Classification Graphs with Large Strong Roman Domination Number

G. Suresh Singh[#1], Narges Mohsenitonekaboni[*2]
[#]Professor and Head, Department of Mathematics, Kariavattom Campus,
Thiruvananthapuram 695581, Kerala, India.
[*]Research scholar, Department of Mathematics, Kariavattom Campus,
Thiruvananthapuram 695581, Kerala, India.

Abstract *— the function $f : V \to \{0,1,2,3\}$ is named a Strong Roman dominating function on a graph G whenever for each vertex u with $f(u)=0$ there exists an adjacent vertex v to u such that $f(v)=3$. Also for each vertex u with $f(u)=1$ there should be at least one adjacent vertex v to u weight 2. The weight of an SRDF is denoted as $f(V)$ and $f(V)=\sum_{u\in V} f(u)$. Strong Roman domination number of a graph G which we denoted by $\gamma_{SR}(G)$ is minimum weight of an SRDF on G. In this paper, we characterize all connected graphs of order n with Strong Roman domination numbers $2n-2$, $2n-3$, $2n-4$ and $2n-5$. Also we present the properties of connected graph of order n with Strong Roman domination number $2n-6$.*

Keywords — *Strong Roman domination number, Strong Roman domination function.*

I. INTRODUCTION

For all terminologies and notations related to graph theory which is not provided here, we follow [2, 3, 7]. In this paper, G is a *graph* with the set of *vertices* $V=V(G)$ and the set of *edges* $E=E(G)$. The *order* and the *size* of a graph G are denoted by $|V|=n$ and $|E|=n$, respectively. The set of all vertices which are adjacent to a vertex v is called *open neighbourhood* of vertex v and denoted by $N(v)$. The *closed neighbourhood* of the vertex v is defined by $N[v]=N(v)\cup\{v\}$. The degree of a vertex $v\in V(G)$ is defined as $\deg(v)=|N(v)|$. The *minimum degree* and *maximum degree* of a graph are denoted by $\delta=\delta(G)$ and $\Delta=\Delta(G)$, respectively. The set of vertices $A\subseteq V(G)$ is a *dominating set* if every vertex v not in A is adjacent to at least one vertex in A. The minimum cardinality of any dominating set of G is the *domination number* of G and is denoted by $\gamma(G)$. A dominating set A in G with $|A|=\gamma(G)$ is called a $\gamma(G)-set$. A set A of vertices in a graph G is a *total dominating set* of G if every vertex of G is adjacent to some vertex in A. The minimum cardinality of a total dominating set of G is the *total domination number* of G and is denoted by $\gamma_t(G)$. A total dominating set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)-set$, see [4]. A set is *independent* (or *stable*) if no two vertices in it are adjacent. An *independent dominating set* of G is a set that is both dominating and independent in G. The *independent domination number* of G, denoted by $i(G)$, and is the minimum size of an independent dominating set. An independent dominating set of G of size $i(G)$ is called an $i-set$, see [1]. The function $f:V(G)\to\{0,1,2,3\}$ is named an SRDF on G whenever for each vertex u with weight 0 there exists an adjacent vertex v such that $f(v)=3$. Also each vertex with weight 1 has at least an adjacent vertex of weight 2. *Strong Roman domination number* of graph G that denoted by $\gamma_{SR}(G)$ is minimum weight of an SRDF on G. The concept of Strong Roman domination is introduced by K. Selvakumar et al. [5]. One of the interests in the discussion of domination and its parameters dependent, determine the graph that is the number of that parameter is Large. For example, M. A. Henning's classification of all graphs with total domination number [4]. W. C. Shiu et al. characterize Triangle-free graphs with large independent domination number [1].

In this paper, we characterize all connected graph of order n with $\gamma_{SR}(G)\in\{2n-1, 2n-2, 2n-3, 2n-4, 2n-5\}$ and also we show that if G is a connected graph of order n with $\gamma_{SR}(G)=2n-6$, then $2\le\Delta(G)\le 3$ add $6\le n\le 8$. Now, we use the following theorem to prove the rest of theorems.

Theorem 1[6]: For any cycle and path of order $n \geq 3$, we have

$$\gamma_{SR}(P_n) = \gamma_{SR}(C_n) = \begin{cases} n & if \quad n \equiv 0 \pmod 3 \\ n+1 & if \quad n \not\equiv 0 \pmod 3 \end{cases}.$$

Theorem 2: Let G be a connected graph of order n. Hence $\gamma_{SR}(G) = 2n-1$ if and only if $G = K_2$.

Proof: Assume that G is a connected graph of order n. If $\gamma_{SR}(G) = 2n-1$, then we show that $G = K_2$. For it is enough to show that $\Delta(G) = 1$.

On the contrary suppose that $\Delta(G) \geq 2$ and v is a vertex with degree $\Delta(G)$. In this case, it is clearly the function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = N(v), V_1 = \phi$, $V_2 = V(G) - N[v]$ and $V_3 = \{v\}$ is an SRDF on graph G. Therefore

$$\begin{aligned} \gamma_{SR}(G) &\leq 2|V_2| + 3|V_3| \\ &= 2(n - \Delta - 1) + 3 \\ &= 2n - 2\Delta + 1. \end{aligned}$$

Since $\Delta(G) \geq 2$, we have

$$\begin{aligned} \gamma_{SR}(G) &\leq 2n - 2\Delta + 1 \\ &= 2n - 4 + 1 \\ &= 2n - 3. \end{aligned}$$

Which is a contradiction.

Therefore $\Delta(G) \leq 1$. Now, suppose that $\Delta(G) = 1$. Since G is a connected graph, then either $G = K_1$ or $G = K_2$. If $G = K_1$ then $\gamma_{SR}(G) = 2 > 2n-1$, which is a contradiction.

Therefore $\Delta(G) = 1$ implies $G = K_2$.

Conversely, if $G = K_2$, then $\gamma_{SR}(G) = 3 = 2n-1$. □

Theorem 3: There is no connected graph G of order n with $\gamma_{SR}(G) = 2n-2$.

Proof: Let G be a connected graph of order n. If $\Delta(G) = 1$, then $G = K_1$ and thus $\gamma_{SR}(G) = 2 > 2n-2$.

Now, suppose that $\Delta(G) = 1$. In this case, $G = K_2$ and thus $\gamma_{SR}(G) = 3 > 2n-2$.

Therefore, we can assume that $\Delta(G) \geq 2$. In this case, the function $h = (V_0, V_1, V_2, V_3)$, where $V_0 = N(v), V_1 = \phi$, $V_2 = V(G) - N[v]$ and $V_3 = \{v\}$ is an SRDF on graph G. Hence

$$\begin{aligned} \gamma_{SR}(G) &\leq h(V) \\ &= 2|V_2| + 3|V_3| \\ &= 2(n - \Delta - 1) + 3 \\ &= 2n - 2\Delta + 1. \end{aligned}$$

Since $\Delta(G) \geq 2$, we have

$$\begin{aligned} \gamma_{SR}(G) &\leq 2n - 2\Delta + 1 \\ &= 2n - 4 + 1 \\ &= 2n - 3 \\ &= 2n - 2. \end{aligned}$$

Thus in any case there is no connected graph G with $\gamma_{SR}(G) = 2n-2$. □

In the previous theorem we proved that there is no connected graph with $\gamma_{SR}(G) = 2n-2$ but if $G = 2K_2$, then $\gamma_{SR}(G) = 6 = 2n-2$.

In the next theorem we characterize all disconnected graphs with $\gamma_{SR}(G) = 2n-2$.

Theorem 4: Let G be a disconnected graph. Therefore $\gamma_{SR}(G) = 2n-2$ if and only if $G = 2K_2 \cup \overline{K}_r$ such that $r = n-4$.

Proof: First, we show that $\Delta(G) = 1$. On the contrary, suppose that $\Delta(G) \neq 1$, therefore $\Delta(G) = 0$ or $\Delta(G) \geq 2$.

If $\Delta(G) = 0$, then since G is a disconnected graph, $G = \overline{K}_n$ for $n \geq 2$. In this case, $\gamma_{SR}(G) = 2n$ which is a contradiction. Therefore $\Delta(G) \neq 0$.

Now, suppose that $\Delta(G) \geq 2$ and $\deg(v) = \Delta(G)$ for a vertex $v \in V(G)$. Similar to the previous theorem, we consider the function $h = (V_0, V_1, V_2, V_3)$, where $V_0 = N(v), V_1 = \phi$, $V_2 = V(G) - N[v]$ and $V_3 = \{v\}$. It is clearly h is an SRDF on graph G and $\gamma_{SR}(G) \leq 2n-3$, which is a contradiction.

Therefore $\Delta(G)=1$. Hence G is a graph obtained from copies of K_2 and isolated vertices. Since G is a disconnected graph and $\Delta(G)=1$, G has at least three vertices. Assume that G has m induced sub-graphs K_2. For solving the problem it is enough to show that $m=2$. On the contrary, suppose that $m \neq 2$. We consider following two cases:

Case 1: $m=1$.

In this case, $G=K_2 \cup \overline{K}_2$. Suppose that $V(K_2)=\{x,y\}$, hence the function $f=(V_0,V_1,V_2,V_3)$, where $V_0=\{y\}$, $V_1=\phi$, $V_2=V(G)-\{x,y\}$ and $V_3=\{x\}$ is an SRDF on graph G. Therefore

$$\begin{aligned}\gamma_{SR}(G) &\leq 2|V_2|+3|V_3|\\ &=2(n-2)+3\\ &=2n-1\\ &>2n-2,\end{aligned}$$

Which is a contradiction.

Case 2: $m>2$.

In this case, G has at least three induced sub-graphs K_2. Suppose that $\{x,y\}$, $\{u,v\}$ and $\{w,z\}$ are vertex sets of those three induced sub-graphs. We define the function $g=(V_0,V_1,V_2,V_3)$, where $V_0=\{y,v,z\}$, $V_1=\phi$, $V_2=V(G)-\{x,y,u,v,z,w\}$ and $V_3=\{x,u,w\}$ is an SRDF on graph G. Therefore

$$\begin{aligned}\gamma_{SR}(G) &\leq g(V)\\ &=2|V_2|+3|V_3|\\ &=2(n-6)+9\\ &=2n-3\\ &<2n-3,\end{aligned}$$

which is a contradiction.

Hence $m=2$ and thus $G=2K_2 \cup \overline{K}_r$ where $r=n-4$.

The converse part is obvious. □

Theorem 5: Let G be a connected graph of order n. Hence $\gamma_{SR}(G)=2n-3$ if and only if $G=(P_3,P_4,C_3,C_4)$.

Proof: First, we show that $\Delta(G)=2$. On the contrary, suppose that $\Delta(G)>2$ and the vertex $v \in V(G)$ has maximum degree $\Delta(G)$. We consider the function $f=(V_0,V_1,V_2,V_3)$, where $V_0=N(v)$, $V_1=\phi$, $V_2=V(G)-N[v]$ and $V_3=\{v\}$ is an SRDF on graph G. It is obvious that f is an SRDF on graph G. Therefore

$$\begin{aligned}\gamma_{SR}(G) &\leq f(V)\\ &=2|V_2|+3|V_3|\\ &=2(n-\Delta-1)+3\\ &=2n-2\Delta+1\\ &<2n-4+1\\ &=2n-3,\end{aligned}$$

which is a contradiction. Thus $\Delta(G)\leq 2$.

If $\Delta(G)=1$, then $G=K_2$. In this case, $\gamma_{SR}(G)=3>2n-3$ which is a contradiction. Then $\Delta(G)\neq 1$.

If $\Delta(G)=0$, then $G=K_1$ and $\gamma_{SR}(G)=2\neq 2n-3$.

Thus $\Delta(G)=2$. Hence G is a path or a cycle. Based on Theorem 1 if $n\equiv 0 \pmod 3$, then $\gamma_{SR}(G)=n$. Therefore

$$n=\gamma_{SR}(G)=2n-3$$

$$\Rightarrow \quad n=3.$$

It follows that $G=P_3$ or $G=C_3$. Now, suppose that $n\not\equiv 0 \pmod 3$. In this case, $\gamma_{SR}(G)=n+1$, then $n+1=\gamma_{SR}(G)=2n-3$, where we conclude that $n=4$ and thus $G=P_4$ or $G=C_4$.

The converse part is obvious from Theorem 1. □

Theorem 6: Let G be a connected graph of order n. Then $\gamma_{SR}(G)=2n-4$ if and only if $G\in\{P_5,C_5\}$.

Proof: Suppose that G be a connected graph of order n with $\gamma_{SR}(G)=2n-4$. We consider the vertex $v\in V(G)$ with maximum degree in G. In this case, the function $f=(V_0,V_1,V_2,V_3)$, where $V_0=N(v), V_1=\phi$, $V_2=V(G)-N[v]$ and $V_3=\{v\}$ is an SRDF on graph G. It is obvious that f is an SRDF on graph G. Therefore

$$\begin{aligned}2n-4&=\gamma_{SR}(G)\\&\le f(V)\\&=2(n-\Delta-1)+3\\&=2n-2\Delta+1.\end{aligned}$$

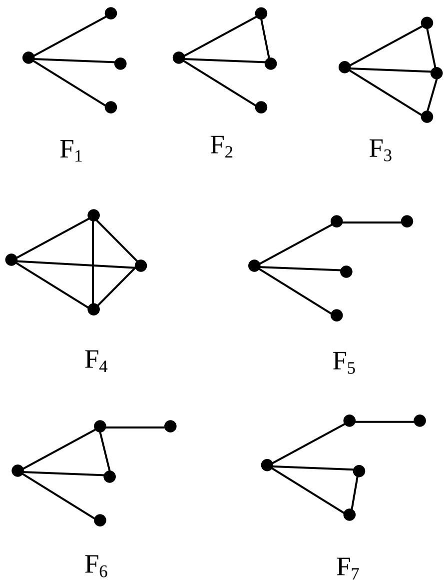
From which we can conclude that

$$\begin{aligned}2n-4&\le 2n-2\Delta+1,\\-5&\le -2\Delta,\\2\Delta&\le 5,\\\Delta&\le 2.\end{aligned}$$

Based on proof of previous theorem it is easy to show that $\Delta(G)=2$. Thus G is a path or a cycle.
If $n\equiv 0\pmod 3$, then based on Theorem 1 we have $\gamma_{SR}(G)=n$. Thus $n=\gamma_{SR}(G)=2n-4$, from which we get $n=4$ which is a contradiction with $n\equiv 0\pmod 3$.
Hence $n\not\equiv 0\pmod 3$. In this case, based on Theorem 1, we have $\gamma_{SR}(G)=n+1$. Then $n+1=2n-4$ and thus $n=5$. Therefore, G is either a path P_5 or a cycle C_5.

Conversely, we assume that $G\in\{P_5,C_5\}$ then based on Theorem 1 we have $\gamma_{SR}(G)=6=2n-4$. □

Let Ƒ be family of graphs shown in Figure 1.



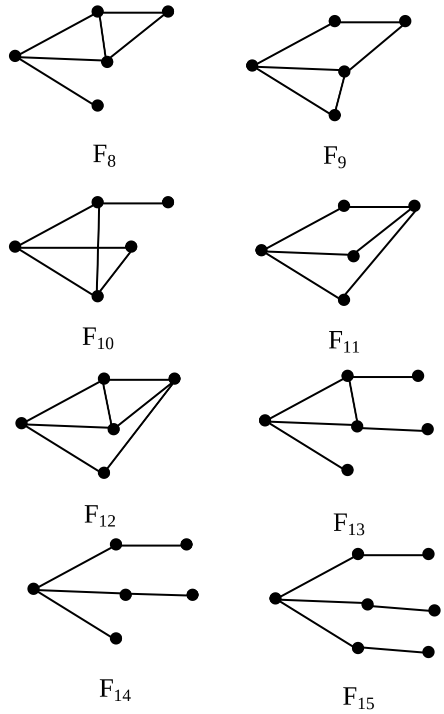
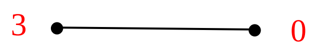


Figure 1. Family of Ƒ.

Theorem 7: Let G be a connected graph of order n. Then $\gamma_{SR}(G)=2n-5$ if and only if $G\in$ Ƒ.

Proof: Suppose that G is a connected graph with $\gamma_{SR}(G)=2n-5$. We consider the vertex $v\in V(G)$ with degree $\Delta(G)$.
If $\Delta(G)=0$, then $G=K_1$ and thus $\gamma_{SR}(G)=2\ne 2n-5$ which is a contradiction.
Now, assume that $\Delta(G)=1$. In this case, $G=K_2$. Therefore, $\gamma_{SR}(G)=3\ne 2n-5$ which is a contradiction.
Hence $\Delta(G)\ge 2$. If $\Delta(G)=2$, then G is a cycle or a path.
Also, if $n\equiv 0\pmod 3$, then based on Theorem 1 we have $\gamma_{SR}(G)=n$. Therefore, $n=\gamma_{SR}(G)=2n-5$ and thus $n=5$ which is a contradiction since $n\equiv 0\pmod 3$.
Hence $n\not\equiv 0\pmod 3$. In this case, based on Theorem 1 we have $\gamma_{SR}(G)=n+1$ and thus $n+1=2n-5$ then $n=6$ which is a contradiction since $n\not\equiv 0\pmod 3$.
Therefore, $\Delta(G)\ge 3$. We define an SRDF function for graph G as follows:
Let $f=(V_0,V_1,V_2,V_3)$,



where $V_0 = N(v), V_1 = \phi$, $V_2 = V(G) - N[v]$ and $V_3 = \{v\}$. Therefore

$$\begin{aligned} 2n-5 &= \gamma_{SR}(G) \\ &\le f(V) \\ &= 2(n-\Delta-1)+3 \\ &= 2n-2\Delta+1. \end{aligned}$$

So, $2n-5 \le 2n-2\Delta+1$ which implies that $-6 \le -2\Delta$.

Therefore, $\Delta(G) \le 3$. Already we have, $\Delta(G) \ge 3$. Hence $\Delta(G) = 3$. Assume that $N(v) = \{x, y, z\}$.

If $V(G) - N[v] = \phi$, then the function $h = (N(v), \phi, \phi, \{v\})$ is a $\gamma_{SR}(G) -$ *fuction* and thus $\gamma_{SR}(G) = 3 = 2n-5$. Now, suppose that there is no edge between x, y and z then $G = F_1$. If there is only one edge between x, y and z then $G = F_2$ and in case if there is two edges between x, y and z then, $G = F_3$. If there exists three edges between x, y and z, then $G = F_4$.

Now, suppose that $V(G) - N[v] \ne \phi$. We claim that the set $V(G) - N[v]$ has at most three elements.

To prove this claim, let $V(G) - N[v]$ has at least four vertices. We consider $\{a, b, c, d\} \subseteq V(G) - N[v]$.

If there exists an edge between both vertices of $V(G) - N[v]$ like a and b, then the function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = N(v) \cup \{b\}$, $V_1 = \phi$, $V_2 = V(G) - (N[v] \cup \{a, b\})$ and $V_3 = \{a, v\}$ is an SRDF on graph G. Therefore

$$\begin{aligned} 2n-5 &= \gamma_{SR}(G) \\ &\le f(V) \\ &= 2|V_2| + 3|V_3| \\ &= 2(n-\Delta-3)+6 \\ &= 2n-2\Delta. \end{aligned}$$

Thus $-5 \le -2\Delta(G)$ and consequently $-2\Delta(G) \le 5$. Therefore, $\Delta(G) \le 2$ which is a contradiction since $\Delta(G) = 3$.

Hence there is no edge between the vertices in $V(G) - N[v]$. In this case, since G is a connected graph, each vertex of $V(G) - N[v]$ is adjacent to at least one vertex of $N(v)$. On the other hand, since $V(G) - N[v]$ has at least four elements and $N(v)$ has three elements, there exists at least one vertex of $N(v)$ adjacent to at least two vertices in $V(G) - N[v]$. Without loss of generality, suppose that $x \in N(v)$ has at least two neighbourhood vertices in $V(G) - N[v]$. In this case, we define an SRDF function as follows: Define $f = (V_0, V_1, V_2, V_3)$, where $V_0 = (N(v) \cup N(x)) - \{x\}$, $V_1 = \phi$, $V_2 = V(G) - (N[v] \cup N[x])$ and $V_3 = \{x, v\}$ is an SRDF on graph G. Therefore

$$\begin{aligned} 2n-5 &= \gamma_{SR}(G) \\ &\le f(V) \\ &= 2|V_2| + 3|V_3| \\ &= 2(n-\Delta-|N(v)|)+6 \\ &\le 2(n-\Delta-3)+6 \\ &= 2n-2\Delta. \end{aligned}$$

Hence $2n-5 \le 2n-2\Delta$ from which we conclude that $\Delta(G) \le 2$ which is a contradiction, since $\Delta(G) = 3$. So, $V(G) - N[v]$ has at most three elements.

Now, we consider the following cases:

Case 1: $V(G) - N[v]$ has exactly one vertex. We assume that $V(G) - N[v] = \{a\}$. Since G is connected, a is adjacent to at least one vertex of $N(v)$. Without loss of generality, suppose that $x \in N(a)$.

If a is adjacent to just the vertex x, since $\Delta(G) \le 3$, we get x is adjacent to at most one of the vertices y and z. Now, we consider the following sub-cases:

Sub-case 1: y is not adjacent to z.

In this case, if x is not adjacent to any of the vertices y and z, then $G = F_5$.

Now, suppose that x is adjacent to one of the vertices y or z. Therefore if y and z are not neighbourhoods. So $G = F_6$.

Sub-case 2: y is adjacent to z.

In this case, if x is not adjacent to any of the vertices y and z, then $G = F_7$. And, if x is adjacent to exactly one vertex of the set of $\{y, z\}$, then $G = F_8$.

Now, assume that a is adjacent with exactly two vertices of the set $\{x, y, z\}$. Without loss of generality, suppose that $N(a) = \{x, y\}$. We consider the following sub-cases:

Sub-case 1: x is not adjacent to y.

In this case, if z is adjacent to exactly one vertex x or y, then $G = F_9$. On the other hand, if z is adjacent to both vertices x and y, then $G = F_{10}$.

Sub-case 2: x is adjacent to y.

In this case, z cannot be adjacent to one of the vertices x and y. Otherwise, G has a vertex of degree 4 which is a contradiction since $\Delta(G) = 3$. Therefore $G = F_8$.

Now, suppose that a is adjacent with all vertices x, y and z.

If x is adjacent to the both vertices y and z, then $\deg(x) = 4$ which is a contradiction since $\Delta(G) = 3$. Therefore, x is adjacent to at most one of vertices y and z.

Same as before we can show that y is adjacent to at most one of the vertices x and z.

Also, z is adjacent to at most one of the vertices x and y.

Therefore, there exists at most an edge between the vertices x, y and z. So $G = F_{12}$. If there does not exist any edge between vertices x, y and z, then $G = F_{11}$.

Case 2: Assume that the set $V(G) - N[v]$ has two elements.

Set $V(G) - N[v] = \{a, b\}$. Earlier we have shown that a is not adjacent to b.

Now, suppose that x is adjacent to the both the vertices a and b. In this case, the function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = \{a, b, y, z\}$, $V_1 = \phi$, $V_2 = \phi$ and $V_3 = \{x, y\}$ is an SRDF on graph G. Therefore

$$\begin{aligned} 2n - 5 = \gamma_{SR}(G) &\le f(V) \\ &= 2|V_2| + 3|V_3| \\ &= 6. \end{aligned}$$

But since $n = 6$, there is a contradiction.

Therefore, x is adjacent to at most one of vertices a and b. Same as before, each vertex of $\{y, z\}$ is adjacent to at most one vertex of $\{a, b\}$.

On the other hand, since G is connected, a is adjacent to at least one vertex of $N(v)$.

Without loss of generality, assume that a is adjacent to x. Also since G is connected and a is not adjacent to b, the vertex b is adjacent to one vertex of $N(v)$. Since earlier we have shown that x is adjacent to at most one of vertices in $\{a, b\}$ and the vertex x is adjacent to the vertex a, we get x cannot be adjacent to b. Therefore, the vertex b is adjacent to at least one of two vertices y and z.

Without loss of generality, suppose that b is adjacent to y is not a neighbourhood of a. We consider the following sub-cases:

Sub-case 1: x is adjacent to y.

In this case, since $\Delta(G) = 3$, we get x and y are not adjacent to z. As in the previous case, we know that the vertex z is adjacent to at most one of the vertices a and b.

Now, if z is not adjacent to any of the vertices a and b, then $G = F_{13}$. Otherwise, suppose that z is adjacent to exactly one vertex of $\{a, b\}$.

Without loss of generality, suppose that z is adjacent to the vertex b. We define the function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = \{x, y, z, a\}$, $V_1 = \phi$, $V_2 = \phi$ and $V_3 = \{x, b\}$. It is obvious that f is an SRDF on graph G, thus $\gamma_{SR}(G) \le f(V) = 6$. Hence $\gamma_{SR}(G) < 2n - 5 = 7$ which is a contradiction. Therefore, z is not adjacent to any of the vertices a and b as discussed earlier.

Sub-case 2: x is not adjacent to y.

In this case, if z is adjacent to both vertices x and y. Since $\Delta(G) = 3$, the vertex z cannot be adjacent to any of the vertices a and b. Now, we define the function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = \{v, a, b, z\}$, $V_1 = \phi$, $V_2 = \phi$ and $V_3 = \{x, y\}$. It is obvious that the function f is an SRDF on graph G. Thus $\gamma_{SR}(G) \le f(V) = 6$ which is a contradiction, since $\gamma_{SR}(G) < 2n-5$. Therefore, z is not adjacent to both vertices x and y.

Now, suppose that z is adjacent to exactly one vertex of x and y.

Without loss of generality, suppose that z is adjacent to y. In this case, the function $f = (V_0, V_1, V_2, V_3)$ where $V_0 = \{v, x, b, z\}, V_1 = \phi, V_2 = \phi$ and $V_3 = \{a, y\}$ is an SRDF on graph G. Therefore

$$\gamma_{SR}(G) \le f(V) = 6 < 2n-5,$$

which is a contradiction to the assumption. Therefore, z cannot be adjacent to any of the vertices x and y.

If z is adjacent to the vertex a, then the function $f = (V_0, V_1, V_2, V_3)$ where $V_0 = \{v, x, z, b\}, V_1 = \phi$, $V_2 = \phi$ and $V_3 = \{a, y\}$ is an SRDF on graph G. Hence

$$\gamma_{SR}(G) \le f(V) = 6 < 2n-5,$$

which is a contradiction. Therefore, z is not adjacent to the vertex a.

And same as before we can show that z cannot be adjacent to b. Hence $G = F_{14}$.

Case 3: Now, assume that $V(G) - N[v] = \{a, b, c\}$.

Based on previous theorem, there is no edge between all vertices of the set $\{a, b, c\}$ and each vertex of $N(v)$ is adjacent to at most one vertex of the set $\{a, b, c\}$. Since G is connected and there is no edge between the vertices of the set $\{a, b, c\}$, we get each vertex of the set $\{a, b, c\}$ has a neighbourhood in $N(v)$.

If there exists a vertex of $\{a, b, c\}$ which is adjacent to at least two vertices of $N(v)$, then there exists a vertex of $N(v)$ which is adjacent to at least two vertices of the set $\{a, b, c\}$ which is a contradiction. Therefore, each vertex of $\{a, b, c\}$ is adjacent to exactly one vertex of $N(v)$.

Without loss of generality, assume that a is adjacent to x, b to y and c to z. Now, since $\Delta(G) = 3$, the vertex x is adjacent to at most one vertex of $\{y, z\}$. Also, y is adjacent to at most one vertex of $\{x, z\}$ and the vertex z is adjacent to at most one vertex of $\{y, z\}$. Therefore, there exists at most one edge between $\{x, y, z\}$.

Suppose that there exists exactly an edge between $\{x, y\}$. We define the function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = \{v, y, a, c\}, V_1 = \phi$, $V_2 = \{b\}$ and $V_3 = \{x, z\}$ is an SRDF on graph G. Thus

$$\gamma_{SR}(G) \le f(V) = 2|V_2| + 3|V_3| = 8 = 2n-5,$$

which is a contradiction. Thus there is no edge between the vertices x, y and z. Hence $G = F_{15}$.

The converse part is obvious. □

In the following theorem we investigate the properties of connected graph G of order n with $\gamma_{SR}(G) = 2n-6$.

Theorem 8: Let G be a connected graph of order n with $\gamma_{SR}(G) = 2n-6$. Hence the following results hold:

- **a)** $2 \le \Delta(G) \le 3$.
- **b)** $N(v)$ has at most two neighborhoods outside $N(x)$.
- **c)** The set $V(G) - N[v]$ has at least two elements and at most four elements.

Proof:

a) Let $\Delta(G) > 3$ and v be a vertex of G with degree $\Delta(G)$. Hence the function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = N(v)$, $V_1 = \phi$, $V_2 = V(G) - N[v]$ and $V_3 = \{v\}$ is an SRDF on graph G. Therefore

$$\begin{aligned} 2n - 6 = \gamma_{SR}(G) &\leq f(V) \\ &= 2|V_2| + 3|V_3| \\ &= 2(n - \Delta - 1) + 3 \\ &= 2n - 2\Delta + 1. \end{aligned}$$

Since $\Delta(G) \geq 4$,

$$\begin{aligned} 2n - 6 = \gamma_{SR}(G) &\leq f(V) \\ &= 2|V_2| + 3|V_3| \\ &= 2n - 2\Delta + 1 \\ &\leq 2n - (2 \times 4) + 1 \\ &= 2n - 7, \end{aligned}$$

which is a contradiction. Thus $\Delta(G) \leq 3$. On the other hand, if $\Delta(G) = 0$ or $\Delta(G) = 1$, since G is connected, then $G = K_1$ or $G = K_2$, respectively. But in each case $\gamma_{SR}(G) \neq 2n - 6$ which is a contradiction. Thus $\Delta(G) \geq 2$. Hence $2 \leq \Delta(G) \leq 3$.

b) On the contrary assume that there exists a vertex $x \in N(v)$ such that x has three neighborhoods a, b and c outside $N(v)$. In this case, the function $f = (V_0, V_1, V_2, V_3)$, where $V_0 = N(v) \cup \{a, b, c\}$, $V_1 = \phi$, $V_2 = V(G) - (N[v] \cup \{a, b, c\})$ and $V_3 = \{x, v\}$ is an SRDF on graph G. Therefore

$$\begin{aligned} 2n - 6 = \gamma_{SR}(G) &\leq f(V) \\ &= 2|V_2| + 3|V_3| \qquad (1) \\ &= 2(n - \Delta - 4) + 6 \\ &= 2n - 2\Delta - 2. \end{aligned}$$

Since $2 \leq \Delta(G) \leq 3$, if $\Delta(G) = 2$, then it is obvious that the vertex x is adjacent to at most one vertex outside $N(v)$ which is a contradiction. So, we can suppose that $\Delta(G) = 3$. In this case, based on (1) we have

$$\begin{aligned} 2n - 6 = \gamma_{SR}(G) &\leq 2n - 2\Delta - 2 \\ &= 2n - 8, \end{aligned}$$

which is a contradiction. Thus, each vertex of $N(v)$ has at most two neighborhoods in outside $N(v)$.

c) First, we show that the set $V(G) - N[v]$ has at least two elements. It is enough to show that the set $V(G) - N[v]$ is neither empty nor it contains a single element.

Let $V(G) - N[v] = \phi$. Based on (1) we have $2 \leq \Delta(G) \leq 3$.

If $\Delta(G) = 2$, then $G = P_3$ or $G = C_3$. But based on Theorem 1 in both cases we have

$$\begin{aligned} \gamma_{SR}(G) &= 3 \\ &\neq 2n - 6, \end{aligned}$$

which is a contradiction.

Therefore, assume that $\Delta(G) = 3$. In this case, the function f where the vertex v has weight 3 and the other vertices have weight 0 is an SRDF on graph G. Hence $\gamma_{SR}(G) = 3$. But, since $n = 4$, we get $2n - 6 \neq 3$ which is a contradiction with $\gamma_{SR}(G) = 2n - 6$.

Therefore, the set $V(G) - N[v]$ has at least two elements.

Now, we show that the set $V(G) - N[v]$ has at most four elements. On the contrary suppose that $V(G) - N[v]$ has at least five vertices. Already, we have that $2 \leq \Delta(G) \leq 3$. Therefore

$$\begin{aligned} n &= |V(G)| \\ &= |N[v] \cup (V(G) - N[v])| \\ &= |N[v]| + |V(G) - N[v]| \\ &\geq (\Delta + 1) + 5. \end{aligned}$$

Thus if $\Delta(G) = 2$, then $n \geq 8$. And if $\Delta(G) = 3$, then $n \geq 9$.

Assume that $\Delta(G)=2$, then G is either a path or a cycle.

If $n\equiv 0\ (\mathrm{mod}\ 3)$, then $\gamma_{SR}(G)=n$ and thus $2n-6=n$ which indicate that $n=6$ which is contradiction with $n\geq 8$.

If $n\not\equiv 0\ (\mathrm{mod}\ 3)$, then $\gamma_{SR}(G)=n+1$ and thus $2n-6=n+1$ which implies that $n=7$ which is a contradiction with $n\geq 8$.

Now, assume that $\Delta(G)=3$. In this case, we show that there exists at most an edge between the vertices $V(G)-N[v]$. On the contrary we assume that there exist at least two edges between the vertices of $V(G)-N[v]$. Therefore, at least one of the following cases occurs:

Case 1: There exists a vertex $a\in V(G)-N[v]$ such that a is adjacent to two vertices b and c of $V(G)-N[v]$.

In this case, the function $f=(V_0,V_1,V_2,V_3)$, where $V_0=N(v)\cup\{b,c\}$, $V_1=\phi$, $V_2=V(G)-(N[v]\cup\{a,b,c\})$ and $V_3=\{v,a\}$ is an SRDF on graph G. Therefore, since $\Delta(G)=3$, we have

$$\begin{aligned}2n-6=\gamma_{SR}(G)&\leq f(V)\\&=2|V_2|+3|V_3|\\&=2(n-\Delta-4)+6\\&=2n-8,\end{aligned}$$

which is a contradiction.

Case 2: There exist two edges ab and cd in graph G, such that $\{a,b,c,d\}\subseteq V(G)-N[v]$.

In this case, the function $f=(V_0,V_1,V_2,V_3)$, where $V_0=N(v)\cup\{b,d\}$, $V_1=\phi$, $V_2=V(G)-(N[v]\cup\{a,b,c,d\})$ and $V_3=\{v,a,c\}$ is an SRDF on graph G. Therefore, since $\Delta(G)=3$, we have

$$\begin{aligned}2n-6=\gamma_{SR}(G)&\leq f(V)\\&=2|V_2|+3|V_3|\\&=2(n-\Delta-5)+9\\&=2n-7,\end{aligned}$$

which is a contradiction. Therefore, if $\Delta(G)=3$, then there exists at most an edge between all vertices of $V(G)-N[v]$.

Now, suppose that $\Delta(G)=3$ and $V(G)-N[v]$ has at least five elements. Hence $n\geq 9$. We put $N(v)=\{x,y,z\}$.

If there is no edge between all vertices of $V(G)-N[v]$, then since G is a connected graph, we get each vertex of $V(G)-N[v]$ should have adjacent to at least one vertex in $N(v)$. In this case, the function $f=(V_0,V_1,V_2,V_3)$, where $V_0=V(G)-N(v)$, $V_1=\phi$, $V_2=\phi$ and $V_3=N(v)$ is an SRDF on graph G. So

$$\begin{aligned}2n-6=\gamma_{SR}(G)&\leq f(V)\\&=9,\end{aligned}$$

which is a contradiction with $n\geq 8$.

Now, suppose that there exists an edge between all vertices $V(G)-N[v]$. We put $ab\in E(G)$ such that $\{a,b\}\subseteq V(G)-N[v]$. Since G is connected, there exists at least one of the vertices a or b is adjacent to a vertex in $N(v)$.

Without loss of generality, suppose that a is adjacent to x. In this case, the function $f=(V_0,V_1,V_2,V_3)$, where $V_0=V(G)-\{a,b,x,y,z\}$, $V_1=\phi$, $V_2=\{b\}$ and $V_3=\{x,y,z\}$ is an SRDF on graph G. Therefore

$$\begin{aligned}2n-6=\gamma_{SR}(G)&\leq f(V)\\&=2|V_2|+3|V_3|\\&=11,\end{aligned}$$

which is a contradiction with $n\geq 9$.

Thus $V(G)-N[v]$ has at most four vertices. $\square$

Corollary: If G is a connected graph of order n with $\gamma_{SR}(G)=2n-6$, then $6\le n\le 8$.

Proof: Assume that $v\in V(G)$ and $\deg(v)=\Delta(G)$. Based on previous Theorem $2\le\Delta(G)\le 3$, we have $2\le|V(G)-N[v]|\le 4$. If $\Delta(G)=3$, then

$$\begin{aligned} n &= |V(G)| \\ &= |N[v]\cup(V(G)-N[v])| \\ &= |N[v]|+|V(G)-N[v]| \\ &\le(\Delta+1)+4 \\ &= 4+4 \\ &= 8, \end{aligned}$$

and also

$$\begin{aligned} 6 &= 4+2 \\ &\le(\Delta+1)+|V(G)-N[v]| \\ &= |N[v]|+|V(G)-N[v]| \\ &= n. \end{aligned}$$

Hence, if $\Delta(G)=3$, then $6\le n\le 8$.

Now, suppose that $\Delta(G)=2$. Same as before we can show that $G=\{P_6, P_7, C_6, C_7\}$ and thus if $\Delta(G)=2$, then $6\le n\le 7$.

Hence in both cases $6\le n\le 8$. $\square$

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