Sandwich Type of Results for ϕ-like Functions using Subordination

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Abstract: *In this paper, we introduce a new subclass of ϕ -like functions and derive certain Sandwich type results that unify the work of earlier researchers.*

Keywords: Analytic function, Differential subordination, Differential superordination, best dominant, ϕ -like functions, best subordinant, and Sandwich type results.

I. INTRODUCTION

Denote by $\mathcal{H}$ the class of analytic functions in the open unit disc E= {z: z $\in C$ and |z|<1}.If a $\in C$ and n $\in$ N, let $\mathcal{H}$ [a, n] be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ... \quad (1)$$

The class of all normalized analytic functions is denoted by $\mathcal{A}$ and is given by

$$\mathcal{A}=\{f \in \mathcal{H} : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ , } f(0) = 0 \text{ and } f'(0) = 1\} \quad (2)$$

Let S be the subclass of $\mathcal{A}$ consisting of all analytic and univalent functions in E. The classes of starlike functions of order α , convex functions of order α , where α is a real number ($0 \le \alpha < 1$) and strongly starlike functions of order α , in E respectively are analytically defined as

$$S^*(\alpha) = \{ f \in \mathcal{A}: \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \text{ , } z \in E\} \quad (3)$$

$$K(\alpha) = \{f \in \mathcal{A}: \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \text{ , } z \in E\} \quad (4)$$

$$ST= \{f \in \mathcal{A}: \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}, (0 < \alpha \le 1) \text{ , } z \in E\} \quad (5)$$

We shall use S^* and K to denote $S^*(0)$ and K (0) respectively, which are the classes of univalent starlike w.r.t. origin and univalent convex functions.

Let f and g be analytic functions in the open unit disc E. If there exist a Schwarz function w analytic in E with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in$ E such that $f(z) = g(w(z))$, $z \in$ E, then f is said to be subordinate to g in E and written as $f \prec g$. In particular if g is univalent in E, the above subordination is equivalent to $f(0) = g(0)$ and f (E) $\subseteq$ g(E).

Let $p(z)$ analytic function in the unit disc E. Assume that $\psi : C^3 \times E \to C$ and $h(z)$ be univalent in E. Then $p(z)$ is said to be a solution of the differential subordination if

$$\psi\left(p(z), zp'(z), z^2 p''(z); z\right) \prec h(z) . \quad (6)$$

If $p(z) \prec q(z)$ for all $p(z)$ satisfying (6) then the univalent function $q(z)$ is called dominant of the solution of differential subordination. A dominant $\breve{q}(z)$ that satisfies $\breve{q}(z) \prec q(z)$ for all dominants $q(z)$ of (6) is said to be the best dominant.

Suppose that $\psi : C^3 \times E \to C$ and $p(z)$ be analytic and univalent in E. Let $h(z)$ be analytic in E. Then $p(z)$ is called a solution of the differential superordination if

$$h(z) \prec \psi\left(p(z), zp'(z), z^2 p''(z); z\right) . \quad (7)$$

If $q(z) \prec p(z)$ for all $p(z)$ satisfying (7) then $q(z)$ is called a subordinant of the solution of differential superordination. A univalent subordinant $\breve{q}(z)$ that satisfies $q(z) \prec \breve{q}(z)$ for all subordinants q (z) of (7) is said to be the best subordinant.

Definition 1.1 [3]:

An analytic function $f \in \mathcal{A}$ is said to be Φ -like function if there exists an analytic function Φ in a domain containing f (E), with $\Phi(0) = 0$ and $\Phi'(0) > 0$ such that

$$\Re\left(\frac{zf'(z)}{\Phi(f(z))}\right) > 0, z \in E . \quad (8)$$

Brickman [3] was the first person who introduced this concept and established that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is Φ -like for some Φ . In case, if Φ is the identity function and a rotation of λ then the function f is starlike and spiral like of $\arg(\lambda)$ respectively.

E.g. $f(z)=z$, $f(z)=\frac{z}{1-z}$ are Φ -like functions in E when $\Phi(w)=w$.

The following is the more general class of Φ -like functions introduced and studied by Ruscheweyh [14].

Definition 1.2 [14]:

Let Φ be an analytic function in a domain containing $f(E)$, with $\Phi(0)=0$, and $\Phi(w)\neq 0$ for $w\in f(E)\backslash\{0\}$. Let q be a fixed analytic function in E, with $q(0)=1$. A normalized analytic function $f\in\mathcal{A}$ is said to be Φ -like function with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))}\prec q(z),\ z\in E. \tag{9}$$

When $\Phi(w)=w$, the class of all Φ -like functions with respect to q is denoted by $S^*(q)$.

Ravichandran et al.[12,Th.2.2, p.139] have obtained sufficient condition for functions to be Φ -like with respect to q. Siregar et al. [16] introduced the new class $\Phi-H_b$ of Φ like functions of Koebe type satisfying

$$\Re\left\{\frac{zf_b'(z)}{\Phi f_b(z))}\left[1+\frac{\alpha zf_b''(z)}{f_b'(z)}+\frac{\alpha z[f_b'(z)-(\Phi(f_b(z)))']}{\Phi(f_b(z))}\right]\right\}>0, \tag{10}$$

for z $\in$ E, where

$$f_b(z):=\frac{z}{(1-z^n)^b}\ (b\geq 0; n\in\square) \tag{11}$$

Siregar et al. [16] derived sufficient condition for starlikeness of the class $\Phi-H_b$ of n-fold symmetric function of Koebe type. For the class $\mathcal{H}$, which f in (2) and $\Phi(w)=w$, Kamali and Srivastava in [6] have investigated the sufficient conditions for the starlikeness of n-fold symmetric function of Koebe type.

Motivated essentially by the above mentioned work, in this paper, we are defining a new class of Φ -like functions and obtain certain Sandwich type result that unifies some known results of starlike functions.

II. PRELIMINARIES

The following definition and lemmas are needed in proving our main results.

Definition 2.1([8], p.21, Definition 2.2b).

Let $\mathbb{Q}$ be the class of all analytic and univalent functions p(z) on $\overline{E}\setminus\mathcal{B}(p)$, where

$\mathcal{B}(p)=\left\{\xi\in\partial E: \lim_{z\to\xi} p(z)=\infty\right\}$, and are such that $p'(\xi)\neq 0$ for $\xi\in\partial E\setminus\mathcal{B}(p)$.

Lemma 2.2([7], p.132, Theorem3.4h).

Assume that $q(z)$ be univalent in the unit disc E and let η and ϕ be analytic functions in a domain $D\supset q(E)$ with $\phi(w)\neq 0$ when $w\in q(E)$. Define

$$Q(z)=zq'(z)\phi(q(z)) \quad\text{and}\quad h(z)=\eta(q(z))+Q(z) \tag{12}$$

Let one of the conditions satisfies:
1. h (z) is convex,
2. $Q(z)$ is starlike and univalent.

Also $\Re\left(\frac{zh'(z)}{Q(z)}\right)>0, z\in E$ .If p is analytic in E, with $p(0)=q(0)$, $p(E)\subseteq D$ and

$$\eta(p(z))+zp'(z)\phi(p(z))\prec\eta(q(z))+zq'(z)\phi(q(z)) \tag{13}$$

Then $p(z)\prec q(z)$ and q is the best dominant.

Lemma 2.3([4], p.28, Cor.3.1).

Suppose that q (z) be univalent in E. Let μ and ϑ be analytic in a domain $D\supset q(E)$ with $\vartheta(w)\neq 0$, when $w\in q(E)$. Define

$$Q(z)=zq'(z)\vartheta(q(z)), h(z)=\mu(q(z))+Q(z) \tag{14}$$

Assume that $Q(z)$ is starlike and univalent in E and $\Re\left(\frac{\mu'(q(z))}{\vartheta(q(z))}\right)>0$, for all $z\in E$. If $p\in\mathcal{H}[q(0), 1]\cap\mathbb{Q}$, with $p(E)\subseteq D$ and

$$\mu(q(z))+zq'(z)\vartheta(q(z))\prec\mu(p(z))+zp'(z)\vartheta(p(z)), \tag{15}$$

then $q(z)\prec p(z)$ and q is the best subordinant.

We now define the following class of functions.

Definition 2.4: Denote by $\Phi-\psi_\lambda^\alpha(z)$ the class of functions $f\in\mathcal{A}$, satisfying

$$\Re\left\{\left(\frac{zf'(z)}{\Phi(f(z))}\right)^\alpha\left[1+\frac{\lambda zf''(z)}{f'(z)}+\frac{\lambda z[f'(z)-(\Phi(f(z)))']}{\Phi(f(z))}\right]\right\}>0, \tag{16}$$

for $z\in E$,where α and λ are complex numbers . Here the power is taken with its principal value.

Remarks: We have the following inclusion relationships and known classes:

1. When $\Phi(w)=w$, $\alpha=1$, $\lambda=0$, $\Phi-\psi_0^1(z)=S^*(0)$.
2. When $\Phi(w)=w$, $\alpha=0,\lambda=1$, $\Phi-\psi_1^0(z)$ is the class of convex functions introduced by Goodman [5].
3. When $\Phi(w)=w$, $\alpha=1,\lambda=1$, $\Phi-\psi_1^1(z)\subset ST(1/2)$, studied by Ramesha et al.[11].
4. When $\Phi(w)=w$, $\alpha=1,\lambda=1$, $\Phi-\psi_1^1(z)\subset ST(\delta)$, where $\delta<1/2$, studied by Nunokawa et al.[9].
5. When $\Phi(w)=w$, $\alpha=1$, $\Phi-\psi_\lambda^1(z)\subset S^*$, studied by Kamali and Srivastava [6].
6. When $\alpha=1,\lambda=0$, $\Phi-\psi_0^1(z)$ is the class of Φ-like functions introduced by Brickman [3].
7. When $\alpha=1,\lambda\neq 0$, $\Phi-\psi_\lambda^1(z)$ is the class of functions $\Phi-H$ studied and investigated by Siregar et al. [16].

III. MAIN RESULTS

By making use of lemma 2.2, we prove the following result.

Theorem 3.1. Let $f\in\mathcal{A}$ satisfying $f(z)\neq 0(z\in E)$. Also let the function $q(z)$ be univalent in E with $q(0)=1$ and $q(z)\neq 0$ such that

$$1.\ \Re\left(1+(\alpha-1)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right)>0 \tag{17}$$

$$2.\ \Re\left(\begin{array}{l}1+(\alpha-1)\dfrac{zq'(z)}{q(z)}+\dfrac{zq''(z)}{q'(z)}+\\ \left(\dfrac{1-\lambda}{\lambda}\right)\alpha+(\alpha+1)q(z)\end{array}\right)>0 \tag{18}$$

for all $z\in E$. If $f\in\mathcal{A}$ satisfies

$$\left(\frac{zf'(z)}{\Phi(f(z))}\right)^\alpha\left[\begin{array}{l}1+\dfrac{\lambda zf''(z)}{f'(z)}+\\ \dfrac{\lambda z[f'(z)-(\Phi(f(z)))']}{\Phi(f(z))}\end{array}\right]\prec h(z) \tag{19}$$

where $h(z)=\lambda q^{\alpha+1}(z)+(1-\lambda)q^\alpha(z)+\lambda zq'(z)q^{\alpha-1}(z)$ (20)

α and λ are complex numbers such that $\lambda\neq 0$, then $\dfrac{zf'(z)}{\Phi(f(z))}\prec q(z)$ and q is the best dominant.

Proof. Let $p(z)$ be the function defined for all $z\in E$ by $p(z)=\dfrac{zf'(z)}{\Phi(f(z))}$.

(21)

Then the function $p(z)$ is analytic in E with $p(0)=1$.

Consider

$$\left(\frac{zf'(z)}{\Phi(f(z))}\right)^\alpha\left[1+\frac{\lambda zf''(z)}{f'(z)}+\frac{\lambda z[f'(z)-(\Phi(f(z)))']}{\Phi(f(z))}\right]=$$

$$(p(z))^\alpha\left[1+\lambda\left(\frac{zp'(z)}{p(z)}-1\right)+\lambda p(z)\right]$$

$$=\lambda p^{\alpha+1}(z)+(1-\lambda)p^\alpha(z)+\lambda zp'(z)p^{\alpha-1}(z) \tag{22}$$

Define the functions η and ϕ as $\eta(w)=w^\alpha(1-\lambda+\lambda w)$ and $\phi(w)=\lambda w^{\alpha-1}$ then the functions η and ϕ are analytic in a domain $D=\mathbb{C}\setminus\{0\}$ and $\varphi(w)\neq 0$, $w\in D$. By defining the functions Q and h as follows: $Q(z)=\lambda zq'(z)q^{\alpha-1}(z)$ and

$h(z)=\eta(q(z))+Q(z)$

$=\lambda q^{\alpha+1}(z)+(1-\lambda)q^\alpha(z)+\lambda zq'(z)q^{\alpha-1}(z)$.

By using equation (22) in (19), we have

$$\lambda p^{\alpha+1}(z)+(1-\lambda)p^\alpha(z)+\lambda zp'(z)p^{\alpha-1}(z)$$
$$\prec\lambda q^{\alpha+1}(z)+(1-\lambda)q^\alpha(z)+\lambda zq'(z)q^{\alpha-1}(z) \tag{23}$$

A simple computation gives

$$\frac{zQ'(z)}{Q(z)}=1+(\alpha-1)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)},$$

$$\frac{zh'(z)}{Q(z)}=1+\frac{zq''(z)}{q'(z)}+(\alpha-1)\frac{zq'(z)}{q(z)}+(\alpha+1)q(z)+\left(\frac{1-\lambda}{\lambda}\right)\alpha.$$

By the conditions 1 and 2, we have that Q is starlike in E and $\Re\left(\dfrac{zh'(z)}{Q(z)}\right)>0$, $z\in E$. Thus conditions of lemma 2.2 are satisfied. Therefore, the proof follows from lemma 2.2.

Remark3.1. By taking $\alpha=1$, we obtain [12, Th.2.1, p.139] as a special case of Theorem 3.1.

Remark3.2. By taking $\Phi(w)=w,\alpha=1$ and $q(z)=\dfrac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, in Theorem 3.1we have the result of Ravichandran and Jayamala [13].

Remark3.3. By taking $\Phi(w)=w$ we obtain [17, Th.3.1, p.32] as a special case of Theorem 3.1.

Remark3.4. If we consider the dominant $q(z)=\dfrac{1+(1-2\nu)z}{1-z}$, $0\le\nu<1$ a little calculation shows that this dominant satisfies the conditions of Theorem 3.1and for some particular choices of Φ,α,λ and ν we get the following cases.

1. When $\Phi(w)=w,\alpha=1$, in Theorem 3.1 we get the result [17, Cor.4.1, p.34] and also for $\Phi(w)=w,\alpha=1,\lambda=1$ with above $q(z)$ the result of Kwon [10] is obtained.

2. For $\Phi(w)=w, \alpha=1, \lambda=\frac{1}{2}$ in Theorem 3.1 and $\nu=0$ in $q(z)$ we obtain:

   If $f\in\mathcal{A}, \frac{zf'(z)}{f(z)}\neq 0$ in E, satisfies

   $$\left(\frac{zf'(z)}{f(z)}\right)\left[1+\frac{1}{2}\frac{zf''(z)}{f'(z)}\right]\prec\frac{1+2z}{(1-z)^2}=F(z), \text{ then } f\in S^*.$$

3. By taking $\Phi(w)=w, \alpha=-1$ we obtain [17, Cor.4.2, p.35] as a special case of Theorem 3.1.

4. By taking $\Phi(w)=w, \alpha=0$ we obtain [17, Cor.4.3, p.35] as a special case of Theorem 3.1.
5. By taking $\Phi(w)=w, \alpha=0, \lambda=1$ in Theorem 3.1and $\nu=\frac{1}{2}$ in $q(z)$ we obtain [17, Cor.4.4, p.36].

Remark 3.5. If we consider the dominant $q(z)=\frac{1+az}{1-z}, \quad -1<a\leq 1$a little calculation shows that this dominant satisfies the conditions of Theorem 3.1 and for some particular choices of Φ,α,λ we get the following cases.

a) When $\Phi(w)=w, \alpha=1, \lambda(0<\lambda\leq 1)$ a real number in Theorem 3.1 we get the result [17, Cor.4.5, p.36].

b) By taking $\Phi(w)=w, \alpha=-1, \lambda$ is a real number such that $\lambda\in(-\infty,0)\bigcup[1,\infty)$ we obtain [17, Cor.4.6, p.36] as a special case of Theorem 3.1.

c) By taking $\Phi(w)=w, \alpha=0$ and λ is a complex number we obtain [17, Cor.4.7, p.36] as a special case of Theorem 3.1.

d) For $\lambda=1$, with the above choices as in (a), (b), (c) we get the results of Singh and Gupta [15].

By making use of lemma 2.3, we obtain the following result.

Theorem 3.2. Let $f\in\mathcal{A}$ satisfying $f(0)=0$. Also let the function $h(z)$ be convex univalent in E and $h\in\mathcal{H}[q(0),1]\bigcap\mathbb{Q}$. Assume that $q, q(z)\neq 0$ be univalent in E such that

1. $$\Re\left(1+(\alpha-1)\frac{zq'(z)}{q(z)}+\frac{zq''(z)}{q'(z)}\right)>0$$

(24)

2. $$\Re\left((\alpha+1)q(z)+\left(\frac{1-\lambda}{\lambda}\right)(\alpha)\right)>0$$

(25)

for all $z\in E$. Assume that

$$\left(\frac{zf'(z)}{\Phi(f(z))}\right)^{\alpha}\left[1+\frac{\lambda zf''(z)}{f'(z)}+\frac{\lambda z[f'(z)-(\Phi(f(z)))']}{\Phi(f(z))}\right]$$ is univalent in E and satisfies the differential superordination

$$h(z)=\vartheta(q(z))+\lambda zq'(z)q^{\alpha-1}(z)$$
$$\prec\left(\frac{zf'(z)}{\Phi(f(z))}\right)^{\alpha}\left[1+\frac{\lambda zf''(z)}{f'(z)}+\frac{\lambda z[f'(z)-(\Phi(f(z)))']}{\Phi(f(z))}\right]$$

(26)

where α and λ are complex numbers such that $\lambda\neq 0$, then $q(z)\prec\frac{zf'(z)}{\Phi(f(z))}$ and q is the best subordinant.

Proof. Let $p(z)$ be the function defined for all $z\in E$ by $p(z)=\frac{zf'(z)}{\Phi(f(z))}$.

(27)

Then the function $p(z)$ is analytic in E with $p(0)=1$. Define the functions μ and ϑ as $\mu(w)=w^{\alpha}(1-\lambda+\lambda w)$ and $\vartheta(w)=\lambda w^{\alpha-1}$ then the functions μ and ϑ are analytic in a domain $D=\mathbb{C}\backslash\{0\}$ and $\vartheta(w)\neq 0$, $w\in D$.By defining the functions Q and h as follows:

$Q(z)=\lambda zq'(z)q^{\alpha-1}(z)$ and $h(z)=\mu(q(z))+Q(z)$
$=\lambda q^{\alpha+1}(z)+(1-\lambda)q^{\alpha}(z)+\lambda zq'(z)q^{\alpha-1}(z)$.

The superordination (3.26) becomes:

$$\lambda q^{\alpha+1}(z)+(1-\lambda)q^{\alpha}(z)+\lambda zq'(z)q^{\alpha-1}(z)$$
$$\prec \lambda p^{\alpha+1}(z)+(1-\lambda)p^{\alpha}(z)+\lambda zp'(z)p^{\alpha-1}(z)$$

(28)

We also observe that

$$\frac{\mu'(q(z))}{\vartheta(q(z))}=(\alpha+1)q(z)+\left(\frac{1-\lambda}{\lambda}\right)\alpha.$$

The use of lemma 2.3 along with (28) completes the proof on the same lines as in case of Theorem. 3.1.

Remark3.6. By taking $\Phi(w)=w$ in Theorem 3.2, we obtain [17, Th.3.2, p.33] as a special case.

Remark3.7. By taking $\Phi(w)=w, \alpha=1$ in Theorem 3.2, we have the result of R.M. Ali et.al. [1, Th.2.2, p.89].

Remark3.8. By taking $\Phi(w)=w, \alpha=-1, \lambda=1$ in Theorem 3.2, we obtain [1, Th.2.10, p.93] as a special case.

Combining Theorem 3.1 and Theorem 3.2 we get the following sandwich theorem.

Theorem 3.3. Let $q_i(z)\neq 0\,(i=1,2)$ be convex univalent in E such that q_1 satisfies the conditions 1 and 2 of Theorem 3.2 and q_2 follows the conditions 1 and 2 of Theorem 3.1. Define $h_i(z)(i=1,2)$ by $h_i(z)=\lambda q_i^{\alpha+1}(z)+(1-\lambda)q_i^{\alpha}(z)+\lambda z q_i'(z)q_i^{\alpha-1}(z)$ and suppose that $f\in\mathcal{A}$, satisfies

$\dfrac{zf'(z)}{\Phi(f(z))}\in\mathcal{H}[q(0),1]\cap\mathcal{Q}$ and

$\left(\dfrac{zf'(z)}{\Phi(f(z))}\right)^{\alpha}\left[1+\dfrac{\lambda zf''(z)}{f'(z)}+\dfrac{\lambda z[f'(z)-(\Phi(f(z)))']}{\Phi(f(z))}\right]$ is univalent in E ,where α and λ are complex numbers such that $\lambda\neq 0$ then

$$h_1(z)\prec\left(\frac{zf'(z)}{\Phi(f(z))}\right)^{\alpha}\left[\begin{array}{c}1+\dfrac{\lambda zf''(z)}{f'(z)}+\\ \dfrac{\lambda z[f'(z)-(\Phi(f(z)))']}{\Phi(f(z))}\end{array}\right]\prec h_2(z)$$

(29)

implies $q_1(z)\prec\dfrac{zf'(z)}{\Phi(f(z))}\prec q_2(z)$ .Further q_1,q_2 are respectively the best subordinant and best dominant.

Remark3.9. By taking $\Phi(w)=w,\alpha=1$ in Theorem 3.3, we obtain [1, Cor.2.3, p.90] as a special case.

Remark3.10. By taking $\Phi(w)=w,\alpha=-1,\lambda=1$ in Theorem 3.3, we obtain [1, Th.2.11, p.93] as a special case.

Remark3.11. By taking $\alpha=1$ in Theorem 3.3 we obtain [2, Th.4.2, p.8] as a special case.

Remark3.12. By taking $\Phi(w)=w$ in Theorem 3.3, we obtain [17, Th.3.3, p.34] as a special case.

Remark3.13. For the selection of $q_1(z)=1+az$ and $q_2(z)=1+bz$, $0<a<b$, in Theorem 3.3 and for some parti- cular choices of Φ , α,λ we obtain the results discussed in [17, Cor.5.1, p.37;Cor.5.2,p.38; Cor.5.3,p.38].

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