# A New-Mean Type Variant of Newton´s Method for Simple and Multiple Roots

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***Abstract-*** *A new variant of Newton's method based on heronian-mean for multiple root has been developed and their convergence properties have been discussed. In addition to numerical tests verifying the theory, a comparison of the results for the proposed method and some of the existing ones have also been given. Convergence analysis shows that the efficiency index of proposed method is 1.442, which is better than Newton´s method (1.414).*

***Keywords-*** *Newton's method, Iteration function, Order of convergence, Function evaluations, Efficiency index*.

## 1. INTRODUCTION

In Applied Mathematical Sciences, many of the nonlinear and transcendental problems, of the form $f(x)=0$, are complex in nature. Since it is not always possible to obtain its exact solution by usual algebraic process, therefore numerical iterative methods such as Newton's method, Secant method are often used to obtain the approximate solution of such problems. These methods can also be used to find local maxima and local minima of functions, as these extrema are the roots of the derivative function. Many optimization problems also lead to solve the system of nonlinear equation- f ind: $x,\ x \varepsilon \varphi$ , such that $\min F(x) = f^{2}(x)$, where $\varphi$ is the solution space. This master function is positive definite and has a global minimum at each of the roots. When the minimization of $F(x)$ is 0, the corresponding $x$ is the exact solution.

Now considering the problem of finding a real zero of a function $f: R \rightarrow R$. This zero can be determined as a fixed point of some iteration function g by means of the one-point iteration method $x_{n+1} = g(x_n)$, $n = 0, 1, \cdots$, where $x_0$ is the starting value, The best known and the most widely used example of these types of methods is the classical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \cdots. \qquad (1)$$

It converges quadratically to simple zeros and linearly to multiple zeros. But their convergence and performance characteristics are highly sensitive to the initial guess of the solution supplied to the methods. In the literature, some of its modifications have been introduced in order to accelerate it or to get a method with a higher order of convergence at the expense of additional evaluations of functions, derivatives and changes in the points of iterations. All these modifications are in the direction of increasing the local order of convergence with the view of increasing their efficiency indices. Ford et. al. [2] and Gerlach [3] described accelerated Newton's method. The method developed by Fernando et al. [1], called as trapezoidal Newton's method or arithmetic mean Newton's method, suggests for some other variants of Newton's method. Frontini et al [8] developed new modifications of Newton's method to produce iterative methods with third order of convergence and efficiency index 1.442. With the same efficiency index, Ozban [6], and Traub [10] developed a third order method requiring one function and two first derivatives evaluations per iteration. Chen [9] described some new iterative formulae having third order convergence. Lukic, et al. [7] described Geometric mean Newton method for simple and multiple roots. This paper is concerned with the iterative method for finding a simple and multiple roots. Method presented in this paper requires three functions evaluations per iteration and it is compared with that of T. Lukic [7].

## 2. DEFINITIONS

***Definition 1****:* Considering the problem of numerical approximation of a real root α of the non linear equation $f(x)=0, f: D \subseteq R \rightarrow R$. The root α is said to be simple if $f(\alpha)=0$ and $f'(\alpha) \neq 0$. If $f(\alpha) = f'(\alpha) = \cdots = f^{m-1}(\alpha) = 0$ and $f^{m}(\alpha) \neq 0$ for $m \geq 1$ then $\alpha$ is of multiplicity $m$.

***Definition 2****:* (See [4]) If the sequence $\{x_n / n \geq 0\}$ tends to a limit α in such a way that

$$\lim_{x_n \to \alpha} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^p} = C = \left| g^{(p)}(\alpha) \right| / p! \qquad (2)$$

for some $C \neq 0$ and $p \geq 1$ , then the order of convergence of the sequence is said to be $p$, and $C$ is known as *asymptotic error constant*.

When $p=1, p=2$ or $p=3$, the sequence is said to convergence *lineally*, *quadratically* and *cubically respectively*. The value of $p$ is called the *order of convergence* of the method which produces the sequence $\{x_n: n \geq 0\}$ . Let $e_n = x_n - \alpha$ then the relation $e_{n+1} = C{e_n}^p + O({e_n}^{p+1})$ is called the error equation for the method, $p$ being the order of convergence.

***Definition 3****:* See [4] *Efficiency index* is simply defined as $p^{1/m}$ where $p$ is the order of the method and $m$ is the number of functions evaluations required by the method (units of work per iteration). Therefore the efficiency index of Newton´s method is 1.414 and iterative methods with order of convergence three has efficiency index 1.442

## 3. DESCRIPTION OF THE METHOD

Let α be a simple zero of a sufficiently differentiable function $f$ and consider the numerical solution of the equation $f(x) = 0$. It is clear that

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \quad (3)$$

Approximating $f'$ by $f'(x_n)$ on the interval $[x_n, x]$, we get the value $(x - x_n)f'(x_n)$ for the integral in (3) and then putting $x = \alpha$, we obtain, $0 \approx f(x_n) + (\alpha - x_n)f'(x_n)$ and hence, a approximation for α is given by equation (1), is called Newton's method for $n = 0, 1, \cdots$. On the other hand, if we approximate the integral in (3) by the trapezoidal rule and then on putting $x = \alpha$, we obtain,

$$0 \approx f(x_n) + (1/2)(\alpha - x_n)(f'(x_n) + f'(\alpha)).$$

Therefore, a approximation $x_{n+1}$ for α is given by:

$$x_{n+1} = x_n - 2f(x_n)/\{f'(x_n) + f'(x_{n+1})\}.$$

If the $(n+1)^{th}$ value of Newton's method is used on the right-hand side of this equation to overcome the implicit problem, then

$$x_{n+1} = x_n - 2f(x_n)/\{f'(x_n) + f'(y_n)\},$$

where $y_n = x_n - f(x_n)/f'(x_n)$, for $n = 0,1,2,\cdots$, named as trapezoidal Newton's method (see, Fernando et al. [1]). Rewriting the this equation as

$$x_{n+1} = x_n - \frac{f(x_n)}{(f'(x_n)+f'(y_n))/2}, \quad n = 0, 1, \cdots \quad (4)$$

So, this variant of Newton’s method can be viewed as arithmetic mean of $f'(x_n)$ and $f'(y_n)$ instead of $f'(x_n)$ in Newtońs method defined by (1), was called as *arithmetic mean Newton*’s *method*.
We know that *Heronian-mean* of two number *a* and *b* is given by

$$HM(a,b) = \frac{1}{3}(2A + G) = \frac{1}{3}\{(a+b) + \sqrt{(ab)}\}. \quad (5)$$

In (4) if we use the heronian-mean instead of arithmetic mean, we get,

$$x_{n+1} = x_n - \frac{3f(x_n)}{f'(x_n)+f'(y_n)+(sign(f'(x_0))\sqrt{f'(x_n)f'(y_n)}}, \quad (6)$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$.

## 4. CONVERGENCE ANALYSIS

***Theorem* 1***:* Let $\alpha \in I$ be a simple zero of a differentiable function *f:* $R \to R$ for an open interval *I*. If $x_0$ is sufficiently close to α, then the methods defined by (6) has third order convergence.

***Proof.*** Since $\alpha \in I$ is a simple zero of $f$, then, we have,

$$f(x_n) = f'(\alpha)\,[\,e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)],$$

where $C_j = \left(\frac{1}{j!}\right) f^{(j)}(\alpha)/f'(\alpha)$,

$$f'(x_n) = f'(\alpha)\,[1 + 2C_2 e_n + 3C_3 e_n^2$$
$$+ 4C_4 e_n^3 + O(e_n^4)]. \quad (7)$$

Now, from second equation of (6), we get, $y_n = \alpha + C_2 e_n^2 + (2C_3 - 2C_2^2)e_n^3 + O(e_n^4)$. Therefore, we have,

$$f'(y_n) = f'(\alpha)[1 + 2C_2^2 e_n^2$$
$$+ 4C_2(C_3 - C_2^2)e_n^3 + O(e_n^4)]. \quad (8)$$

From equations (7) and (8), we find that,

$$\sqrt{f'(x_n)f'(y_n)} = f'(\alpha)[1 + C_2 e_n + \frac{1}{2}(C_2^2 + 3C_3)e_n^2$$
$$+ 2(C_2C_3 - C_2^3 + 4C_4)e_n^3 + O(e_n^4)].$$

On using this approximation in equation (6), the error equation

$$e_{n+1} = \left(\frac{5}{6}C_2^2 + \frac{3}{6}C_3\right)e_n^3 + O(e_n^4).$$

This shows that method has third order convergence.
***Theorem* 2.** - Let $\alpha \in I$ be a multiple zero of a differentiable function $f: I \subset R \to R$ for an open interval $I$. If $x_0$ is sufficiently close to α, then the method defined by (6) has first order convergence.

***Proof.*** Since $\alpha \in I$ is a multiple zero of $f$. $f(\alpha) = 0$, $f'(\alpha) = 0$, $f''(\alpha) = 0$, $f^{m-1}(\alpha) = 0$ and $f^m(\alpha) \neq 0$,therefore on by expanding $f(x_n)$ and $f'(x_n)$ about α, we get,

$$f(x_n) = f(\alpha + e_n) = \frac{1}{m!}e_n^m f^{(m)}(\alpha)[1 + C_2 e_n$$
$$+ C_3 e_n^2 + C_4 e_n^3 + O(e_n^4)], \quad (9)$$

$$f'(x_n) = f'(\alpha + e_n) = \frac{1}{(m-1)!}e_n^{m-1} f^{(m)}(\alpha)[1$$
$$+ \frac{(m+1)}{m}C_2 e_n + \frac{(m+2)}{m}C_3 e_n^2 + O(e_n^3)], \quad (10)$$

where $e_n = x_n - \alpha$, and

$$C_j = f^{(m+j-1)}(\alpha)/\{(m+1)\cdots(m+j-1)f^{(m)}(\alpha)\}.$$

From equation (9) and (10), we get,

$$f(x_n)/f'(x_n) = \frac{1}{m}\left[e_n + C_2\left(1 - \frac{m+1}{m}\right)e_n^2 + O(e_n^3)\right].$$

Using this approximation in second equation of (6), we get,

$$y_n = \alpha + \left(1 - \frac{1}{m}\right) e_n + C_2 \frac{1}{m}\left(\frac{m+1}{m} - 1\right) e_n^2 + O(e_n^3).$$

Therefore, we get,

$$f'(y_n) = \frac{A e_n^{m-1}}{(m-1)!} f^{(m)}(\alpha) \left[1 + \frac{(m^2-1)}{m^2} C_2 e_n + \left\{\frac{(m+1)}{m^3} C_2^2 + \frac{(m+2)(m-1)^2}{m^3} C_3\right\} e_n^2 + O(e_n^3)\right] \quad (11)$$

where

$$A = [(1 - \frac{1}{m}) + C_2 \frac{1}{m}(\frac{m+1}{m} - 1)\, e_n + O(e_n^2)\,]^{m-1}.$$

On Multiplying (10) and (11), we get,

$$f'(x_n) f'(y_n) = A e_n^{2(m-1)} \frac{\{f^m(\alpha)\}^2}{(m-1)!^2} \times \left[1 + \frac{(m+1)(2m-1)}{m^2} C_2 e_n + O(e_n^2)\right]. \quad (12)$$

From equation (12) we easily show that $f'(x_n) f'(y_n) > 0$ and $\operatorname{sign}\left(f'(x_0)\right) = \operatorname{sign}(f'(y_n))$ holds for $n = 0,1,2,\cdots$, therefore, we have,

$$\operatorname{sign}(f'(x_0)) \sqrt{f'(x_n) f'(y_n)} = \sqrt{A} \frac{f^{(m)}(\alpha)}{(m-1)!} e_n^{m-1} \times \left[1 + \frac{(m+1)(2m-1)}{2m^2} C_2 e_n + O(e_n^2)\right] \quad (13)$$

On using the approximations given by equations (10), (11) and (13) in equation (6), we get the error equation

$$e_{n+1} = \left(1 - \frac{3}{k\,m}\right) e_n + O(e_n^3),$$

where $k = \left(A + \sqrt{A} + 1\right)$. This shows that method has first order convergence. For sufficiently large $n$, we may approximate the value of $k$ as

$$k \approx \left(\frac{m-1}{m}\right)^{m-1} + \left(\frac{m-1}{m}\right)^{(m-1)/2} + 1$$

for $m > 1$ and sufficiently large $n$.

## 5. NUMERICAL ILLUSTRATION

In this section, we presented the results of some numerical tests to compare the efficiencies of the proposed method. We employed some third order method CN method, AN method of Fernando et al. [1], HN, GN, MN. Numerical computations have been carried out in MATLAB. The stopping criterion has been taken as $|x_{n+1} - \alpha| + |f(x_{n+1})| < 10^{-14}$ and in Table 1 for simple roots following test functions have been used, see [5].

(a) $x^3 + 4x^2 - 10$, α = 1.365230013414097,
(b) $\sin^2 x - x^2 + 1$, α= -1.404491648215341,
(c) $x^3 - 10$, α = 2.154434690031884,
(d) $(x-1)^3 - 1$, α = 2,
(e) $(x-2)^{23} - 1$, α = 3.

All numerical results are in accordance with the theory and the basic advantage of the variants of Newton's method based on *means* or *integration. methods* that they do not require the computation of second- or higher-order derivatives The main characteristic of the HMN are: at least third order of convergence for the simple roots and at least linear order of convergence for the multiple roots. The efficiency index of this method is 1.442 which is greater than Newton´s method 1.414. This method can also be compared with some well known third order method of the same efficiency.

In Table 2, we give the number of iterations (N) and the number of function evaluations (NOFE) required to satisfy the stopping criterion, F denotes that method fails and D denotes for divergence. HMN denotes proposed method. CN denotes for Classical Newton method, AN-Arithmetic mean Newton method, HN Harmonic mean Newton method, GN- Geometric mean Newton method, MN – Mid- point Newton method.

*Example 1*: Consider the equation $f(x) = (x-2)^3 (x+2)^4$.

We start with initial approximations $x_0 = 1$. The results obtained by Newton iteration, Geometric iteration and present iteration are shown in Table 3.

Now from the Table 2 and Table 3, we observe that the proposed method takes lesser number of iterations than the others compared here and example shows that the proposed method requires lesser number of functional evaluations, as well as it works when other method fails. Thus, the Proposed method is not only faster but the cost effecting parameters obtain in examples shows that it has minimum cost among all the methods taken here.

**Conclusion:**

We discussed the convergence analysis and asymptotic error constant of the proposed method for the simple and multiple roots. As far as the numerical results are considered, for the simple roots, in most of the cases Mean type Newton method HMN requires the least number of function evaluations , and in some cases it take the almost same no of function evaluations as the other third order method takes. But it always takes the less no of function evaluations in comparison with Newton method, which can be seen in examples (a), (b), (c), (d), (e). In case of multiple roots for example 1 Geometric Newton’s method either fails or converges to undesired root, but the proposed method always converges to root.

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Table 1

| M | Order | | AEC | | | | |
|---|---|---|---|---|---|---|---|
| | *CN* | *AN, HN, GN, HMN* | *CN* | *AN* | *HN* | *GN* | *HMN* |
| 1 | 2 | 3 | $\frac{f''(\alpha)}{2f'(\alpha)}$ | $\left(\frac{f''(\alpha)}{2f'(\alpha)}\right)^2 + \frac{f'''(\alpha)}{12f'(\alpha)}$ | $\frac{f'''(\alpha)}{12f'(\alpha)}$ | $\frac{1}{2}\left(\frac{f''(\alpha)}{2f'(\alpha)}\right)^2 + \frac{f'''(\alpha)}{12f'(\alpha)}$ | $\frac{5}{6}\left(\frac{f''(\alpha)}{2f'(\alpha)}\right)^2 + \frac{f'''(\alpha)}{12f'(\alpha)}$ |
| 2 | 1 | 1 | 0.50 | 0.33 | 0.25 | 0.29 | 0.32 |
| 3 | 1 | 1 | 0.67 | 0.54 | 0.46 | 0.5 | 0.52 |
| 4 | 1 | 1 | 0.75 | 0.65 | 0.58 | 0.61 | 0.63 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| k | 1 | 1 | $1 - \frac{1}{k}$ | $1 - \frac{2}{k\left\{1+\left(\frac{k-1}{k}\right)^{k-1}\right\}}$ | $1 - \frac{1+\left(\frac{k-1}{k}\right)^{1-k}}{2k}$ | $1 - \frac{\left(\frac{k-1}{k}\right)^{\frac{1-k}{2}}}{k}$ | $1 - \frac{3}{k\left\{\left(\frac{k-1}{k}\right)^{k-1}+\left(\frac{k-1}{k}\right)^{\frac{1-k}{2}}+1\right\}}$ |

Table 2 - Comparison with the third order method for the different examples

| F($x$) | $x_0$ | N | | | | | | NOFE | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | CN | AN | HN | GN | MN | HMN | CN | AN | HN | GN | MN | HMN |
| (a) | -0.5 | 131 | 6 | 65 | F | 10 | 3 | 262 | 18 | 195 | - | 30 | 12 |
| | 1 | 5 | 3 | 3 | 3 | 3 | 2 | 10 | 9 | 9 | 9 | 9 | 8 |
| | 2 | 5 | 3 | 3 | 3 | 3 | 2 | 10 | 9 | 9 | 9 | 9 | 8 |
| (b) | -2 | 5 | 5 | 3 | 6 | 3 | 2 | 10 | 15 | 9 | 18 | 9 | 8 |
| | -3 | 6 | 5 | 3 | 7 | 3 | 2 | 12 | 15 | 9 | 21 | 9 | 8 |
| (c) | -3 | 18 | D | 17 | D | D | 4 | 54 | - | 51 | - | - | 8 |
| | 2 | 4 | 3 | 3 | D | 3 | 2 | 8 | 9 | 9 | - | 9 | 8 |
| | 2.5 | 5 | 3 | 3 | D | 3 | 2 | 10 | 9 | 9 | - | 9 | 8 |
| (d) | 0.1 | 6 | 4 | 3 | 188 | 4 | 3 | 18 | 12 | 9 | 564 | 12 | 12 |
| | 2.5 | 7 | 5 | 4 | 31 | 4 | 2 | 21 | 15 | 12 | 93 | 12 | 8 |
| (e) | 0.5 | F | D | 30 | D | 138 | 15 | - | - | 90 | - | 414 | 60 |
| | 1 | 232 | D | 79 | D | D | 10 | 696 | - | 237 | - | - | 40 |
| | 3.5 | 14 | 9 | 8 | D | 9 | 6 | 42 | 27 | 24 | - | 27 | 24 |

Table 3. Comparison of Proposed method for multiple roots

| Mathod | $x_0$ | n | $x_n$ | $f(x_n)$ |
|---|---|---|---|---|
| Newton iteration | 1.0 | 74 | 1.999999999999955 | -2.310014734547614e-038 |
| | | 75 | 1.999999999999970 | -6.895439418326135e-039 |
| Geometric iteration | 1.0 | - | Failed | - |
| Proposed iteration | 1.0 | 47 | 1.999999999999975 | -4.262399603859903e-039 |
| | | 48 | 1.999999999999987 | -6.361362534618141e-040 |