LA-Noetherian in a Generalized LA-Ring

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Abstract

The present study introduces the notion of an LA-Noetherian in an LA-ring and a generalized LA-ring. Moreover, it extends the notion of ideal in an nLA-ring and LA-module over LA-ring and its substructure to LA-Noetherian.

Keywords: LA-ring, LA-modules, LA-submodules, nLA-ring, Ideals.

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1 <u>*Introduction*</u>

The concept of Right Almost semigroup, Left Almost semigroup and Almost semigroup was introduced by M. A. Kazim and M. Naseeruddin [2] in 1972. A groupoid G is said to be a Right Almost semigroup (abbreviated as an RA-semigroup) if it satisfies the right invertible law i.e.,

$$a * (b * c) = c * (b * a); \forall a, b, c \in G. \tag{1}$$

Similarly, the concept of Left Almost semigroup (abbreviated as an LA-semigroup) is thought to be a groupoid G if it satisfies the left invertible law i.e.,

$$(a * b) * c = (c * b) * a; \forall a, b, c \in G. \tag{2}$$

The concept of Almost semigroup could not be so unless it satisfies both (1) and (2). But the theory footed parallel to both RA-semigroup and LA-semigroup. So, M. A. Kazim and M. Naseeruddin [2] deal with LA-semigroup. An LA-semigroup generalizes the notion of a commutative semigroup. The notion of LA-semigroup is also known as Abel-Grassmann's groupoid [4] (abbreviated as an AG-groupoid). Furthermore, Q. Mushtaq and M. S. Kamran [3] extended the concept of LA-semigroup to Left Almost group (abbreviated as an LA-group), which are non-associative structures, defining LA-semigroup in such a way that $\exists\ e \in G$ such that $ea = a$, $\forall a \in G$ and for every $a \in G$, $\exists\ a^{-1} \in G$ such that $a^{-1}a = e$. The notion of LA-group is the generalization of a commutative group. S. M. Yusuf [8] extended the non-associative structure with two binary operations $'+'$ and $'.'$. If a non-empty set R together with binary operations $'+'$ and $'.'$ are satisfying the following conditions:-

(1) $(R, +)$ is an LA-group,

(2) $(R, .)$ is an LA-semigroup and

(3) both left and right distributive law holds.

Then, $(R, +, .)$ is called an LA-ring.

T. Shah and I. Rahman [7] introduced the notion of LA-module over an LA-ring and further T. Shah, M. Raees and G. Ali [5] discussed sub-structures of LA-module and also non-similarity of an LA-module to the usual frame of module. The structure introduced by S. M. Yusuf [8] was strengthened by T. Shah and I. Rahman [7] as a generalization of commutative semigroup rings.

T. Shah, Fazal ur Rehman and M. Raees [6] generalized the concept of an LA-ring into a near left almost ring (abbreviated as an nLA-ring). In the same paper they also introduced the concept of ideal, factor nLA-ring, nLA-ring homomorphism, nLA-integral domain and near almost field. Due to structural properties of nLA-ring, it behaves like a commutative ring and a commutative near ring. However, the aim of the present paper is to introduce the notion of Left Almost-Noetherian (abbreviated as an LA-Noetherian) and further extend the notion of ideal defined in an nLA-ring and LA-module over LA-ring and its substructure to LA-Noetherian.

2 *Left Almost Noetherian*

This section introduces the notion of Left Almost Noetherian, Right Almost Noetherian and Almost Noetherian. The ideas are similar to an LA-group and an nLA-ring. Further, it gives some significant theorems and propositions [1] of an LA-Noetherian.

Definition 1. *An nLA-ring R is Left Almost Noetherian (abbreviated as an LA-Noetherian) if it satisfies the ascending chain of left ideals in R is stable. An nLA-ring R is Right Almost Noetherian (abbreviated as an RA-Noetherian) if it satisfies the ascending chain of right ideals in R is stable. An nLA-ring R is Almost-Noetherian if it is both LA-Noetherian and RA-Noetherian.*

Definition 2. *A free LA-module R is one which is isomorphic to an LA-module R of the* *form $\oplus_{i\in I}M_i$, where each $M_i \cong R$. A finitely generated free LA-module R is therefore isomorphic to $R \oplus R \oplus ... \oplus R$, which is denoted by R^n.*

Definition 3. *An nLA-ring R is an LA-Noetherian if it satisfies the following three equivalent conditions:*
(i) Maximal Condition:- Every non-empty set of left ideals in R has a maximal element.
(ii) Ascending Chain Condition:- Every ascending chain of left ideals in R is stationary.
(iii) Finitely Generated Condition:- Every left ideal in R is finitely generated.

The theory runs completely parallel to both left ideals and right ideals in an nLA-ring R. Throughout this paper, consider left ideals in an nLA-ring R. One can obtain RA-Noetherian by replacing left ideals to right ideals in an nLA-ring R.

Theorem 1. *Let M be an LA-module over an LA-ring R. Then the following are equivalent:*
(i) M is an LA-Noetherian over an LA-module R.
(ii) Every non-empty set of LA-submodules of M contains a maximal element.
(iii) Every LA-submodules of M is finitely generated.

Proof. $(i) \Rightarrow (ii)$. Let Σ be any non-empty collection of LA-submodules of M. If we choose any $M_1 \in \Sigma$. If M_1 is a maximal element in Σ, $\Rightarrow (ii)$. Assume that M_1 is not maximal element. So there is some $M_2 \in \Sigma$ such that $M_1 \subset M_2$. If M_2 is maximal element in $\Sigma \Rightarrow (ii)$, if not, we may assume there is an $M_3 \in \Sigma$ such that $M_2 \subset M_3$. Proceeding in this way, we see that if (ii) fails, we have an infinite strictly increasing chain of elements of Σ, which contradict (i).
$(ii) \Rightarrow (iii)$. Suppose that (ii) holds and let M_1 be any LA-submodule of M. Let $\Sigma = \{finitely\ generated\ LA-submodules\ of\ M_1\}$. Since $\{0\} \in \Sigma$, so Σ is non-empty. By (ii), Σ has a maximal element say M_2. If $M_2 \neq M_1$, let $y \in M_1 - M_2$. Since $M_2 \in \Sigma$, the LA-submodule by assumption, M_2 is finitely generated, hence the LA-submodule generated by M_2 and y is finitely generated. This contradict the maximality of M_2. Hence $M_1 = M_2$ is finitely generated.
$(iii) \Rightarrow (i)$. Suppose that (iii) holds and let $M_1 \subseteq M_2 \subseteq M_3$ ... be a chain of an LA-submodules of M. Let $N = \cup_{i=1}M_i$, here N is an LA-submodule of M. By (iii), $N =< x_1, x_2, ..., x_n >$. Since $x_i \in N$, $\forall i$, each x_i belongs to one of the LA-submodules in the chain. Let $l = Max\{1, 2, ..., n\}$. Then $x_i \in M_l$, $\forall i$. Therefore the LA-module they generate is contained in M_l, i.e., $N \subseteq M_l$. Thus $M_l = N = M_k$, $\forall k \geq l$, which implies (i). □

Corollary 1: A finitely generated LA-module over an LA-Noetherian is an LA-Noetherian.

Proposition 1. If I is a left ideal of an LA-Noetherian in an nLA-ring R, then the factor R/I is an LA-Noetherian.

Proposition 2. If M is an LA-Noetherian and φ is a homomorphism of M onto an nLA-ring R, then M is an LA-Noetherian.

Corollary 2: Any homomorphic image of an LA-Noetherian is an LA-Noetherian.

Theorem 2. *The following three statements are equivalent:*
(i) nLA-ring R is an LA-Noetherian.
(ii) Every non-empty set of left ideals of R has a maximal element.
(iii) Every left ideal of R is finitely generated.

Proof. The proof is similar to that of **Theorem 1**. □

Theorem 3. *Let S be an nLA-subring of an nLA-ring R. Suppose that S is an LA-Noetherian and R is finitely generated as an LA-module S, then R is an LA-Noetherian.*

Proof. By ***Corollary 1***, R is an LA-Noetherian LA-module S and all left ideals of R are also an LA-submodules S of R. Since LA-submodules S satisfy the ascending chain condition, so left ideals of R. This complete the proof. □

Theorem 4. *If R is an LA-Noetherian and S is any multiplicatively closed subset of R, then $S^{-1}R$ is an LA-Noetherian.*

Proof. Let I be any left ideal of nLA-ring R. But R is an LA-Noetherian, so

$$I =< x_1, x_2, ..., x_n > for\ some\ x_1, x_2, ..., x_n \in R.$$

It is clear that $S^{-1}I$ is generated by $x_1/1, x_2/1, ..., x_n/1$. Thus all left ideals of $S^{-1}R$ are finitely generated. Hence $S^{-1}R$ is an LA-Noetherian. □

Theorem 5. *If R is an LA-Noetherian. Then the polynomial ring $R[x]$ is an LA-Noetherian.*

Proof. Let I be a left ideal in $R[x]$ and H ={leading coefficients of elements in I}. It is easy to show that H is a left ideal of R, so

$$H =< a_1, a_2, a_3, ..., a_n > for\ some\ a_1, a_2, ..., a_n \in R, \tag{3}$$

since R is an LA-Noetherian.
For each $i = 1, 2, ..., n$, let f_i be an element of I whose leading coefficient is a_i and let K be maximum of the degrees of $f_1, f_2, ..., f_n$. For each $d \in \{0, 1, 2, ..., K - 1\}$ and let

$$H_d = \{\text{leading coefficients of polynomials in } I \text{ of degree d together with } 0\}.$$

Clearly H_d is an ideal in R, so

$$H_d =< b_{d,1}, b_{d,2}, b_{d,3}, ..., b_{d,n_d} > for\ some\ b_{d,1}, b_{d,2}, b_{d,3}, ..., b_{d,n_d} \in R, \tag{4}$$

since R is an LA-Noetherian. Let $f_{d,i}$ be a polynomial in I of degree d with leading coefficients $b_{d,i}$.
We claim that

$$I =< \{f_1, f_2, ..., f_n\} \cup \{f_{d,i} |\ 0 \leq d \leq K-1, 1 \leq i \leq n_d\} > .$$

Clearly, I is finitely generated. As I was arbitrary, it follows that $R[x]$ is an LA-Noetherian. By our construction, the ideal $I^{'} \subseteq I$ on the right above, since all the generators were chosen in I. If suppose $I \neq I^{'}$, then $\exists$ a minimum degree polynomial $f \in I$ such that $f \notin I^{'}$. Let l be the degree and a be the leading coefficient of polynomial f. Now we have the possibility, the degree of polynomial f is greater than $K-1$, i.e., $l > K-1$. Since $a \in H$ we may write, by (3)

$$a = \sum_{i=1}^{n} r_i a_i.$$

If f_i has degree K_i then $r_i x^{l-K_i} f_i$ is an element of $I^{'}$ of degree l with leading coefficient $r_i a_i$. It follows that $f - \sum_{i=1}^{n} r_i x^{l-K_i} f_i \in I - I^{'}$ of degree strictly less than l. This contradict the minimality of polynomial f. Thus it implies that degree of polynomial $f \leq K-1$. $a \in H_d$ for some $d \leq K-1$. By (4) we may write $a = \sum r_i b_{d,i}$ for some $r_i \in R$. It implies $g = \sum r_i f_{d,i}$ is a polynomial in $I^{'}$ of degree d with leading coefficient a. Thus $f - g \in I - I^{'}$ and degree of polynomial $f-g$ is strictly less than l, again contradict the minimality of polynomial f. Hence $R[x]$ is an LA-Noetherian. □

Corollary 3: If R is an LA-Noetherian, so is $R[x_1, x_2, ..., x_n]$ for every $n \geq 1$.

Corollary 4: A near almost field (abbreviated as n-almost field) is always an LA-Noetherian.

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