STUDYOF A GENERATING FUNCTION INVOLVING GENERALISED LAURICELLA FUNCTION

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Abstract.*Recently, Generating function of the extended second Appllehypergeometricfunction was introduced by S. D.Purohit, R.K.Parmar and K.S. Nishar, here in the present investigations we established some generating functions involving generalised extendedLauricellafunction*

$F_A(a,b_1,b_2,...,b_n;c_1,c_2,...,c_n;x_1,x_2,...,x_n;p;\mu)$. *Further we develop some certain interesting special cases*

Key words*: Extended Beta function, extendedhypergeometric function, Lauricellafunction.*

1. INTRODUCTION

Some extensions of the well-known special functions have been considered by several authors (See [1–6]). In 1994, Chaudhry and Zubair [1] presented the following extension of gamma function.

$$\Gamma(x;p)=\int_0^\infty t^{x-1}\exp(-t,-t-pt^{-1})dt, \qquad (\mathrm{Re}(x)>0,\ \mathrm{Re}\ (p)>0). \tag{1}$$

In 1997, Chaudhry et al.[2] presented the following extension of Euler's Beta function.

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}e\,\mathrm{xp}\left(\frac{-p}{t(1-t)}\right)dt, \ \left(\mathrm{R}(p)>0;\ \mathrm{R}(x)>0;\ \mathrm{R}(y)>0\right). \tag{2}$$

In 2004, Chaudhry et al. [3] used $B(x,y;p)$ to extend the hypergeometric functions as follows.

$$F_p(a,b;c;z)=\frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n B(b+n,c-b;p)\frac{z^n}{n!},$$

$$\left(|z|<1;\mathrm{R}(c)>\mathrm{R}(b)>0;\mathrm{R}(p)>0\right). \tag{3}$$

In 2010, Özarslan and Özergin [7] extended the familiar Appellhypergeometric function using $B(x,y;p)$ as given below.

$$F_2(a,b,b';c,c';x,y;p)=\sum_{m,n=0}^{\infty}(a)_{m+n}\frac{B(b+m,c-b;p)B(b'+n,c'-b';p)}{B(b,c-b)B(b',c'-b')}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$\left(|x|+|y|<1;\mathrm{Re}(p)>0\right). \tag{4}$$

In 2011, Lee et al.[9], presented the generalisation of extended beta function (2) and hypergeometric function (3) as follows:

$$B(x,y;p;\mu)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(\frac{-p}{t^{\mu}(1-t)^{\mu}}\right)dt \quad ,$$

$$\left(R(x),R(y)>0,R(p)>0,R(\mu)>0\right). \tag{5}$$

and

$$F_{p;\mu}(a,b;c;z)=\frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n\,B(b+n,c-b;p;\mu)\frac{z^n}{n!} \quad ,$$

$$\left(|z|<1,R(c)>R(b)>0,R(p)>0,R(\mu)>0\right). \tag{6}$$

Recently R. K. Parmar (2014) generalised the extended Appellhypergeometric function by using $B(x,y,p;\mu)$,

$$F_2(a,b,b';c,c';x,y;p;\mu)=\sum_{m,n=0}^{\infty}(a)_{m+n}\frac{B(b+m,c-b;p;\mu)B(b'+n,c'-b';p;\mu)}{B(b,c-b)B(b',c'-b')}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$\left(|x|+|y|<1;R(p)>0,R(\mu)>0\right). \tag{7}$$

Here we also use $B(x,y,p;\mu)$ in the firstLauricella function such as

$$F_A^{(n)}\left[a,b_1,\ldots b_n;c_1,\ldots c_n;x_1,\ldots,x_n;p;\mu\right]=$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1+\ldots+m_n}B(b_1+m_1;c_1-b_1;p;\mu)B(b_2+m_2;c_2-b_2;p;\mu)\ldots B(b_n+m_n;c_n-b_n;p;\mu)}{B(b_1;c_1-b_1)B(b_2;c_2-b_2)\ldots B(b_n;c_n-b_n)}\left(\frac{(x_1)^{m_1}}{m_1!}\cdots\frac{(x_n)^{m_n}}{m_n!}\right),$$

$$\text{where }\left(|x_1|+\ldots+|x_n|<1\right). \tag{8}$$

When $\mu=1$, equations (5), (6),(7) yields extended Beta function, Gauss'shypergeometricandAppellhypergeometric functions [5,8] respectively.If we put $\mu=1, p\to 0$, in equation (9), we get the Lauricella function.

2. MAIN RESULT

Here we obtain some generating function for the generalised extendedLauricella function $F_A^{(n)}\left[a,b_1,\ldots b_n;c_1,\ldots c_n;x_1,\ldots x_n;p;\mu\right]$ which is defined above.

Theorem-1: The following function hold true

$$\sum_{k=0}^{\infty}(\gamma)_k\, F_A^{(n)}\left[\gamma+k,b_1,....b_n;c_1,...c_n;x_1,...x_n;p;\mu\right]\frac{t^k}{k!}$$

$$=(1-t)^{\gamma}\, F_A^{(n)}\left[\gamma,b_1,....b_n;c_1,...c_n;\frac{x_1}{(1-t)},\frac{x_2}{(1-t)}...,\frac{x_n}{(1-t)};p;\mu\right],$$

where $\{R(p)>0,\ R(\mu)>0,\ \gamma\in C\ and\ |t|<1\}$.

Proof: Using definition (8) in the left hand side,we have

$$\sum_{k=0}^{\infty}(\gamma)_k\, F_A^{(n)}\left[\gamma+k,b_1,....b_n;c_1,...c_n;x_1,...x_n;p;\mu\right]\frac{t^k}{k!}$$

$$=\sum_{k=0}^{\infty}(\gamma)_k\frac{t^k}{k!}$$

$$\left(\sum_{m_1,...,m_n=0}^{\infty}\frac{(\gamma+k)_{m_1+...+m_n}\,B(b_1+m_1;c_1-b_1;p;\mu)B(b_2+m_2;c_2-b_2;p;\mu)...B(b_n+m_n;c_n-b_n;p;\mu)}{B(b_1,c_1-b_1)B(b_2,c_2-b_2)...B(b_n,c_n-b_n)}\frac{(x_1)^{m_1}}{m_1!}..\frac{(x_n)^{m_n}}{m_n!}\right)$$

Changing order of summation

$$\sum_{m_1,...,m_n=0}^{\infty}\frac{B(b_1+m_1;c_1-b_1;p;\mu)B(b_2+m_2;c_2-b_2;p;\mu)...B(b_n+m_n;c_n-b_n;p;\mu)}{B(b_1,c_1-b_1)B(b_2,c_2-b_2)...B(b_n,c_n-b_n)}$$

$$\left(\sum_{k=0}^{\infty}(\gamma)_k(\gamma+k)_{m_1+...+m_n}.\frac{(x_1)^{m_1}}{m_1!}...\frac{(x_n)^{m_n}}{m_n!}\right)\frac{t^k}{k!}$$

Using the identity $(\gamma)_k(\gamma+k)_{m_1+m_2+...+m_n}=(\gamma)_{m_1+m_2+...+m_n}(\gamma+m_1+m_2+...+m_n)_k$ in above equation, we have

$$=\left[\sum_{m_1,...,m_n=0}^{\infty}(\gamma)_{m_1+m_2+...+m_n}\frac{B(b_1+m_1;c_1-b_1;p;\mu)B(b_2+m_2;c_2-b_2;p;\mu)...B(b_n+m_n;c_n-b_n;p;\mu)}{B(b_1,c_1-b_1)B(b_2,c_2-b_2)...B(b_n,c_n-b_n)}\right.$$

$$\left..\frac{(x_1)^{m_1}}{m_1!}..\frac{(x_n)^{m_n}}{m_n!}.\left(\sum_{k=0}^{\infty}(\gamma+m_1+m_2+...+m_n)_k\frac{t^k}{k!}\right)\right]$$

$$=\left[\sum_{m_1,...,m_n=0}^{\infty}(\gamma)_{m_1+m_2+...+m_n}\frac{B(b_1+m_1;c_1-b_1;p;\mu)B(b_2+m_2;c_2-b_2;p;\mu)...B(b_n+m_n;c_n-b_n;p;\mu)}{B(b_1,c_1-b_1)B(b_2,c_2-b_2)...B(b_n,c_n-b_n)}\right.$$

$$\left..\frac{(x_1)^{m_1}}{m_1!}...\frac{(x_n)^{m_n}}{m_n!}(1-t)^{-(\gamma+m_1+m_2+...+m_n)}\right]$$

$$=(1-t)^{-\gamma}\,F_A^{n}(\gamma,b_1,b_2,...,b_n;c_1,c_2,...,c_n;x_1,x_2,...,x_n;p;\mu)$$

Theorem-2: For $\left(R(p)>0, R(\mu)>0, \gamma \in C \,\&\, |t|<1\right)$

$$\sum_{k=0}^{\infty}(\gamma)_k\, F_A^n\left(-k, b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; x_1, x_2, ..., x_n; p; \mu\right)$$

$$=(1-t)^{-\gamma}\, F_A^n\left(\gamma, b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; \frac{-x_1 t}{(1-t)}, \frac{-x_2 t}{(1-t)}, ..., \frac{-x_n t}{(1-t)}; p; \mu\right)$$

Proof: using above definition of Lauricella function (8) and changing order of summation, we get

$$\sum_{m_1,...,m_n=0}^{\infty}\frac{B(b_1+m_1; c_1-b_1; p;\mu)B(b_2+m_2; c_2-b_2; p;\mu)...B(b_n+m_n; c_n-b_n; p;\mu)}{B(b_1, c_1-b_1)B(b_2, c_2-b_2)....B(b_n, c_n-b_n)}.$$

$$\left(\sum_{k=0}^{\infty}(\gamma)_k\,(-k)_{m_1+...+m_n}\cdot\frac{(x_1)^{m_1}}{m_1!}...\frac{(x_n)^{m_n}}{m_n!}\right)\frac{t^k}{k!}$$

Using following factorial function property in above theorem

$$\frac{(-k)_{m_1+m_2+...m_n}}{k!}=\frac{(-1)^{m_1+m_2+...+m_n}}{[k-(m_1+m_2+...+m_n)]!}\text{, we get}$$

$$\sum_{m_1,...,m_n=0}^{\infty}\frac{B(b_1+m_1; c_1-b_1; p;\mu)B(b_2+m_2; c_2-b_2; p;\mu)...B(b_n+m_n; c_n-b_n; p;\mu)}{B(b_1, c_1-b_1)B(b_2, c_2-b_2)...B(b_n, c_n-b_n)}$$

$$\left(\sum_{k=0}^{\infty}(\gamma)_k\,(-1)^{m_1+m_2+...+m_n}\frac{(x_1)^{m_1}}{m_1!}...\frac{(x_n)^{m_n}}{m_n!}\right)\frac{t^k}{[k-(m_1+m_2+...+m_n)]!}$$

If we put, $k=k+m_1+m_2+....+m_n$, we get the required result.

Corollary:

$$for\left(R(p)>0, R(\mu)>0, \gamma\in C\,\&\,|t|<1\right),$$

$$\sum_{k=0}^{\infty}F_A^n\left(-k, b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; x_1, x_2, ..., x_n; p; \mu\right)t^k$$

$$=e^t\,\phi_p(b_1; c_1; -x_1 t, \mu)\phi_p(b_2; c_2; -x_2 t, \mu)...\phi_p(b_n; c_n; -x_n t, \mu)$$

Proof: By replacing $\gamma\to\frac{\gamma}{k}$ in the theorem (2), then we have

$$=\left(1-\frac{t}{\gamma}\right)^{-\gamma}\sum_{m_1,...,m_n=0}^{\infty}(\gamma)_{m_1+m_2+...+m_n}\frac{B(b_1+m_1; c_1-b_1; p;\mu)B(b_2+m_2; c_2-b_2; p;\mu)...B(b_n+m_n; c_n-b_n; p;\mu)}{B(b_1, c_1-b_1)B(b_2, c_2-b_2)...B(b_n, c_n-b_n)}$$

$$\times\left(\frac{-x_1 t/\gamma}{(1-t/\gamma)}\right)^{m_1}\frac{1}{m_1!}\left(\frac{-x_2 t/\gamma}{(1-t/\gamma)}\right)^{m_2}\frac{1}{m_2!}\times....\left(\frac{-x_n t/\gamma}{(1-t/\gamma)}\right)^{m_n}\frac{1}{m_n!}$$

$$=\left(1-\frac{t}{\gamma}\right)^{-\gamma+m_1+...+m_n}\sum_{m_1,...,m_n=0}^{\infty}(\gamma)_{m_1+m_2+...+m_n}\left(\frac{1}{\gamma}\right)^{m_1+m_2+...+m_n}$$

$$.\frac{B(b_1+m_1; c_1-b_1; p;\mu)B(b_2+m_2; c_2-b_2; p;\mu)...B(b_n+m_n; c_n-b_n; p;\mu)}{B(b_1, c_1-b_1)B(b_2, c_2-b_2)...B(b_n, c_n-b_n)}$$

$$.\frac{(-x_1 t)^{m_1}}{m_1!}\frac{(-x_2 t)^{m_2}}{m_2!}...\frac{(-x_n t)^{m_n}}{m_n!},$$

usingbinomial theorem**,t**aking limit $|\gamma| \to \infty$, and using the property

$(\gamma)_{m_1+m_2+...+m_n}(\gamma+m_1+m_2+...+m_n)_k = (\gamma+k)_{m_1+m_2+...+m_n}(\gamma)_k$ and

$Lim_{|\gamma|\to\infty}\left((\gamma)_n (x/\gamma)^n\right) = x^n, \left(n \in N_0\right)$, we get desired result.

3. SPECIAL CASES

Case-I: if we put $\mu = 1$ in theorem 1 and theorem 2 above results are reduced to the extended Lauricella function.

$$\sum_{k=0}^{\infty}(\gamma)_k F_A^{(n)}[\gamma+k, b_1,....b_n; c_1,...c_n; x_1,...x_n; p]\frac{t^k}{k!}$$
$$=(1-t)^{\gamma} F_A^{(n)}\left[\gamma, b_1,....b_n; c_1,...c_n; \frac{x_1}{(1-t)}, \frac{x_2}{(1-t)}..., \frac{x_n}{(1-t)}; p\right],$$

where $\{R(p) > 0,\ \gamma \in C \ and \ |t| < 1\}$.

and

$$for\left(R(p) > 0, \gamma \in C \,\&\, |t| < 1\right),$$
$$\sum_{k=0}^{\infty}(\gamma)_k F_A^n(-k, b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; x_1, x_2, ..., x_n; p)\frac{t^k}{k!}$$
$$=(1-t)^{-\gamma} F_A^n\left(\gamma, b_1, b_2, ..., b_n; c_1, c_2, ..., c_n; \frac{-x_1 t}{(1-t)}, \frac{-x_2 t}{(1-t)}, ..., \frac{-x_n t}{(1-t)}; p\right).$$

Case-2: If we put $\mu = 1, p \to 0$, in theorem 1 and 2, the above results arereducedtotheLauricella function.

Case-3: If we put $x_3 = x_4 = ... = x_n = 0$,then results are converted into generalised extended second Appell function.

Case-4: If we put $x_2 = x_3 = x_4 = ... = x_n = 0$ then results are true for generalised extended Gauss hypergeometric function.

4.CONCLUSION

In this paper,we have investigated and evaluate some interesting generating functions involving generalised extended Lauricella functionin the main result section .We have also developed some special cases which are unique by themselves. Theorems proved in the present paper are general in character and can be applied to other similar problems.

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