Cartesian product and Neighbourhood Polynomial of a Graph

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Abstract-*The Cartesian product $G = G_1 \times G_2$ of any two graphs G_1 and G_2 has been studied widely in graph theory ever since the operation has been introduced. $G = G_1 \times G_2$ gives insight to the structural property of G, if G_1 and G_2 are known. The neighbourhood polynomial plays a vital role in describing the neighbourhood characteristics of the vertices of a graph.*

In this study neighbourhood polynomial of the Cartesian product of certain classes of graphs are calculated and tried to characterize the nature of neighbourhood polynomial.

Keywords -*Cartesian product, Neighbourhood polynomial.*

I. INTRODUCTION

1.1 Cartesian product- Cartesian product of any two graphs G_1 and G_2 of order m and n, respectively is defined as $G = G_1 \times G_2$. The vertex set V of G is $V_1 \times V_2$, where $V_1 = \{u_1, u_2, \ldots u_m\}$and $V_2 = \{v_1, v_2, \ldots v_n\}$ are the vertex sets of G_1 and G_2 respectively. Any two vertices (u_i, v_j) and (u_k, v_l) are adjacent in G if either $u_i = u_k$ and v_j adjacent to v_l in G_2 or $v_j = v_l$ and u_i adjacent to u_k in G_1 [2].

1.2 Neighbourhood complex and polynomial- A complex on a finite set $\mathcal{X}$ is a collection $\mathcal{C}$ of subsets of $\mathcal{X}$, closed under certain predefined restriction. Each set in $\mathcal{C}$ is called the face of the complex. In the neighbourhood complex $\mathcal{N}(G)$ of a graph G, $\mathcal{X} = V(G)$, and faces are subsets of vertices that have a common neighbour. In [1] the neighbourhood polynomial of a graph G, is defined as

$$neigh_G(x) = \sum_{u \in \mathcal{N}(G)} x^{|u|}.$$

For example consider C_4 with vertices $\{a, b, c, d\}$. The neighbourhood complex $\mathcal{N}(C_4)$ of C_4 is $\{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, d\}\}$. Since the empty set trivially has a common neighbour, each of the single vertices has a neighbour, the sets $\{a, c\}, \{b, d\}$ has two common neighbours (one is sufficient), but no three vertices have a common neighbour. The associated neighbourhood polynomial of C_4 is $neigh_{C_4}(x) = 1 + 4x + 2x^2$.

Similarly, the neighbourhood polynomials of certain standard graphs are as follows:

- K_n - $neigh_{K_n}(x) = (1 + x)^n - x^n$.
- P_n - $neigh_{P_n}(x) = 1 + nx + (n - 2)x^2$.
- C_n - $neigh_{C_4}(x) = \begin{cases} 1 + nx + nx^2, n \neq 4 \\ 1 + nx + 2x^2, n = 4 \end{cases}$

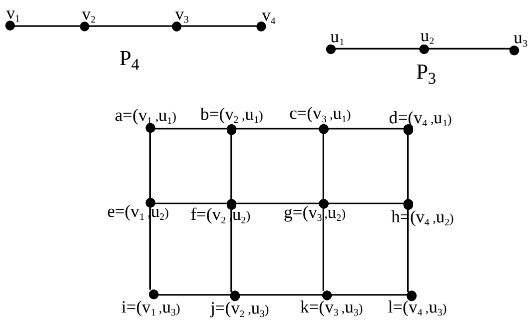
.

Here the neighbourhood polynomial for the graphs planar grids, ladder graphs, torus grids, prisms, which are formed by the Cartesian product of two graphs are calculated. Also tried to give one characteristic property of the neighbourhood polynomial of the graph G, where $G = G_1 \times G_2$.

II. MAIN RESULTS

Theorem 2.1 The neighbourhood polynomial of planar grid is
$1 + mnx + [4mn - 4(m + n) + 2]x^2 + [4mn - 6(m + n) + 8]x^3 + (m - 2)(n - 2)x^4$.

Proof: The planar grid is obtained by the Cartesian product of P_m and P_n. Consider
$G = P_4 \times P_3$.



G = P$_4$ X P$_3$

Fig.1

The empty set trivially has a common neighbour. Each of the $12[=(4\times 3)=(m\times n)]$ vertices has a neighbour.

The following $12[=(m-1)2(n-1)=3\times 2\times$ 2 two element subsets $b,e,\ a,f,\ c,f,\ b,g,\ d,g,\ c,h,$ $\{f,i\}$, $\{e,j\}$, $\{g,j\}$, $\{f,k\}$, $\{h,k\}$, $\{g,l\}$ have two common neighbours and the $10[=((m-2)n+$ $n-2m=2\times 3+1\times 4$ two element subsets $a,c,\ b,d,$ $\{a,i\}$, $\{b,j\}$, $\{e,g\}$, $\{c,k\}$, $\{f,h\}$, $\{d,l\}$, $\{i,k\}$, $\{j,l\}$ have one common neighbour.

Also the $14[=2[(m-2)(n-1)+(m-1)(n-$ $2=2[(2\times 2)+(3\times 1)]$ three element subsets $a,f,c,$ $\{b.g,d\}$, $\{e,b,g\}$, $\{f,c,h\}$, $\{e,j,g\}$, $\{f,k,h\}$, $\{i,f,k\}$, $\{j,g,l\}$, $\{a,f,i\},\{b,g,j\}$, $\{c,h,k\}$, $\{b,e,j\}$, $\{c,f,k\}$, $\{d,g,l\}$, have got a single common neighbour and there are $2[=$ $m-2n-2=2\times 1$ four element subsets b,e,g,j and $\{c,f,h,k\}$ having a single common neighbour.

Thus the neighbourhood polynomial of $G=P_4\times P_3$ is $1+12x+22x^2+14x^3+2x^4$. In general for any m and n, if $G=P_m\times P_n$,

$$neigh_G(x)=1+mnx+[4mn-4(m+n)+$$
$$2x2+4mn-6m+n+8x3+m-2(n-2)x4.$$

Theorem 2.2 The neighbourhood polynomial of ladder graph L_n is
$neigh_{L_n}(x)=1+2mx+2(2m-3)x^2+2(n-2)x^3$.

Proof: We have,

$$neigh_{P_m\times P_n}(x)=1+mnx+[4mn-$$
$$4m+n+2x2+4mn-6m+n+8x3+$$
$$(m-2)(n-2)x^4.$$

Ladder graph $L_n=P_m\times K_2$. When n is replaced by 2 in the above equality (since $P_2\simeq K_2$), we get

$$\begin{aligned} neigh_{P_m\times K_2}(x) &= 1+2mx \\ &\quad +[8m-4(m+2)+2]x^2 \\ &\quad +[8m-6(m+2)+8]x^3+ \\ &\quad (m-2)(2-2)x^4 \\ &= 1+2mx+2(2m-3)x^2+ \\ 2(m-2)x^3. \end{aligned}$$

Thus we have, $neigh_{L_n}(x)=1+2mx+2(2m-3)x^2+2(n-2)x^3$.

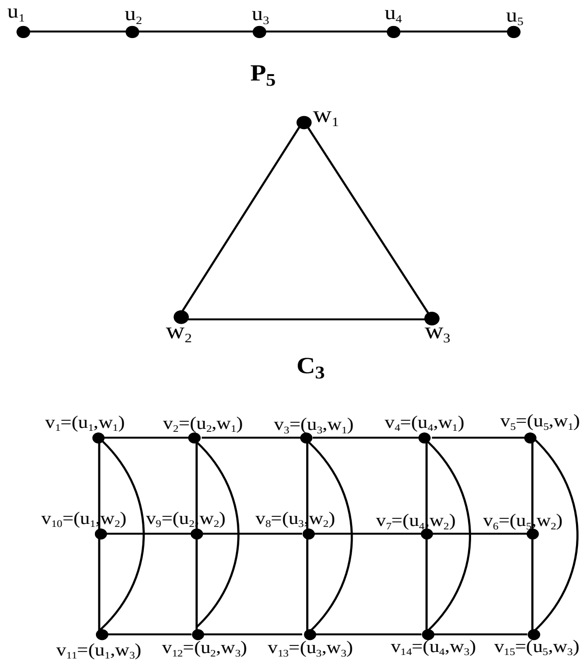
Theorem 2.3 The neighbourhood polynomial of torus grid is
$1+mn(x+4x^2+4x^3+x^4), m,n\neq 4$.

Proof: The torus grid $C_m\times C_n$, consists of mn vertices and each of these mn vertices are of degree 4. Also the neighbourhood complex $\mathcal{N}(G)$ of a graph G, consists of empty set , each of the mn single vertices has a neighbour, $4mn[=$ $2m-1n-1+(m-2n+n-2m+2m+2n+2m-1+2$ $(n-1)+2]$, two element subsets having at least one common neighbour, $4mn[=2(m-2)(n-$ $1+m-1n-2+2n-2+m-2+22m-2+2+22n-2$ $+2+8]$, three element subsets having one common neighbour, and finally, mn subsets of four elements have one common neighbour. Also no vertices of five or more elements have any common neighbour.

Hence the neighbourhood polynomial of $C_m\times C_n$ is

$$neigh_{C_m\times C_n}(x)=1+mn(x+4x^2+4x^3+x^4), m,n\neq 4.$$

Example 1
Consider $G=C_5\times C_3$.



G= P_5 X C_3

Fig. 2

The neighbourhood complex $\mathcal{N}(G)$ of graph G with vertex labelings as given in the figure (figure 2) is,

$\mathcal{N}(G) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}, \{j\}, \{k\}, \{l\}, \{m\}, \{n\}, \{o\}, \{a,i\}, \{b,h\}, \{c,g\}, \{d,f\}, \{e,g\}, \{d,h\}, \{c,i\}, \{b,j\}, \{j,l\}, \{i,m\}, \{h,n\}, \{g,o\}, \{f,n\}, \{g,m\}, \{h,l\}, \{i,k\}, \{a,c\}, \{b,d\}, \{c,e\}, \{h,j\}, \{g,i\}, \{f,h\}, \{k,m\}, \{l,n\}, \{m,o\}, \{a,k\}, \{b,l\}, \{c,m\}, \{d,n\}, \{e,o\}, \{a,j\}, \{b,i\}, \{c,h\}, \{d,g\}, \{e,f\}, \{j,k\}, \{i,l\}, \{h,m\}, \{g,n\}, \{f,o\}, \{a,d\}, \{j,g\}, \{k,n\}, \{b,e\}, \{f,i\}, \{l,o\}, \{a,l\}, \{b,m\}, \{c,n\}, \{d,o\}, \{e,n\}, \{d,m\}, \{c,l\}, \{b,k\}, \{a,o\}, \{e,k\}, \{a,f\}, \{k,f\}, \{e,j\}, \{j,o\}, \{a,c,i\}, \{b,d,h\}, \{c,e,g\}, \{b,h,j\}, \{c,g,i\}, \{d,f,h\}, \{h,j,l\}, \{g,i,m\}, \{f,h,n\}, \{i,k,m\}, \{h,l,n\}, \{g,m,o\}, \{a,i,k\}, \{b,h,l\}, \{c.g,m\}, \{d,f,n\}, \{e,g,o\}, \{d,h,n\}, \{c,i,m\}, \{b,j,l\}, \{a,f,i\}, \{f,i,k\}, \{e,j,k\}, \{a,j,o\}, \{e,g,j\}, \{g,j,o\}, \{a,f,o\}, \{e,f,k\}, \{a,i,l\}, \{c,i,l\}, \{b,h,m\}, \{d,h,m\}, \{c,g,n\}, \{e,g,n\}, \{b,j,k\}, \{d,f,o\}, \{b,i,k\}, \{b,i,m\}, \{c,h,l\}, \{c,h,n\}, \{d,g,m\}, \{d,g,o\}, \{a,j,l\}, \{e,f,n\}, \{a,f,k\}, \{e,j,o\}, \{a,c,l\}, \{b,d,m\}, \{c,e,n\}, \{b,k,m\}, \{c,l,n\}, \{d,m,o\}, \{b,e,j\}, \{b,e,k\}, \{a,d,f\}, \{a,d,o\}, \{f,k,n\}, \{e,k,n\}, \{j,l,o\}, \{a,l,o\}, \{b,e,j,k\}, \{a,ci,l\}, \{b,d,h,m\}, \{c,e,g,n\}, \{a,d,f,o\}, \{e,g,j,o\}, \{d,f,h,n\}, \{c,g,i,m\}, \{b,h,j,l\}, \{a,f,i,k\}, \{a,j,l,o\}, \{b,i,k,m\}, \{c,h,l,n\}, \{d,g,m,o\}, \{e,f,k,n\}\}$.

No five or more vertices have a common neighbour.

Hence the neighbourhood polynomial of $G = C_5 \times C_3$ is,

$neigh_G(x) = 1 + 15x + 60x^2 + 60x^3 + 15x^4$.

Corollary 2.4 The neighbourhood polynomial of torus grid is

$neigh_{C_m \times C_n} =$

$$\begin{cases} 1 + 4mx + 14mx^2 + 16mx^3 + 4mx^2, n = 4 \\ 1 + 16x + 48x^2 + 64x^3 + 16x^4, m = n = 4 \end{cases}.$$

Proof: Consider two cases separately.

Case1: If $n = 4$, in the neighbourhood complex $\mathcal{N}(G)$ of graph G, where $G = C_m \times C_4$, of the $4mn$, subsets of two elements having at least one common neighbour of $C_m \times C_n$,the $2m$ two element subsets are same as those of $m(n-2)$,two element subsets. Thus discarding these repeated $2m$ entries, the number of faces with two elements is $(4mn - 2m) = 16m - 2m(since\ n = 4) = 14m$.

The number of three element and four element subsets having at least one common neighbour remains the same with $C_m \times C_n$, for any m and n.

Hence the neighbourhood polynomial of $C_m \times C_4$ is

$neigh_{C_m \times C_4}(x) = 1 + 4mx + 14mx^2 + 16mx^3 + 4mx^2$.

Case 2: When $m = n = 4$ in $C_m \times C_{n,}$ on considering the number of faces with two elements, $m(n-2)$ entries coincide with $2m$ entries and $n(m-2)$entries with $2n$ entries. Therefore the cardinality of faces with two elements are $(4mn - 2m - 2n = 4m2 - 2m - 2m$

$= 48\ (since\ m = n = 4)$

Cardinality of all other faces do remains same as that of general $C_m \times C_n$.

Thus the neighbourhood polynomial of $C_4 \times C_4$ is

$neigh_{C_4 \times C_4}(x) = 1 + 16x + 48x^2 + 64x^3 + 16x^4$.

Theorem 2.5 The neighbourhood polynomial of prism $G = P_n \times C_m$ is,

$$neigh_G(x) = 1 + mnx + 4m(n-1)x^2 + 2m(2n-3)x^3 + m(n-2)x^4, m \neq 4$$

Proof: Consider $G = P_n \times C_m$. From the definition of Cartesian product and the neighbourhood complex of a graph, we have the empty set trivially has a common neighbour and each of the mn vertices has at least one neighbour. The number of two element subsets having at least one common neighbour is

$4m(n-1)[= 2(n-1)(m-1) + m(n-2) + n(m-2) + 2n + 2(n-2) + 2]$, for $m \neq 4$ Also the three element and four element subsets having common neighbour are $(4mn - 6)$ and $m(n-2)$ respectively, on both cases of $m \neq 4$ and $m = 4$.

Hence,

$$neigh_G(x) = 1 + mnx + 4m(n-1)x^2 + 2m(2n-3)x^3 + m(n-2)x^4, m \neq 4$$

Example 2

Consider $G = P_5 \times C_3$, (Fig. 3)

The neighbourhood complex $\mathcal{N}(G)$ of graph G is,

$\mathcal{N}(G) = \phi$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$, $\{v_6\}$, $\{v_7\}$, $\{v_8\}$, $\{v_9\}$, $\{v_{10}\}$, $\{v_{11}\}$, $\{v_{12}\}$, $\{v_{13}\}$, $\{v_{14}\}$, $\{v_{15}\}$, $\{v_1, v_9\}$, $\{v_2, v_8\}$, $\{v_3, v_7\}$, $\{v_4, v_6\}$, $\{v_5, v_7\}$, $\{v_4, v_8\}$, $\{v_3, v_9\}$, $\{v_2, v_{10}\}$, $\{v_{10}, v_{12}\}$, $\{v_9, v_{13}\}$, $\{v_8, v_{14}\}$, $\{v_7, v_{15}\}$, $\{v_6, v_{14}\}$, $\{v_7, v_{13}\}$, $\{v_8, v_{12}\}$, $\{v_9, v_{11}\}$, $\{v_1, v_{10}\}$, $\{v_2, v_9\}$, $\{v_3, v_8\}$, $\{v_4, v_7\}$, $\{v_5, v_6\}$, $\{v_{10}, v_{11}\}$, $\{v_9, v_{12}\}$, $\{v_8, v_{13}\}$, $\{v_7, v_{14}\}$, , $\{v_1, v_3\}$, $\{v_2, v_4\}$, $\{v_3, v_5\}$, $\{v_{10}, v_8\}$, $\{v_9, v_7\}$, $\{v_8, v_6\}$, $\{v_{11}, v_{13}\}$, $\{v_{12}, v_{14}\}$, $\{v_{13}, v_{15}\}$, $\{v_1, v_{12}\}$, $\{v_2, v_{13}\}$, $\{v_3, v_{14}\}$, $\{v_4, v_{15}\}$, $\{v_5, v_{14}\}$, $\{v_4, v_{13}\}$, $\{v_3, v_{12}\}$, $\{v_2, v_{11}\}$, $\{v_1, v_{11}\}$, $\{v_2, v_{12}\}$, $\{v_3, v_{13}\}$, $\{v_4, v_{14}\}$, $\{v_5, v_{15}\}$, $\{v_1, v_3, v_9\}$, $\{v_2, v_4, v_8\}$, $\{v_3, v_5, v_7\}$, $\{v_2, v_8, v_{10}\}$, $\{v_3, v_7, v_9\}$, $\{v_4, v_6, v_8\}$, $\{v_8, v_{10}, v_{12}\}$, $\{v_7, v_9, v_{13}\}$, $\{v_6, v_8, v_{14}\}$, $\{v_9, v_{11}, v_{13}\}$, $\{v_8, v_{12}, v_{14}\}$, $\{v_7, v_{13}, v_{15}\}$, $\{v_1, v_9, v_{11}\}$, $\{v_2, v_8, v_{12}\}$, $\{v_3, v_7, v_{13}\}$, $\{v_4, v_6, v_{14}\}$, $\{v_5, v_7, v_{15}\}$, $\{v_4, v_8, v_{14}\}$, $\{v_3, v_9, v_{13}\}$, $\{v_2, v_{10}, v_{12}\}$, $\{v_1, v_{10}, v_{12}\}$, $\{v_2, v_9, v_{11}\}$, $\{v_2, v_9, v_{13}\}$, $\{v_3, v_8, v_{12}\}$, $\{v_3, v_8, v_{14}\}$, $\{v_4, v_7, v_{13}\}$, $\{v_4, v_7, v_{15}\}$, $\{v_5, v_6, v_{14}\}$, $\{v_2, v_{10}, v_{11}\}$, $\{v_1, v_9, v_{12}\}$, $\{v_3, v_9, v_{12}\}$, $\{v_2, v_8, v_{13}\}$, $\{v_4, v_8, v_{13}\}$, $\{v_3, v_7, v_{14}\}$, $\{v_5, v_7, v_{14}\}$, $\{v_4, v_6, v_{15}\}$, $\{v_2, v_{11}, v_{13}\}$, $\{v_3, v_{12}, v_{14}\}$, $\{v_4, v_{13}, v_{15}\}$, $\{v_1, v_3, v_{12}\}$, $\{v_2, v_4, v_{13}\}$, $\{v_3, v_5, v_{14}\}$, $\{v_1, v_3, v_9, v_{12}\}$, $\{v_2, v_4, v_8, v_{13}\}$, $\{v_3, v_5, v_7, v_{14}\}$, $\{v_4, v_6, v_8, v_{14}\}$, $\{v_3, v_9, v_7, v_{13}\}$, $\{v_2, v_8, v_{10}, v_{12}\}$, $\{v_2, v_9, v_{11}, v_{13}\}$, $\{v_3, v_8, v_{12}, v_{14}\}$, $\{v_4, v_7, v_{13}, v_{15}\}\}$.

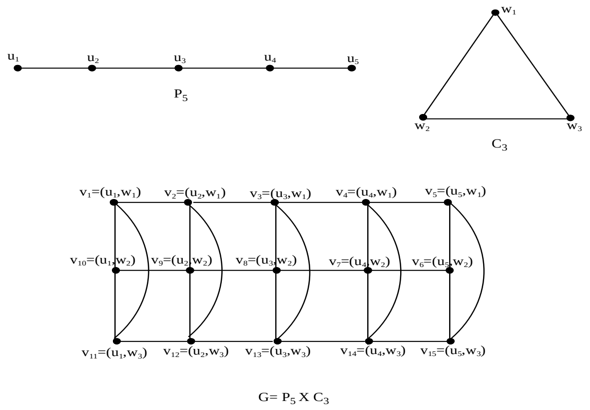


Fig. 3

In this $G = P_n \times C_m$, where $n = 5, m = 3$, each of the $15[= n \times m(5 \times 3)]$ single vertices has a neighbour.

48 $[= \{2(m-1)(n-1) + m(n-2) +$ *nm−2+2n+2n−2+2=4mn−1]*, subsets of vertices with two elements has got at least one common neighbour. Also

$42[= \{2(m-2)(n-1) + 2(m-1)(n-2) +$ *22n−2+2+ 2n−2=4mn−6m*] and

$9[= m(n-2)]$, subsets of vertices with three and four elements respectively.

Thus the neighbourhood polynomial of $G = P_5 \times C_3$ is,

$neigh_G(x) = 1 + 15x + 48x^2 + 42x^3 + 9x^4$.

Corollary 2.6 The neighbourhood polynomial of $P_n \times C_m$ is

$1 + mnx + [4m(n-1) - 2n]x^2 + 2m(2n -$ *3x3+mn−2x4,m=4*.

Proof: It has been calculated that there are

$2(n-1)(m-1) + m(n-2) + n(m-2) + 2n + 2(n-2) + 2 = 4m(n-1)$ two element subsets of vertices in the neighbourhood complex of $P_n \times C_m$. When $m = 4$, of the above two element subsets of vertices $n(m-2)$ vertices coincide with $2n$ vertices. Hence in effect, the number of two element subsets of vertices is $4m(n-1) - 2n$. The number of three and four element subsets of vertices remains the same on both $P_n \times C_n$ and $P_n \times C_4$.

Hence the result follows.

Theorem 2.7 Let $G = G_1 \times G_2$, where G_1 and $G_2$are any two graphs of order m and n respectively. Then $deg(neigh_G(x)) = \Delta(G_1) + \Delta(G_2)$.

Proof:Let $(u_1, u_2, \dots u_m) \in V(G_1)$, and $(v_1, v_2, \dots v_n) \in V(G_2)$

If $G = G_1 \times G_2$, G has mn vertices and from the definition of Cartesian product it follows that the degree of each vertex $w_k = (u_i, v_j) \in V(G)$ is $d(w_k) = d(u_i) + d(v_j)$. Hence there exists at least one vertex $w_g = (u_k, v_l) \in V(G)$, with maximum degree in G.

Since, $d(w_g) = d(u_k) + d(v_l)$,
the sum is maximum only if

$$d(u_k) = \Delta(G_1) \text{ and } d(v_l) = \Delta(G_2).$$

That is, $d(w_g) = \Delta(G_1) + \Delta(G_2)$.

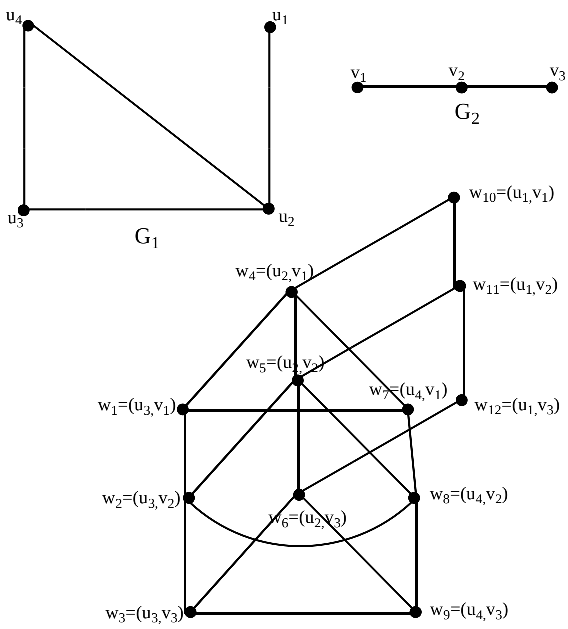
In the neighbourhood complex $\mathcal{N}(G)$ of graph G, the vertices are vertices of graph and the faces are subsets of vertices having a common neighbour. Consider a vertex w_k of G with degree $d(w_k)$. The $d(w_k)$ vertices adjacent to w_k, forms complexes with one element, two elements, three elements,... $d(w_k)$ elements (since they have at least w_k as a common neighbour) and no $(d(w_k) + 1)$, vertices can have w_k as a common neighbour. Thus there exists a face with maximum cardinality with respect to a vertex having the maximum degree, and that cardinality is equal to the degree of that vertex.

Also the neighbourhood polynomial of G is defined as $neigh_G(x) = \sum_{u \in \mathcal{N}(G)} x^{|u|}$. Hence the degree of $neigh_G$ is the maximum cardinality of the face in neighbourhood complex. Thus, if w_g is a vertex with maximum degree in G with $d(w_g) = \Delta(G_1) + \Delta(G_2)$, then,

$$deg(neigh_G(x)) = \Delta(G_1) + \Delta(G_2).$$

Example 3

Consider the figure 4.



$G = G_1 \times G_2$

Fig. 4

The neighbourhood complex $\mathcal{N}(G)$ of graph G is,
$\mathcal{N}(G) = \{\phi, \{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}, \{w_5\}, \{w_6\}, \{w_7\}, \{w_8\}, \{w_9\}, \{w_{10}\}, \{w_{11}\}, \{w_{12}\}, \{w_2, w_4\}, \{w_2, w_7\}, \{w_4, w_7\}, \{w_1, w_3\}, \{w_1, w_5\}, \{w_1, w_8\}, \{w_3, w_5\}, \{w_3, w_8\}, \{w_5, w_8\}, \{w_2, w_6\}, \{w_2, w_9\}, \{w_6, w_9\}, \{w_1, w_7\}, \{w_1, w_{10}\}, \{w_5, w_7\}, \{w_5, w_{10}\}, \{w_7, w_{10}\}, \{w_2, w_8\}, \{w_2, w_{11}\}, \{w_4, w_6\}, \{w_4, w_8\}, \{w_4, w_{11}\}, \{w_6, w_8\}, \{w_6, w_{11}\}, \{w_8, w_{11}\}, \{w_3, w_9\}, \{w_3, w_{12}\}, \{w_5, w_9\}, \{w_5, w_{12}\}, \{w_9, w_{12}\}, \{w_1, w_4\}, \{w_2, w22_5\}, \{w_7, w_9\}, \{w_3, w_6\}, \{w_{10}, w_{12}\}, \{w_2, w_4, w_7\}, \{w_1, w_3, w_5\}, \{w_1, w_3, w_8\}, \{w_1, w_5, w_8\}, \{w_3, w_5, w_8\}, \{w_2, w_6, w_9\}, \{w_1, w_5, w_7\}, \{w_1, w_5, w_{10}\}, \{w_1, w_7, w_{10}\}, \{w_5, w_7, w_{10}\}, \{w_2, w_6, w_4\}, \{w_2, w_4, w_8\}, \{w_2, w_4, w_{11}\}, \{w_2, w_6, w_8\}, \{w_2, w_6, w_{11}\}, \{w_2, w_8, w_{11}\}, \{w_4, w_6, w_8\}, \{w_4, w_6, w_{11}\}, \{w_4, w_8, w_{11}\}, \{w_6, w_8, w_{11}\}, \{w_3, w_5, w_9\}, \{w_3, w_5, w_{12}\}, \{w_3, w_9, w_{12}\}, \{w_5, w_9, w_{12}\}, \{w_1, w_4, w_8\}, \{w_2, w_5, w_7\}, \{w_2, w_5, w_9\}, \{w_5, w_7, w_9\}, \{w_2, w_7, w_9\}, \{w_3, w_6, w_8\}, \{w_5, w_{10}, w_{12}\}, \{w_1, w_3, w_5, w_8\}, \{w_1, w_5, w_7, w_{10}\}, \{w_2, w_4, w_6, w_8\}, \{w_2, w_4, w_6, w_{11}\}, \{w_2, w_6, w_8, w_{11}\}, \{w_4, w_6, w_8, w_{11}\}, \{w_2, w_4, w_8, w_{11}\}, \{w_3, w_5, w_9, w_{12}\}, \{w_2, w_5, w_7, w_9\}, \{w_2, w_4, w_6, w_8, w_{11}\}\}$.

The only five element complex $\{w_2, w_4, w_6, w_8, w_{11}\}$, has the common neighbour w_5, and from the figure we have $d(w_5) = 5$.

Also the neighbourhood polynomial of G is,
$neigh_G(x) = 1 + 12x + 35x^2 + 31x^3 + 9x^4 + x^5$,and $deg(neigh_G(x)) = 5$.

Reference

[1] Jason I. Brown, Richard J. Nowakowski, "The neighbourhood polynomial of a graph", Australian journal of Combinatorics, Volume 42(2008), Pages 55-68.

[2] G.Suresh Singh, "Graph Theory", PHI Learning Private Limited, New Delhi, 2010.