Frames and its applications

P.Kalyani

Department of Mathematics, Aurora's Technological and Research Institute,
Uppal, Hyderabad

Abstract:- *Different types of frames and operators are defined with examples and applications of each of the type of frames are explained*

1. Introduction

Frames are generalization of bases .D. Han and D. R. Larson have developed a number of basic concepts of operator theoretic approach to frame theory in C* algebra. Peter G Casazza presented a tutorial on frame theory and he suggested the major directions of research in frame theory. Radu V. Balan and Peter G. Casazza have analyzed decomposition of a normalized tight frame and obtained identities for frames. A. Najati and A. Rahimi have developed the generalized frame theory and introduced methods for generating g-frames of a C* algebra.

1.1 Banach Algebra :- A Banach Algebra is a complex Banach space A together with an associative and distributive multiplication such that λ(ab)= λ(a) λ(b)
$\|ab\| \leq \|a\|\|b\|$ $\forall$a,b$\in A$ and $\lambda \in C$

For any x,$x^1, y, y^1 \in A$ we have $\|xy - x^1y^1\| \leq$

$\|x\|\|y - y^1\|$+$\|x - x^1\|$

The algebra A is said to be commutative if ab=ba$\forall a, b \in A$

1.2 Definition (Involution of an algebra):- Let A be a Banach algebra .An involution on A is a map

*:A→A such that

1. a^{*^*}=a

2. $(\lambda a + \mu b)^*$ =$\bar{\lambda}$a*+$\bar{\mu}$b*

3.$(ab)^*$=b*a*

1.3 Definition:- (C* algebra) If A is a Banach algebra with involution and also $\|aa*\|$=$\|a\|^2$ then A is called a c* algebra.

Example:- C(X) let X be a compact space and C(X) is a Banach space of all complex valued functions on X with norm $\|f\|$ =$sup_{x\in X}|f(x)|$ Multiplication on C(X) is defined as pointwise ief.g(x)=f(x)g(x)

And involution by complex conjugation f*(x)=$\overline{f(x)}$

2. G- frame and g-frame operator

Throughout this paper {A_j j∈J} will denote a sequence of C* algebras
Let L (A, A_j) be a collection of bounded linear operators from A toA_j and
{Δ_j ∈ L (A , A_j) ; j ∈ J} we obtain some characterization of g-frame operator. They are the generalizations of results of frame operator.

2.1 Definition: - A sequence of operators{Δ_j}$_{j\in J}$ is said to be g-frame for C* algebra A with respect to sequence of C* algebras {A_j , j ∈ J} if there exists two constants $0 < A \leq B < \infty$ for any vector f ∈ H,
A $\|\bar{f}\|^2 \leq \sum_{j\in J} \|\Delta_j \bar{f}\|^2 \leq$ B$\|f^2\|$ where $\bar{f}$(x) = f* (x)

The above inequality is called a g-frame inequality. The numbers A, B are called the lower frame bound and upper frame bound respectively.

2.2 Definition: -A g-frame for $\{\Delta_j\}_{j\in J}$ is said to be g-tight frame if A=B, then we have
A $\|\bar{f}\|^2 = \sum_{j\in J} \|\Delta_j\, \bar{f}\|^2$ for all f* ∈ A
2.3 Definition: - A g-frame $\{\Delta_j\}_{j\in J}$ for A is said to be a g-normalized tight frame for A if A = B = 1. Then we have $\|\bar{f}\|^2 = \sum_{j\in J} \|\Delta_j\, \bar{f}\|^2$ for all f ∈ H

2.4 Definition: -Let $\{\Delta_j\}_{j\in J}$ be a g-frame for c* algebra. G-frame operator
S^g : A → A is defined as
$S^g f = \sum_{j\in J} \Delta_j^* \Delta_j$ f* for all f* ∈ A
2.5 **Definition 2.1.** A sequence $\{f_j\}_{j\in J}$ of vectors in a Hilbert space H is called a frame if there exists two constants $0 < A \le B < \infty$, such that

$$A\,\|f\|^2 \le \sum_{j\in J} \left| < f, f_j > \right|^2 \le B\,\|f\|^2 \quad \text{for all } f \in H.$$

The above inequality is called a frame inequality. The numbers A and B are called the lower and upper frame bounds respectively.

The following definitions and theorems from [1,4] are useful in our discussion.

2.6 Definition:- Let l_2 be the space of all sequences of scalars x= (a_n) for which $\|x\|_2 = \left(\sum_n |a_n|^2\right)^{\frac{1}{2}} < \infty$, and $\{f_j\}_{j\in J}$ is a frame for Hilbert space H. A synthesis operator $T : l_2 \to H$ is defined as $Te_j = f_j$, where $\{e_j\}$ is an orthonormal basis for l_2.

2.7 Definition :- Let $\{f_j\}_{j\in J}$ be a frame for H and $\{e_j\}$ be an orthonormal basis for l_2. Then the analysis operator $T^* : H \to l_2$ is the adjoint of synthesis operator T and is defined as $T^* f = \sum_{j\in J} < f, f_j > e_j$ for all f ∈ H.

2.8 Definition :-Let $\{f_j\}_{j\in J}$ be a frame for Hilbert space H. A frame operator $S = TT^* : H \to H$ is defined as $S f = \sum_j < f, f_j > f_j$ for all f ∈ H.

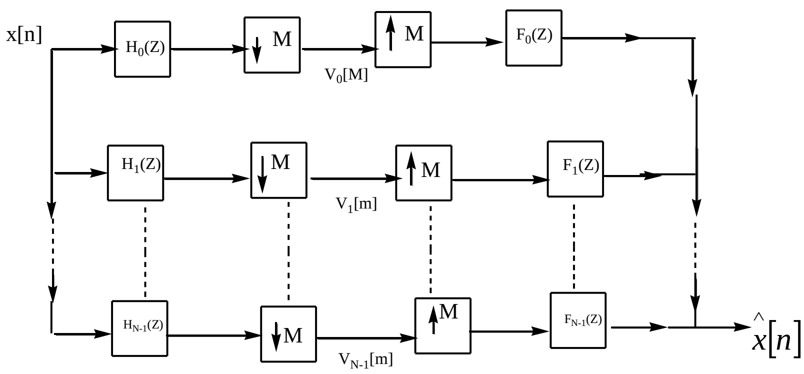
2.9 Definition:- Filter banks: We consider a N-channel FB[fig 1] with the subsampling by the integer factor M in each channel, so that $\hat{x}[n] = x[n]$ where $x[n]$ and $\hat{x}[n]$ denote the input and reconstructed signasl respectively. The transfer function of the analysis and synthesis filters are H(z) and F(z) (k= 0,1,2,……….N-1) with corresponding impulse responses $h_k[n]$ and $f_k[n]$ respectively. The sub band signals are given by

$$v_k[m] = \sum_{n=-\infty}^{\infty} x[n] h_k[mM - n] \text{, k= 0,1,2……….N-1} \qquad (1)$$

and reconstructed signal is

$$\hat{x}[n]=\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty}v_k[m]\,f_k[n-mM],\ \text{k= 0,1,2……….N-1} \tag{2}$$

fig-1



Analysis filter bank

Syntheiss filter bank

N-Channel filter bank

The polyphase decomposition of the analysis filters $H_k(z)$ are

$$H_k(z)=\sum_{n=0}^{M-1}z^{-n}E_{k,n}\left(z^M\right),\ \text{k=0,1,2……….N-1} \tag{3}$$

Where $E_{k,n}(z)=\sum_{m=\infty}^{\infty}h_k[mM-n]z^{-m}$, k=0,1,2……….N-1, n=0,1,2……….M-1

is the n[th] polyphase component of the k[th] analysis filter $H_k[z]$.

The $N\times M$ analysis polyphase matrix is defind as

$$E(z)=\begin{bmatrix} E_{0,0}(z) & E_{0,1}(z) & - & - & E_{0,M-1}(z) \\ E_{1,0}(z) & E_{1,1}(z) & - & - & E_{1,M-1}(z) \\ - & - & - & - & - \\ - & - & - & - & - \\ E_{N-1,0}(z) & E_{N-1,1}(z) & - & & E_{N-1,M-1}(z) \end{bmatrix}$$

The synthesis filters $F_k[z]$ can be decomposed as

$$F_k(z)=\sum_{n=0}^{M-1}z^{-n}R_{k,n}\left(z^M\right)\ ,\ \text{k= 0,1,2……….N-1} \tag{4}$$

With the synthesis polyphase components

$$R_{k,n}(z)=\sum_{m=-\infty}^{\infty}f_k[mM+]z^{-m}\ ,\ \text{k=0,1,2……….N-1, n=0,1,2……….M-1}$$

The $M \times N$ synthesis polyphase matrix is defind as

$$R(z) = \begin{bmatrix} R_{0,0}(z) & R_{1,0}(z) & - & - & R_{N-1,0}(z) \\ R_{0,1}(z) & R_{1,1}(z) & - & - & R_{N-1,1}(z) \\ - & - & - & - & - \\ - & - & - & - & - \\ R_{0,M-1}(z) & R_{1,M-1}(z) & - & & R_{N-1,M-1}(z) \end{bmatrix}$$

Definition 3.1. The set $\{h_{k,m}[n]\}$ with $h_{k,m}[n] = h_k^*[mM - n]$, k= 0,1,2,………N-1 is called a Uniform filter bank frame(UFBF) for $l^2(z)$ if there exists two constants $0 < A \leq B < \infty$, such that

$$A\|x\|^2 \leq \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \left|\langle x, h_{k,m} \rangle\right|^2 \leq B\|x\|^2 \quad \forall \quad x[n] \in l^2(z)$$

Definition 3.2. Let $\{h_{k,m}[n]\}$ with $h_{k,m}[n] = h_k^*[mM - n]$, k= 0,1,2,…………N-1, be a Uniform filter bank frame(UFBF) for $l^2(z)$. A Uniform filter bank Synthesis operator (UFBSO) denoted by T_F and is defined as

$T_F(x[n]) = \langle x[n], h_{k,m}[n] \rangle$, k= 0,1,2,……..N-1 for all $x[n] \in l^2(z)$.

Definition 3.3. Let $\{h_{k,m}[n]\}$ with $h_{k,m}[n] = h_k^*[mM - n]$, k= 0,1,2,……… N-1, be a Uniform filter bank frame(UFBF) for $l^2(z)$. A Uniform filter bank Analysis operator (UFBAO) denoted by T_F^* which is the adjoint operator of T_F and is defined as $(T_F^* v)[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} v_k[m] h_{k,m}[n]$, k= 0,1,2,……..N-1.

Definition 3.4. Let $\{h_{k,m}[n]\}$ with $h_{k,m}[n] = h_k^*[mM - n]$, k= 0,1,2,…………N-1, be a Uniform filter bank frame(UFBF) for $l^2(z)$. A Uniform filter bank Frame operator (UFBFO) denoted by S_F and is defined as

$$(S_F x)[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle h_{k,m}[n] \text{ for all } x[n] \in l^2(z).$$

Definition 3.5. A frame $\{h_{k,m}[n]\}$ is said to be Uniform filter bank tight frame (UFBTF) for $l^2(z)$ if A=B i,e $A\|x\|^2 = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \left|\langle x, h_{k,m} \rangle\right|^2$ for all $x[n] \in l^2(z)$.

Definition 3.6. A frame $\{h_{k,m}[n]\}$ is said to be Uniform filter bank normalized tight frame (UFBNTF) for $l^2(z)$ if A=B=1 i,e

$$\|x\|^2 = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \left|\langle x, h_{k,m} \rangle\right|^2 \text{ for all } x[n] \in l^2(z).$$

Uniform filter bank normalized tight frame is also called Uniform filter bank paraunitary frame.

Definition 3.7. Let $\{h_{k,m}[n]\}$ with $h_{k,m}[n]=h_k^*[mM-n]$, k= 0,1,2,…………N-1, be a Uniform filter bank frame(UFBF) for $l^2(z)$ with frame operator S_F. Then the sequence of functions $\{f_{k,m}[n]\}=\{\overline{h_{k,m}}[n]\}=\{(S_F^{-1}h_{k,m})[n]\}$ is called Uniform filter bank dual of $\{h_{k,m}[n]\}$.

4. Applications of frames

Frames are powerful tools to generalize matrix inversion for general vector space possibly infinite dimensional. Genenralizes bases for redundant systems Every linear operator with bounded inverse we can get frame from it.

4.1 Signal expansion:

The main application of frame theory is signal expansion. The following theorem states that every signal x∈H can be extended into a frame. The expansion of coeffiecient can be chosen as the inner products of x with canonical dual frame elements.

Theorem 4.1.1 : Let $\{g_k\}_{k\in K}$ and $\{\bar{g}_k\}_{k\in K}$ be canonical dual frames for Hilbert space H.Every signal x∈ H can be decomposed as follows

x= T*$\bar{T}$x = $\sum_{k=1}^{\infty}\langle x,\bar{g}_k\rangle g_k$

x== $\bar{T}$*Tx = $\sum_{k=1}^{\infty}\langle x,g_k\rangle \bar{g}_k\langle \quad \langle$

note that equivalently we have

$\bar{T}$*T = = T*$\bar{T}$ =I_H

Proof:- We have

T*$\bar{T}$x= $\sum_{k=1}^{\infty}\langle x,\bar{g}_k\rangle g_k$

=$\sum_{k=1}^{\infty}\langle x,S^{-1}\bar{g}_k\rangle g_k$

= $\sum_{k=1}^{\infty}\langle S^{-1}x,g_k\rangle \bar{g}_k$

= SS^{-1}x

This proves that T*$\bar{T}$ = I_H

The proof of $\bar{T}$*T = I_H can be proved similiarly.

Theorem 4.1.2:- Let $\{g_k\}_{k\in K}$ be a frame for a Hilbert space H. The frame $\{g_k\}_{k\in K}$ is a tight frame with frame bound A if and only if its corresponding frame operator satisfies S=AI_H or equivalently if

X= $\frac{1}{A}\sum_{k\in K}\langle x,g_k\rangle\, g_k$ for all x in H.

Proof:- First observe that S=AI_H is equivalent to Sx= AI_Hx=Ax for all x∈ H which is equivalent to definition of frame operator

To prove tightness of $\{g_k\}_{k\in K}$ consider $\langle Sx,x\rangle$ = A$\langle x,x\rangle$ for all x∈H

$\langle\langle S-AI_H\rangle x,x\rangle$=0 for all x∈ H

Which implies S=AI_H

To prove S=AI_H implies tightness of $\{g_k\}_{k\in K}$ we take the innerproduct with x on both the sides we obtain

$\langle x, x\rangle = \frac{1}{A} \sum_{k\in K}\langle x, g_k\rangle\langle g_{k,}x\rangle$ This is equivalent to

A $\|x\|^2 = \sum_{k\in K}|\langle x, g_k\rangle|^2$

Which shows that $\{g_k\}_{k\in K}$ is a tight frame for H with frame bound equal to A.

The practical importantance of tight frame lies in the fact that they make the computation of the canocical dual frame which in the general case requires inversion of an operator and application of this inverse to all frame elements.

4.2 Mercedes-Benz frame:- The Mercedes-Benz frame is given by following three vectors in R^2

$$g_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ g_2 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{-1}{2} \end{bmatrix}\ g_3 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -1/2 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}}{2} & -1/2 \\ \frac{\sqrt{3}}{2} & -1/2 \end{bmatrix}$$ The adjoint T^* of the analysis operator is given by

$$T^H = \begin{bmatrix} 0 & \frac{-\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ 1 & -1/2 & -1/2 \end{bmatrix}$$

Therefore the frame operator S is represented by

$$S = T^H T = \begin{bmatrix} 0 & \frac{-\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ 1 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\frac{\sqrt{3}}{2} & -1/2 \\ \frac{\sqrt{3}}{2} & -1/2 \end{bmatrix}$$

$$= \frac{3}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{3}{2} I_2$$

Hence S= A I_2 with A=3/2

Hence by the above theorem $\{g_1, g_2, g_3\}$ is a tight frame.

4.3 Sampling theorem:-

Consider signal x(t) in the space of square inferable functions L^2.In general we cannot expect this signal to be uniquely specified by its samples $\{x(kT)\}_{k\in Z}$ where T is sampling period .If a signal is strictly band limited that is its Fourier transform Vanishes outside a certain finite interval and if T is chosen small enough then the samples $\{x(kT)\}_{k\in Z}$ do uniquely specify signal and we can construct x(t) from $\{x(kT)\}_{k\in Z}$ perfectly.The process of obtaining samples $\{x(kT)\}_{k\in Z}$ from the continuous time signal x(t) is called Analog digital conversion.

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