Singular Value Decomposition & Few Application

[1]Jajimogga Raghavendar, [2] V.Dharmaiah
[1]Research Scholar,Department of Mathematics,Rayalaseema University
Kurnool, Andhra Pradesh,India
[2]Department of Mathematics, Retd. Professor, Osmania University, Hyderabad, Telangana-500007

Abstract—*Singular Value Decomposition (SVD) is a tool for teaching linear algebra geometrically. In Linear algebra SVD is very usefulin many cases in problem solving. Some of the applications of SVD are,Image processing, Population related problems, Least square approximation in Numerical methods, Dimension reduction, Low rank data's storage, Education related problems, Data composition. This paper discuss about fewer applications.*

Keywords—*SVD, Least Square Approximation, Education & Population Related Problems*.

1. Introduction

This section introduces the hanger stretcher and aligner matrices. Any matrix can be written as the product of hanger, stricter and an aligner matrix. A 2D perpendicular frame consists of perpendicular circular unit vector which can be specified by an angleα. As an example consider the unit vectors

$$V_1 = (cos\alpha, sin\alpha)$$
$$V_2 = [\cos\left(\frac{\pi}{2}+\alpha\right), \sin\left(\frac{\pi}{2}+\alpha\right)]$$
$$V_1 . V_2 = 0$$

Using these two perpendicular vectors as column or rows we can define respective hanger or the aligner matrix

1.1 Hanger Matrices

The hanger matrix determined by $H = \begin{bmatrix} cos\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) \\ sin\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) \end{bmatrix}$

Set the angleα for the matrix H to $\alpha = \frac{\pi}{2}$

Consider the ellipse

$$x(\theta) = 2\ cos\theta$$
$$y(\theta) = 1.5\ sin\theta \text{In parametric form}$$

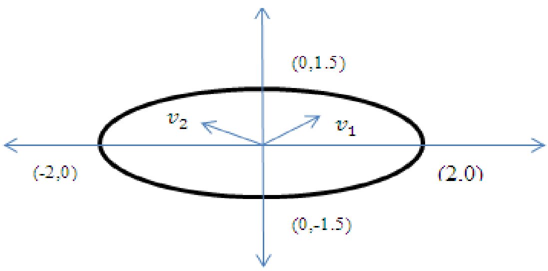

Fig 1

Perpendicular vector $\alpha = \frac{\pi}{4}$together with an ellipse..

The point on the ellipse are defined by the unit vector (1,0) and (0,1)

$$[x(\theta), y(\theta)] = x(\theta)[1,0] + y(\theta)[0,1]$$

∴ This means that $x(\theta)$ units in the direction of (1, 0) and $y(\theta)$ units in the direction of (0,1).

Hung ellipse$(\theta) = x(\theta)V_1 + y(\theta)V_2$

Hung ellipse$(t) = \begin{bmatrix} cos\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) \\ sin\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

i.e; V_1 is playing the former role of (1, 0) and V_2 is playing the former role of (1, 0).

1.2 Aligner matrices for align a curve on the X-Y-axes

Given the perpendicular vector

$$V_1 = (cos\alpha, sin\alpha)$$
$$V_2 = [\cos\left(\frac{\pi}{2}+\alpha\right), \sin\left(\frac{\pi}{2}+\alpha\right)]$$

Aligner $S = \begin{bmatrix} cos\alpha & sin\alpha \\ \cos\left(\frac{\pi}{2}+\alpha\right) & \sin\left(\frac{\pi}{2}+\alpha\right) \end{bmatrix}$

Parametric equation in ellipsis hung ellipse $= x(\theta)V_1 + y(\theta)V_2$

Aligned ellipse $e(\theta) = \begin{bmatrix} cos\alpha & sin\alpha \\ \cos\left(\frac{\pi}{2}+\alpha\right) & \sin\left(\frac{\pi}{2}+\alpha\right) \end{bmatrix} \begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix}$

note that the aligner matrix undoes what the hanger matrix does that is, one is the inverse of the other

$$\begin{bmatrix} cos\alpha & \cos\left(\frac{\pi}{2}+\alpha\right) \\ sin\alpha & \sin\left(\frac{\pi}{2}+\alpha\right) \end{bmatrix} \begin{bmatrix} cos\alpha & sin\alpha \\ \cos\left(\frac{\pi}{2}+\alpha\right) & \sin\left(\frac{\pi}{2}+\alpha\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.2 Diagonal matrices X-Y-streching

Consider the diagonalmatrix $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ which multiplying every point of the unit circle given of parametric form, the result is shown in fig 2.

Where it can be seen that all measurements along the X–axis have been stretching by a factor of 3(the X $_{\text{stretch factor}}$) and all measuring matrix along the Y-axis by a factor of 2 the Y $_{\text{stretch factor}}$)

To invert stretcher matrix D

$$D^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

2. SVD Application

The SVD analyses enhance the teaching of linear algebra concepts and it is explain the definition of the matrix on a curve when the inverse of a matrix does not exists.

2.1 Theorem: The image of the unit circle in $\mathbb{R}^2$ by an invertible 2×2 matrix is an ellipse with center at the origin.

Proof Denote the invertible matrix by $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, ad - bc \neq 0$ (2.1.1)

Then $M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

MMaps the unit circle to $\left\{\begin{pmatrix} u \\ v \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 = 1\right\}$

The equation of this set u, v space is $\left\|M^{-1}\begin{pmatrix} u \\ v \end{pmatrix}\right\|^2 = 1$

Plugging in the form for M^{-1} yields the equation

$$\frac{1}{(ad-bc)^2}\|du - bv, -cu + av\|^2 = 1$$

Expanding yields

$$\frac{1}{(ad-bc)^2}((du-bv)^2 + (-cu+av)^2) = 1$$

Simplify tofind $(d^2+c^2)u^2 - 2(ac+db)uv + (a^2+b^2)v^2 = (ad-bc)^2$

This quadratic equation describes the image of the unit circle so is a bounded closed curve. As a quadratic curve it is a conic section or a degenerate section.That is an ellipse, hyperbola, parabola or degenerated form, a line, two lines, a point, or empty. Since the set contains an infinite number of points(M is a one to one and the circle is infinite) and is compact M is continuous and the circle is compact) the only candidate is ellipse.

2.2 Matrix action on a curve

Consider the unit circle showing n fig 2

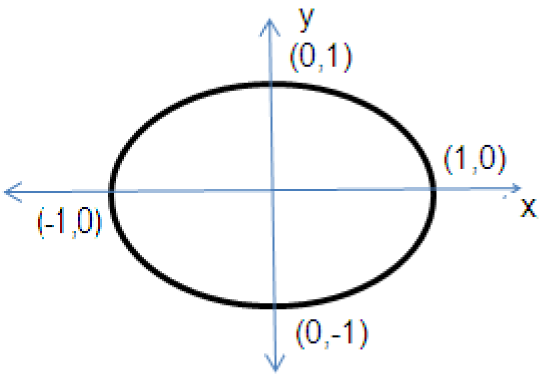

Fig 2

$$\{x(\theta), y(\theta)\} = \{\cos(\theta), \sin(\theta)\}\ 0 \le \theta \le 2\pi$$
$$M = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$
$$U = \begin{bmatrix} 0.7497 & -0.6618 \\ 0.6618 & 07497 \end{bmatrix}$$
$$D = \begin{bmatrix} 2.9208 & 0 \\ 0 & 0.6847 \end{bmatrix}$$
$$V = \begin{bmatrix} 0.2567 & -0.9665 \\ 0.9665 & 0.2567 \end{bmatrix}$$

M=hanger(S_1). Strecher. Alinger(S_2)

After mutipling the unit circle in time the matrix M we obtain an ellipse
Let $M.[x(\theta), y(\theta)]$

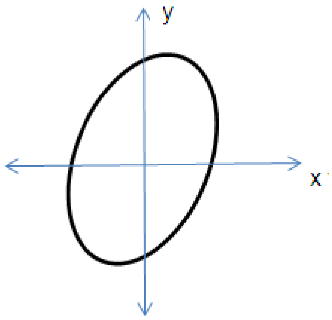

Fig 3

The product of aligned $S_2.\{x(\theta), y(\theta)\}$ is a circle

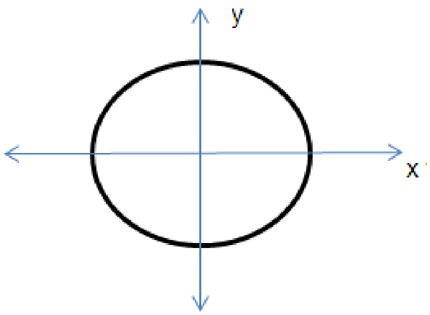

Fig 4

3. Solution of linear least square problem by SVD

Let us consider
$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^{n-1}$Here$(x_i, y_i), i = 1,2..n$
Compare this with interpolation where we choose the degree $n-1$ of the polynomial

To pass through all the points. The generalization of the linear least square problem is to take a linear combination of the form $\{f_1(x), f_2(x), \dots f_n(x)\}$ where $f_1(x), f_2(x), \dots f_n(x)$ are functions of x whose degree are 0, 1,… n-1

$\therefore f(x) = a_1 f_1(x) + \cdots .. + a_n f_n(x)$.

The best approximation f_{approx} is given by a linear combination of the function is the one for which the residual $y_i - f(x_i)$ are smallest. We take the quantity $\sqrt{\sum_{i=1}^{n} |y(_i - f(x_i)|^2}$

The problem of fitting a polynomial of degree n-1 can be written as

$$a_1 + a_2 x_1 + \cdots + a_n x_1^{n-1} \approx y_1$$
$$a_1 + a_2 x_2 + \cdots + a_n x_2^{n-1} \approx y_2$$
$$\vdots$$
$$a_1 + a_2 x_n + \cdots + a_n x_m^{n-1} \approx y_m$$

Which denote an over determined system of linear equation because $m > n$

And x_i, y_i are given data a_i are unknown

In the matrix form $AX \approx Y$

$$A = \begin{bmatrix} 1 & \cdots & x_1^{n-1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & x_m^{n-1} \end{bmatrix} \quad X = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$\therefore$ m- Equations with n unknown

The residual vector is $r = Y - AX$

Using the Euclidean norms $\|U\| = \sqrt{\sum_{i=1}^{m} u_i^2}$

The least square problem becomes $\min_C \|Y - AX\|^2$

Let AX=Y

$$A^T AX = A^T Y$$

Which are known as the normal equation

The least square problem is obtained using SVD analysis of A as shown

We have $A = UDVT$

$$\|Y - AX\|^2 = \|Y - (UDVT)X\|^2$$
$$= \|U^T\|^2 \|Y - (UDVT)X\|^2$$
$$= \|U^T Y - DVTX\|^2$$

De$U^T y$ is D and $V^T X$ and rank of A , we have $\|Y - A\|^2 = \|dDz\|^2 = \left\| \begin{pmatrix} d_1 - \sigma_1 z_1 \\ d_2 - \sigma_2 z_2 \\ \vdots \\ d_r - \sigma_r z_r \end{pmatrix} \right\|$

$$= (d_1 - \sigma_1 z_1)^2 + (d_2 - \sigma_2 z_2)^2 + \cdots + (d_r - \sigma_r z_r)^2$$

We can now uniquely solve $z_i = \frac{d_i}{\sigma_i}$ for $i = 1,2, \dots r$

To reduce this expression to its minimum value, the solution to the least square is obtained for

$$V^T X = Z$$
$$X = Vz$$

Examples

1. The population contain town is shown in the following table

Year	1920	1930	1940	1950	1960
Population	19.96	39.65	58.81	77.21	94.61

We want to fit a polynomial of degree 2 to these points. The polynomial is $f(x) = a_0 + a_1 x + a_2 x^2$

Such that $f(x_i)$ should be as close as possible to y_i where the point $(x_i, y_i); i = 1,2,3,4,5.$

The year (x_i) the population is (y_i)

Our system of equation $AX = Y$ is

$$\begin{bmatrix} 1 & 1920 & 3686400 \\ 1 & 1930 & 37244900 \\ 1 & 1940 & 3763600 \\ 1 & 1950 & 3802500 \\ 1 & 1960 & 3841600 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 19.96 \\ 39.65 \\ 58.81 \\ 77.21 \\ 94.61 \end{bmatrix}$$

The SVD analysis of the matrix

$A = UDV^T$ is obtained using MATLAB software

$$U = \begin{pmatrix} -0.4380 & 0.6356 & 0.5384 & -0.0258 & -0.3371 \\ -0.4425 & 0.3243 & -0.2653 & 0.2809 & 0.7414 \\ -0.4471 & 0.0098 & -0.5345 & -0.6882 & -0.2017 \end{pmatrix}$$

$$D = \begin{pmatrix} 8417008.11 & 0 & 0 \\ 0 & 31.62 & 0 \\ 0 & 0 & 0.000099420 \end{pmatrix}$$

$$V = \begin{pmatrix} -0.0000 & 0.0010 & 1.0000 \\ -0.0005 & 1.0000 & -0.0010 \\ -1.0000 & -0.0005 & 0.0000 \end{pmatrix}$$

$$U^T Y = [-130.65 \quad -57.18 \quad -1.78]^T = d$$

And compute $z = [-0.00000155 \ -1.808 \ -17903.842]^T$Where $z_i = \frac{d_i}{\sigma_i}$, i=1,2,3.

$$V^T X = z$$
$$X = Vz$$

$$X = [-17903.84 \quad 16.10 \quad 0.000000090555]^T$$

4. Conclusion

The singular value decomposition is nearly hundred years old.For the case of square matrices,it was discovered independently by Beltrami in 1873 and Jordan in 1874.

The technique was extended to rectangular matrices by Eckart and Young in the year 1930.

SVD of a matrix is still not widely used in education. This paper may useful for the further application of SVD.

5. References

[1] Alter, O., P. O. Brown, and D. Botstein. 2000. Singular value decomposition for genome-wide expression data processing and modeling. Proc. Natl. Acad. Sci. USA.97:10101–10106. [PMC free article][PubMed]

[2] Berman, H. M., J. Westbrook, Z. Feng, G. Gilliland, T. N. Bhat, H. Weissig, I. N. Shindyalov, and P. E. Bourne. 2000. The Protein Data Bank. Nucleic Acids Res.28:235–242. [PMC free article][PubMed]

[3] Borgstahl, G. E. O., D. R. Williams, and E. D. Getzoff. 1995. 1.4 Ångstrom structure of photoactive yellow protein, a cytosolic photoreceptor: unusual fold, active site, and chromophore. Biochemistry.34:6278–6287. [PubMed]

[4] D. Kahaner, C. Moler, S. Nash, Numerical Methods and Software, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[5] R.D. Skeel, J.B. Keiper, Elementary Numerical Computing with Mathematica, McGraw-Hill, New York, NY, 1993.

[6] W.T. Shaw, J. Tigg, Applied Mathematica: Getting Started, Getting it Done, Addison-Wesley, Reading, MA, 1994.