On Quasi Generalized S Topological Simple Groups

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Abstract: *In this paper we introduce and study the concept of Quasi generalized S topological simple groups. The notion of Quasi generalized S topological simple group and the property of left(right) translation and inversion mapping are discussed and studied.*

Key words: *Quasi S-topological group, $\mathcal{G}$-semi open, $\mathcal{G}$-semi closed, $\mathcal{G}$-semi continous, $\mathcal{G}$-homeomorphism, $\mathcal{G}$-homogeneous, Quasi $\mathcal{G}$-S topological simple group.*

1. INTRODUCTION

The notion of generalized topology was introduced by Csaszar in [2]. Generalized topology is denoted by $\mathcal{G}$-topology[3]. Moiz ud din khan et.al[1] introduced and studied the concept of quasi S-topological group. This motivate us to introduce the notions of quasi generalized S-topological simple group. In this paper quasi generalized S topological simple group is denoted by quasi $\mathcal{G}$-S topological simple group

In this paper we discuss some theorems related to quasi $\mathcal{G}$-S topological simple group. Throught this paper generalized semi open, generalized semi closed, generalized semi closure, generalized semi continous etc. are denoted by $\mathcal{G}$- semi open, $\mathcal{G}$-semi closed, $\mathcal{G}$-semi interior, $\mathcal{G}$-semi closure, $\mathcal{G}$-semi continous etc. respectively.

2. PRELIMINARIES

Definition: 2.1[2] Let X be any set and let $\mathcal{G} \subseteq$ P(X) be a subfamily of power set of X. Then $\mathcal{G}$ is called a generalized topology if $\phi \in \mathcal{G}$ and for any index set I, $\bigcup_{i\in I} O_i \in \mathcal{G}$, $O_i \in \mathcal{G}$, i ∈ I.

Definition: 2.2[3] Let X be generalized topological space and $A \subseteq X$. Then A is said to be generalized semi open if $A \subseteq cl(int(A))$ or equivalently if there exists an $\mathcal{G}$-open set U in X such that $U \subseteq A \subseteq cl(A)$.

$SO(X)$ denotes the collection of all generalized semi open sets in X. $SO(X,x)$ is the collection of all generalized semi open sets in X containing $x \in X$.

The complement of a generalized semi open set is said to be generalized semi-closed.

Definition: 2.3[3] Let X be generalized topological space and $A \subseteq X$. Then the generalized semi closure of A , is the intersection of all generalized semi closed set of X containing A.

Note: 2.4[3] Let X be generalized topological space and $A \subseteq X$. Then

(i). $x \in scl(A) \Leftrightarrow$ for any generalized semi open set U containing x, $U \cap A \neq \phi$.

(ii). Every generalized open(generalized closed) is generalized semi-open(generalized semi-closed).

(iii). The union of any collection of generalized semi-open set is again a generalized semi-open set.

(iv). The intersection of two generalized semi-open sets need not be generalized semi-open.

Definition: 2.5[3] Let X be generalized topological space. A set $U \subseteq X$ is generalized semi neighbourhood of a point $x \in X$ if there exists $A \in SO(X)$ such that $x \in A \subseteq U$.

Note: 2.6[3] Let X be generalized topological space. Then a set $A \subseteq X$ is generalized semi open in $X \Leftrightarrow A$ is a generalized semi neighbourhood of each of its points.

Definition: 2.7 Let X and Y be two generalized topological spaces. A mapping $f: X \to Y$ is called generalized semi-continous if for each $\mathcal{G}$-open set $V \subseteq Y$, the set $f^{-1}(V)$ is generalized semi open in X.

Equivalently, the mapping f is generalized semi continous if for each $x \in X$ and for each $\mathcal{G}$-open neighbourhood V of $f(x)$, there exists a generalized semi open neighbourhood U of x such that $f(U) \subseteq V$.

Definition: 2.8 Let X and Y be two generalized topological spaces. A mapping $f: X \to Y$ is called irresolute if each generalized semi open set $V \subseteq Y$, the set $f^{-1}(V)$ is generalized semi open in X.

Definition: 2.9 Let X and Y be two generalized topological spaces. A mapping $f: X \to Y$ is called a generalized semi open mapping if every $\mathcal{G}$-open subset A of X, $f(A)$ is generalized semi open in Y.

Definition: 2.10[9] Let G be any group. Given $x \in G$,a mapping $l_x: G \to G$ by $l_x(a) = x * a, a \in G$ is called a left translation.

Definition: 2.11[9] Let G be any group. Given $x \in G$,a mapping $r_x: G \to G$ by $r_x(a) = a * x, a \in G$ is called a right translation.

Definition: 2.12[19] A group G is called a simple group if it has no nontrivial normal subgroup.

Definition: 2.13[4] A quasi $\mathcal{G}$-topological simple group G is a simple group which is also $\mathcal{G}$-topological space if the following conditions are satisfied.

*(i).*The left translation $l_x: G \to G$ by $l_x(a) = x * a, a \in G$ is $\mathcal{G}$-continous.

*(ii).*The right translation $r_x: G \to G$ by $r_x(a) = a * x, a \in G$ is $\mathcal{G}$-continous.

*(iii).*The inverse mapping $i: G \to G$ by $i(x) = x^{-1}, x \in G$ is $\mathcal{G}$-continous.

3. QUASI $\mathcal{G}$-S TOPOLOGICAL SIMPLE GROUP

Definition: 3.1 A quasi $\mathcal{G}$-S topological simple group G, is a simple group which is also a $\mathcal{G}$-topological space if the following conditions are satisfied.

*(i).*The left translation $L_x: G \to G$ by $L_x(a) = x * a, a \in G$ is $\mathcal{G}$-semi continous.

*(ii).*The right translation $R_x: G \to G$ by $R_x(a) = a * x, a \in G$ is $\mathcal{G}$-semi continous.

*(iii).*The inverse mapping $i: G \to G$ by $i(x) = x^{-1}, x \in G$ is $\mathcal{G}$-semi continous.

Example: 3.2 Any simple group with the indiscrete or discrete $\mathcal{G}$-topology, is a quasi $\mathcal{G}$-topological simple group.

Example: 3.3 $G = \{-1,1\}$ is a simple group under multiplication. Then we define a generalized topology on G by $\mathcal{G} = \{\varphi, G, \{1\}\}$. Then G is neither quasi $\mathcal{G}$-topological simple group nor quasi $\mathcal{G}$-S topological simple group

Theorem: 3.4 Let $(G, *, \mathcal{G})$ be a quasi $\mathcal{G}$-S topological simple group and β_e be the collection of all semi open neighbourhood at identity e of G. Then

(i). For every $U \in \beta_e$, there is an element $V \in SO(G, e)$ such that $V^{-1} \subseteq U$.

(ii). For every $U \in \beta_e$, there is an element $V \in SO(G, e)$ such that $V * x \subseteq U$ and $x * V \subseteq U$, for each $x \in U$.

Proof: (i). Since $(G, *, \mathcal{G})$ is a quasi $\mathcal{G}$-S topological simple group. Therefore, for every $U \in \beta_e$, there exists $V \in SO(G, e)$ such that $i(V) = V^{-1} \subseteq U$, because the inverse mapping $i: G \to G$ is $\mathcal{G}$-semi continous.

(ii). Since $(G, *, \mathcal{G})$ is a quasi $\mathcal{G}$-S topological simple group. Thus for each $\mathcal{G}$-open set U containing x, there exists $V \in SO(G, e)$ such that $R_x(V) = V * x \subseteq U$. Similarly, $L_x(V) = x * V \subseteq U$.

Theorem: 3.5 Let A be a subset of a quasi $\mathcal{G}$-S topological simple group $(G, *, \mathcal{G})$. Then

$(scl(A))^{-1} \subseteq cl(A^{-1})$.

Proof: Let $x \in (scl(A))^{-1}$. Let U be an $\mathcal{G}$-open neighbourhood of x. Then U^{-1} is a $\mathcal{G}$-semi open neighbourhood of x^{-1}. Since $x^{-1} \in scl(A)$, $U^{-1} \cap A \neq \phi$.

$\Rightarrow U \cap A^{-1} \neq \phi$.

$\Rightarrow x \in cl(A^{-1})$.

$\Rightarrow (scl(A))^{-1} \subseteq cl(A^{-1})$.

Remark: 3.6 (i). $(G, *, \mathcal{G})$ quasi $\mathcal{G}$-S topological simple group. Then $\mathcal{G}^{-1} = \{A \subseteq G : A^{-1} \in \mathcal{G}\}$.[1]

(ii). $\mathcal{G}$ is topology on $G \Rightarrow \mathcal{G}^{-1}$ is also a topology on G.[1]

Theorem: 3.7 Let $(G, \mathcal{G})$ be a quasi $\mathcal{G}$-S topological simple group. If U is $\mathcal{G}$-semi open set in $(G, \mathcal{G})$, then U^{-1} $\mathcal{G}$-semi open set in $(G, \mathcal{G}^{-1})$.

Proof: We know that $\mathcal{G}$ is a generalized topology. Then $\mathcal{G}^{-1}$ is also a generalized topology. Let $U \in SO(G, \mathcal{G})$. Then there exists an $\mathcal{G}$-open set $O \in \mathcal{G}$ such that,

$O \subseteq U \subseteq cl(O)$ (or)

$\Rightarrow O^{-1} \subseteq U^{-1} \subseteq (cl(O))^{-1}$.

$\Rightarrow O^{-1} \subseteq U^{-1} \subseteq cl(O)^{-1}$.

Now $O^{-1} \in \mathcal{G}^{-1}$.

$\Rightarrow U^{-1} \in SO(G, \mathcal{G}^{-1})$.

Hence U^{-1} is $\mathcal{G}$- semi open in $(G, \mathcal{G}^{-1})$.

Theorem: 3.8 Let $(G, *, \mathcal{G})$ be a quasi $\mathcal{G}$-S topological simple group. If A is $\mathcal{G}$-open in G, then $A * B$ and $B * A$ are $\mathcal{G}$-semi open in $(G, *, \mathcal{G})$ for any subset B of G.

Proof: Let $x \in B$ and $z \in A * x$. We show that z is a $\mathcal{G}$- semi interior point of $A * x$.

Let $z = y * x$ for some $y \in A = A * x * x^{-1}$.

$\Rightarrow y = z * x^{-1}$.

Now $R_{x^{-1}}: G \to G$ is $\mathcal{G}$-semi continous. That is, for every $\mathcal{G}$-open set containing $R_{x^{-1}}(z) = z * x^{-1} = y$, there exists a $\mathcal{G}$-semi open set in M_z containing z such that $R_{x^{-1}}(M_z) \subseteq A$.

$\Rightarrow M_z * x^{-1} \subseteq A$.

$\Rightarrow M_z \subseteq A * x$.

$\Rightarrow z$ is $\mathcal{G}$-semi interior point of $A * x$. Thus $A * x$ is $\mathcal{G}$-semi open. This implies,

$A * B = \cup_{x \in B} A * x$ is $\mathcal{G}$-semi open in $(G, *, \mathcal{G})$. Simillarly we can prove that for every $\mathcal{G}$-open set A of G, $B * A$ is $\mathcal{G}$-semi open in a quasi $\mathcal{G}$-S topological simple groups.

4. QUASI $\mathcal{G}$-S HOMEOMORPHISM

Definition: 4.1 A bijective mapping $f: (G, \mathcal{G}_G) \to (H, \mathcal{G}_H)$ is called quasi $\mathcal{G}$-S homeomorphism if it is

$\mathcal{G}$-semi continous and $\mathcal{G}$-semi open.

Theorem: 4.2 Let $(G, *, \mathcal{G})$ be a quasi $\mathcal{G}$-S topological simple group. Then each left(right) translation $L_x: G \to G (R_x: G \to G)$ is a quasi $\mathcal{G}$-S homeomorphism.

Proof: Since $(G, *, \mathcal{G})$ is a quasi $\mathcal{G}$-S topological simple group. Therefore $L_x: G \to G$ is $\mathcal{G}$-semi continous by definition, so it is enough to show that $L_x: G \to G$ is $\mathcal{G}$-semi open. Let V be a $\mathcal{G}$-open set in G. Then by theorem 3.8, $L_g(V) = g * V \in SO(G)$. Hence $L_x: G \to G$ is a $\mathcal{G}$-semi open mapping. Similarly we can prove for right translation(R_x).

Theorem: 4.3 Suppose that a subgroup H of a quasi $\mathcal{G}$-S topological simple group $(G, *, \mathcal{G})$ contains a non empty $\mathcal{G}$-open subset of G. Then H is $\mathcal{G}$-semi open in G.

Proof: By above theorem, for every $g \in H$, $R_x: G \to G$ is quasi $\mathcal{G}$-S homeomorphism. Let U be a non-empty $\mathcal{G}$-open subset of G, with $U \subseteq H$, then for every $g \in H$, then set $R_g(U) = U * g$ is $\mathcal{G}$-semi open in $(G, *, \mathcal{G})$. Now $H = \bigcup\{U * g : g \in H\}$ is $\mathcal{G}$-semi open in G being union of $\mathcal{G}$-semi open sets of G.

5. QUASI $\mathcal{G}$-S HOMOGENEOUS

Definition: 5.1 A $\mathcal{G}$-topological space $(G, \mathcal{G})$ is said to be quasi $\mathcal{G}$-S homogeneous if for all $x, y \in G$, there is a quasi $\mathcal{G}$-S homeomorphism f of the space G onto itself such that $f(x) = y$.

Theorem: 5.2 If $(G, *, \mathcal{G})$ is a quasi $\mathcal{G}$-S topological simple group, then every $\mathcal{G}$-open subgroup of G is also $\mathcal{G}$-semi closed.

Proof: Since $(G, *, \mathcal{G})$ is a quasi $\mathcal{G}$-S topological simple group and H is an $\mathcal{G}$-open subgroup of G. Then any left or right translation, $x * H$ or $H * x$ is $\mathcal{G}$-semi open for each $x \in G$. So $Y = \{x * H : x \in G\}$ of all left cosets of H in G forms a partition of G. This gives $G - H$ is union of $\mathcal{G}$-semi open sets and hence $\mathcal{G}$-semi open. This proves that H is $\mathcal{G}$-semi closed.

Corollary: 5.3 Every quasi $\mathcal{G}$-S topological simple group is a quasi S-homogeneous space.

Proof: Let us take elements x and y in $(G, *, \mathcal{G})$ and put $z = x^{-1} * y$. Since $R_x: G \to G$ is a quasi $\mathcal{G}$-S homeomorphism of $(G, *, \mathcal{G})$ and,

$$\begin{aligned} R_z(x) &= x * z \\ &= x * (x^{-1} * y) \\ &= e * y \\ &= y. \end{aligned}$$

Hence $(G, *, \mathcal{G})$ is quasi $\mathcal{G}$-S homogeneous space.

Theorem: 5.4 Let $f: (G,*,\mathcal{G}_G) \to (H,*,\mathcal{G}_H)$ be a homomorphism of quasi $\mathcal{G}$-S topological simple groups. If f is irresolute at the neutral(identity) element e_G, then f is $\mathcal{G}$-semi continous on G.

Proof: Let $x \in G$ be an arbitrary element. Suppose that W is an $\mathcal{G}$-open neighbourhood of $y = f(x) \in H$. Since the left translation in H is a $\mathcal{G}$-semi continous, there is a $\mathcal{G}$-semi open neighbourhood V of the neutral element e_H of H such that $L_y(V) = y * V \subseteq W$. Since f is irresolute at e_G, $f(U) \subseteq V$, for some $\mathcal{G}$-semi open neighbourhood U of e_G in G.

Now $y * f(U) \subseteq y * V \subseteq W$.

$\Rightarrow f(x) * f(U) \subseteq W$.

$\Rightarrow f(x * U) \subseteq W$.

Since $(G,*,\mathcal{G}_G)$ is a quasi $\mathcal{G}$-S topological simple group, therefore $x * U$ is $\mathcal{G}$-semi open in G. This proves that f is $\mathcal{G}$-semi continous at x. Since x was an arbitrary element of G. This completes the proof.

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