\Phi Z)g(Y,\Phi U)$$
$$+f_2S(X,\Phi Y)g(Z,\Phi U)-2f_2g(Y,\Phi Z)S(X,\Phi U)+(f_1-2f_3)S(X,Z)\eta(Y)\eta(U)$$
$$+(2f_3-f_1)S(X,Y)\eta(Z)\eta(U)-f_3g(Y,U)S(X,\xi)\eta(Z)+f_3g(Z,U)S(X,\xi)\eta(Y)\}=0$$

Putting $Z=U=e_i$ in the above equation and taking summation over $i, 1\le i\le 2n+1$, we get

$$S(X,Y)=\frac{2n(f_1-f_3)(f_1+6f_2+(4n-3)f_3)}{(f_1+3f_2+(2n-2)f_3)}g(X,Y)-\frac{2n(f_1-f_3)(3(2n+1)f_3+\epsilon)}{(f_1+3f_2+(2n-2)f_3)}\eta(X)\eta(Y)$$

which implies that

$$S(X,Y)=\lambda_1 g(X,Y)-\lambda_2\eta(X)\eta(Y)$$

which show that M^{2n+1} is an $\eta-$Einstien manifold. □

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