New Note on Maximal E-Open and E-Semi-Maximal Open Sets in Topological Spaces

Bishnupada Debnath
[#] Research Scholar, Department of Mathematics, Tripura University, Suryamaninagar, Agartala, Pin-799022, Tripura, India.

Abstract— *E. Ekici [8] introduced e-open (resp. e-closed) sets in general topology. Thereafter Nakaoka and Oda ([1] and [2]) initiated the notion of maximal open (resp. minimal closed) sets in topological spaces. In the present work, the author introduces new classes of open and closed sets called maximal e-open sets, minimal e-closed sets, e-semi maximal open and e-semi minimal closed and investigate some of their fundamental properties with example and counter examples.*

Keywords — *δ-open, θ-open, maximal e-open, minimal e-closed, e-semi maximal open and e-semi minimal closed sets.*

I. INTRODUCTION

The notion of generalized closed sets and its dual open sets was introduced by Norman Levine [5] in topological spaces. After him many authors concentrated in this directions and defined various types of generalized closed sets in that spaces. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open sets. One of the most well-known notions and also an inspiration source is the notion of δ-open and θ-open sets introduced by N.V.Velicko [3] in 1968. Since the collection of θ-open sets in a topological space (X,τ) forms a topology τ_θ on X, then the union of two θ-open sets is of course θ-open. Moreover $\tau = \tau_\theta$ if and only if (X,τ) is regular. F. Nakaoka and N. Oda in [1] and [2] introduced the notion of maximal open sets and minimal closed sets. Thereafter, E. Ekici in [8] introduced e-open and e-closed sets. The main purpose of the present paper is to introduce the concept of a new class of open sets called maximal e-open sets, minimal e-closed sets, e-semi maximal open and e-semi minimal closed sets. We also investigate some of their fundamental properties.

II. PRELIMINERY NOTE

Throughout the work ordered pairs (X,τ) and (Y,σ) (or X and Y) will denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X. We denote the interior and the closure of a set A by Int(A) and Cl(A), respectively. Let us recall the following definitions which are useful in the sequel:

Definition 2.1 [3]. Let A be a subset of a space X. A point $x \in X$ is called a θ-cluster point of A if $A \cap Cl(U) \neq \varphi$, for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure of A, denoted by $Cl_\theta(A)$.

Definition 2.2 [3]. A subset A of X is called θ-closed if $A = Cl_\theta(A)$. The complement of a θ-closed set is called θ-open set in X. We denote the collection of all θ-open (respectively, θ-closed) sets by θ-O(X,τ) (respectively, θ-C(X,τ)).

Definition 2.3 [3]. Let A be a subset of a space X. A point $x \in X$ is called a δ-cluster point of A if $A \cap U \neq \varphi$, for every regular open set U of X containing x. The set of all δ-cluster points of A is called the δ-closure of A, denoted by $Cl_\delta(A)$.

Definition 2.4 [3]. A subset A of X is called δ-closed if $A = Cl_\delta(A)$. The complement of a δ-closed set is called δ-open set in X.

We denote the collection of all δ-open (respectively, δ-closed) sets by δ-O(X,τ) (respectively, δ-C(X,τ)).

It is very well-known that the families of all δ-open (resp. θ-open) subsets of (X,τ) are topologies on X which we shall denote by τ_δ (resp.τ_θ). From the definitions it follows immediately that $\tau_\theta \subseteq \tau_\delta \subseteq \tau$. The space (X, τ_δ) is also called the semi-regularization of (X,τ). A space (X,τ) is said to be semi-regular if $\tau_\delta = \tau$ and (X,τ) is regular iff $\tau_\theta = \tau$. It is easily seen that one always has, $A \subseteq Cl(A) \subseteq Cl_\delta(A) \subseteq Cl_\theta(A) \subseteq \overline{A}^\theta$, where, $\overline{A}^\theta$ denotes the closure of A with respect to (X, τ_θ).

Definition 2.5 [1]. A proper nonempty open set U of X is said to be a maximal open set if any open set which contains U is either X or U.

Definition 2.6 [2]. A proper nonempty closed set V of X is said to be a minimal closed set if any closed set contained in V is either φ or V.

The family of all maximal open (resp. minimal closed) sets will be denoted by $M_aO(X)$ (resp. $M_iC(X)$). We define $M_aO(X, x) = \{U : x \in U \in M_aO(X)\}$ and $M_iC(X,x) = \{V : x \in V \in M_iC(X)\}$.

Definition 2.7[6]. A subset A of a topological space (X,τ), is called δ-preopen [2] if $A \subset Int(\delta\text{-}Cl(A))$. The complement of δ-preopen is δ-pre closed set in X.

Definition 2.8[7]. A subset A of a topological space (X,τ), is called δ-semiopen [2] if $A \subset Cl(\delta\text{-}Int(A))$. The complement of δ-semiopen is δ-semi closed set in X.

Definition 2.9[8]. A subset A of a topological space (X,τ), is called e-open [2] if $A \subset Cl(\delta\text{-}Int(A)) \cup Int(\delta\text{-}Cl(A))$. The complement of e-open is e-closed set in X.

3. MAXIMAL e-OPEN SETS AND MINIMAL e-CLOSED SETS

In this section we introduce the notion of maximal e-open set and minimal e-closed sets and investigate some fundamental results with example and counter examples.

Definition 3.1. A proper nonempty e-open set A of X is said to be a maximal e-open set if any e-open set U which contains A is either X or A (i.e. A= U or U =X, whenever, $A \subset U$).

Definition 3.2. A proper nonempty e-closed set B of X is said to be a minimal e-closed set if any e-closed set which is contained in B is either φ or B (i.e. A= U or U = φ, whenever, $A \subset U$).

The family of all maximal e-open (resp. minimal e-closed) sets will be denoted by M_ae-O(X) (resp. M_ie-C(X)). We denote M_ae-O(X, x) = {A : x ∈ A ∈ M_a e-O(X)} and M_i e-C(X, x) = {F : x ∈ F ∈ M_i e-C(X)}.

Theorem 3.3. Let A be a proper nonempty subset A of X. Then A is a maximal e-open set if and only if X\A is a minimal e-closed set.

Proof: Let A be a maximal e-open set. Then $A \subset X$ or $A \subset A$. Hence, $\varphi \subset X\backslash A$ or $X\backslash A \subset X\backslash A$: Therefore by Definition 3.2, X\A is a minimal e-closed set.

Conversely, let X\A be a minimal e-closed set. Then, $\varphi \subset X\backslash A$ or $X\backslash A \subset X\backslash A$. Hence $A \subset X$ or $A \subset A$ which implies that A is a maximal e-open set.

Remark 3.4. The following example shows that **(1)** maximal-open sets and maximal e-open sets are independent. **(2)** Maximal δ-open sets and maximal e-open sets are independent. **(3)** Maximal open sets and maximal δ-open sets are independent.

i,e. Maximal δ-Open sets ↮ Maximal Open sets

⊬ Maximal e-Open sets ⊬

Example 3.5. Let us consider the topological space (X,τ) such that X = {a, b, c, d} and τ = {X,φ, {c}, {c,d}, {a, b}, {a, b,c}}.Throughout careful computation we find that δO(X,τ) = {X,φ, {c,d}, {a, b}}; e-O(X,τ) = {X,φ, {a},{b},{c},{d},{c,d},{a,c}, {a,d}, {b,c}, {b,d}, {a, b}, {a,b,c}, {a,c,d},{a,b,d},{b,c,d}}.

Clearly, **(1)** {c,d} is maximal open which is not maximal e-open and {a,b,d} is maximal e-open which is not maximal open set in X.

(2) {a,b} is maximal δ-open which is not maximal e-open and {b,c,d} is maximal e-open which is not maximal δ-open set in X.

(3) {a,b,c} is maximal open which is not maximal δ-open and {a,b} is maximal δ-open which is not maximal open set in X.

Theorem 3.6. In any topological space $(X,\boldsymbol{\tau})$, if A be a maximal e-open set and B be a e-open set of $(X,\boldsymbol{\tau})$. Then either $A \cup B = X$ or $B \subset A$.

Proof. Let A be a maximal e-open set and B be a e-open set of $(X,\boldsymbol{\tau})$. If $A \cup B = X$, then we are done. But if $A \cup B \neq X$, then we have to prove that $B \subset A$. Now $A \cup B \neq X$ means $B \subset A \cup B$ and $A \subset A \cup B$. Therefore, we have, $A \subset A \cup B$ and A is maximal e-open, then by definition, $A \cup B = X$ or $A \cup B = A$ but $A \cup B \neq X$, then $A \cup B = A$ which implies $B \subset A$.

Theorem 3.7. In any topological space $(X,\boldsymbol{\tau})$, if A and B be maximal e-open sets of $(X,\boldsymbol{\tau})$. Then either $A \cup B = X$ or $B = A$.

Proof. Let A and B be maximal e-open sets of $(X,\boldsymbol{\tau})$. If $A \cup B = X$, then we are done. But if $A \cup B \neq X$, then we have to prove that B = A. Now $A \cup B \neq X$ means $A \subset A \cup B$ and $B \subset A \cup B$. Now, since, $A \subset A \cup B$ and A is a maximal e-open set, then by definition, $A \cup B = X$ or $A \cup B = A$ but $A \cup B \neq X$, therefore, $A \cup B = A$ which implies $B \subset A$. Similarly, if $B \subset A \cup B$ we obtain, $A \subset B$. Therefore A = B.

Theorem 3.8. In any topological space $(X,\boldsymbol{\tau})$, if F be a minimal e-closed set and G be a e-closed set of $(X,\boldsymbol{\tau})$. Then either $F \cap G = \varphi$ or $F \subset G$.

Proof. Let F be a minimal e-closed set and G be a e-closed set of $(X,\boldsymbol{\tau})$. If $F \cap G = \varphi$, then there is nothing to prove. But if $F \cap G \neq \varphi$, then we have to prove that $F \subset G$. Now if $F \cap G \neq \varphi$, then $F \cap G \subset F$ and $F \cap G \subset G$. Since $F \cap G \subset F$ and given that F is minimal e-closed, then by definition $F \cap G = F$ or $F \cap G = \varphi$. But $F \cap G \neq \varphi$, then $F \cap G = F$ which shows that $F \subset G$.

Theorem 3.9. In any topological space $(X,\boldsymbol{\tau})$, if F and G be minimal e-closed sets of $(X,\boldsymbol{\tau})$. Then either $F \cap G = \varphi$ or F = G.

Proof. Let F and G be two minimal e-closed sets of $(X,\boldsymbol{\tau})$. If $F \cap G = \varphi$, then there is nothing to prove.

But if $F \cap G \neq \varphi$, then we have to prove that $F = G$. Now, if $F \cap G \neq \varphi$, then $F \cap G \subset F$ and $F \cap G \subset G$. Since $F \cap G \subset F$ and given that F is minimal e-closed, then by definition $F \cap G = F$ or $F \cap G = \varphi$. But $F \cap G \neq \varphi$, then $F \cap G = F$ which implies that $F \subset G$. Similarly, if $F \cap G \subset G$ and given that G is minimal e-closed, then by definition $F \cap G = G$ or $F \cap G = \varphi$. But $F \cap G \neq \varphi$, then $F \cap G = G$ which implies $G \subset F$. Hence, $F = G$.

Theorem 3.10. In any topological space (X,τ),

(a) Let A be a maximal e-open set of (X,τ) and x an element of $X\backslash A$. Then for any e-open set B containing x, $X\backslash A \subset B$.

(b) Let A be a maximal e-open set of (X,τ). Then, either of the following (i) and (ii) holds:

(i) For each $x \in X\backslash A$ and each e-open set B containing x, $B = X$.

(ii) There exists a e-open set B such that $X\backslash A \subset B$ and $B \subset X$.

(c) Let A be a maximal e-open set of (X,τ). Then, either of the following (i) and (ii) holds:

(i) For each $x \in X\backslash A$ and each e-open set B containing x, we have $X\backslash A \subset B$.

(ii) There exists a e-open set B such that $X\backslash A = B \neq X$.

Proof. (1) Since $x \in X\backslash A$, we have $B \not\subset A$ for any e-open set B containing x.Then, $A \cup B = X$ by Theorem 3.6. Therefore, $X\backslash A \subset B$.

(2) If **(i)** does not hold, then there exists an element x of $X\backslash A$ and a e-open set B containing x such that $B \subset X$. By (1), we have, $X\backslash A \subset B$.

(3) If **(ii)** does not hold, then, by **(1)**, we have $X\backslash A \subset B$ for each $x \in X\backslash A$ and each e-open set B containing x. Hence, we have $X\backslash A \subset B$.

Theorem 3.11. Let A, B, C be maximal e-open sets such that $A \neq B$. If $A \cap B \subset C$, then either $A = C$ or $B = C$.

Proof. Given that $A \cap B \subset C$. If $A = C$, then there is nothing to prove. But if $A \neq C$, then we have to prove $B = C$, Using Theorem 3.7, we have, $B \cap C = B \cap [C \cap X] = B \cap [C \cap (A \cup B)] = B \cap [(C \cap A) \cup (C \cap B)] = (B \cap C \cap A) \cup (B \cap C \cap B) = (A \cap B) \cup (C \cap B)$ [since, $A \cap B \subset C$] $= (A \cup C) \cap B = X \cap B = B$, [since $A \cup C = X$]. This implies $B \subset C$ also from the definition of maximal e-open set it follows that $B = C$.

Theorem 3.12. Let A, B,C be maximal e-open sets which are different from each other. Then $(A \cap B) \not\subset (A \cap C)$.

Proof. Let $(A \cap B) \cap (A \cap C)$. Then, $(A \cap B) \cup (C \cap B) \subset (A \cap C) \cup (C \cap B)$. Hence, $(A \cup C) \cap B \subset C \cap (A \cup B)$. Since by Theorem 3.7, $A \cup C = X$. We have $X \cap B \subset C \cap X$ which implies $B \subset C$. From the definition of maximal e-open set it follows that $B = C$. Contradiction to the fact that A, B and C are different from each other. Therefore $(A \cap B) \not\subset (A \cap C)$.

Theorem 3.13. (1) Let F be a minimal e-closed set of X. If $x \in F$, then $F \subset G$ for any e-closed set G containing x.

(2) Let F be a minimal e-closed set of X. Then $F = \cap \{G : x \in G \in e\text{-}C(X)\}$ for any element x of F.

Proof. (1) Let $F \in M_i e\text{-}C(X,x)$ and $G \in e\text{-}C(X,x)$ such that $F \not\subset G$. This implies that $F \cap G \subset F$ and $F \cap G \neq \varphi$. But since F is minimal e-closed, by Definition 3.2, $F \cap G = F$ which contradicts the relation $F \cap G \subset F$. Therefore, $F \subset G$.

(2) By **(1)** and the fact that F is e-closed containing x, we have, $F \subset \cap \{ G : G \in e\text{-}C(X, x)\} \subset F$. Therefore, we have the result.

Theorem 3.14. Let H and H_α ($\alpha \in \Lambda$, index set) be minimal e-closed sets of (X,τ). If $H \subset \cup_{\alpha\in\Lambda} H_\alpha$, then there exists $\alpha \in \Lambda$ such that $H = H_\alpha$.

Proof. Let H and H_α ($\alpha \in \Lambda$) be minimal e-closed sets such that $H \subset \cup_{\alpha\in\Lambda} H_\alpha$ We have to prove that $H \cap H_\alpha \neq \varphi$. Since if $H \cap H_\alpha = \varphi$, then $H_\alpha \subset X\backslash H$ and hence $H \subset \cup_{\alpha\in\Lambda} H_\alpha \subset X\backslash H$ which is a contradiction. Now as $H \cap H_\alpha \neq \varphi$, then $H \cap H_\alpha \subset H$ and $H \cap H_\alpha \subset H_\alpha$. Since $H \cap H_\alpha \subset H$ and given that F is minimal e-closed, then by definition $H \cap H_\alpha = H$ or $H \cap H_\alpha = \varphi$. But $H \cap H_\alpha \neq \varphi$, then $H \cap H_\alpha = H$ which implies $H \subset H_\alpha$. Similarly, if $H \cap H_\alpha \subset H_\alpha$ and given that H_α is minimal e-closed, then by definition $H \cap H_\alpha = H_\alpha$ or $H \cap H_\alpha = \varphi$. But $H \cap H_\alpha \neq \varphi$, so, $H \cap H_\alpha = H_\alpha$ which implies $H_\alpha \subset H$. Then $H = H_\alpha$.

Theorem 3.15. Let H and H_α ($\alpha \in \Lambda$) be minimal e-closed sets of (X,τ). If $H \neq H_\alpha$, for any $\alpha \in \Lambda$, then $H \cap (\cup_{\alpha\in\Lambda} H_\alpha) = \varphi$.

Proof. Let us assume that $H \cap (\cup_{\alpha\in\Lambda} H_\alpha) \neq \varphi$, then there exists $\alpha \in \Lambda$ such that $H \cap H_\alpha \neq \varphi$.By Theorem 3.9. we have $H = H_\alpha$ which is a contradiction to the fact $H \neq H_\alpha$. Hence $H \cap (\cup_{\alpha\in\Lambda} H_\alpha) = \varphi$.

4. e-SEMI-MAXIMAL OPEN SETS AND e-SEMI-MINIMAL CLOSED SETS

This section introduces the notion of e-semi-maximal open set and e-semi-minimal closed sets

and investigate some of their properties with examples.

Definition 4.1. A subset G in a topological space X is said to be e-semi-maximal open if there exists a maximal e-open set U such that $U \subset G \subset Cl(U)$. The complement of a e-semi-maximal open set is called a e-semi-minimal closed set.

Remark 4.2. Every maximal e-open (resp. minimal e-closed) set is e-semi-maximal open (resp. e-semi-minimal closed).

Example 4.3. Let us consider the topological space (X,τ) such that X = {a, b, c, d} and
τ = {X,φ, {a},{b}, {a, b}}.
We observe that $\delta O(X,\tau)$ = {X,φ, {a},{b}, {a, b}}; e-O(X,τ) = {X, φ, {a},{b},{a,c},{b,d}, {a,d}, {a, b}, {a,b,c}, {a,c,d},{a,b,d}}. Clearly, {a,b,c}, {a,c,d} and {a,b,d} are maximal e-open sets and each set is e-semi-maximal open set in X. Also it is clear that the complement sets {b}, {c} and {d} are each minimal e-closed set and hence it is e-semi-minimal closed set.

The collection of all e-semi-maximal open (resp. e-semi-minimal closed) set of X is denoted by e-SMaO(X) (resp. e-SMaC(X)).

Theorem 4.4. If G is a e-semi-maximal open set of X and $G \subset M \subset Cl(G)$,then M is a e-semi-maximal open set of X.

Proof. Since G is e-semi-maximal open, there exists a maximal e-open set U such that $U \subset G \subset Cl(U)$. Then $U \subset G \subset M \subset Cl(G) \subset Cl(U)$. Hence$U \subset M \subset Cl(U)$. Thus M is e-semi-maximal open.

Theorem 4.5. A subset B of a topopological space X is e-semi-minimal closed if and only if there exists a minimal e-closed set G in X such that $Int(G) \subset B \subset G$.

Proof. Suppose B is e-semi-minimal closed in X. By Definition 4.1, X\B is e-semi-maximal open in X. Therefore, there exists a maximal e-open set U such that $U \subset X\backslash B \subset Cl(U)$, which implies $Int(X\backslash U) = X\backslash Cl(U) \subset B \subset X\backslash U$.Let us take G = X\U, so that G is a minimal e-closed set such that $Int(G) \subset B \subset G$.
Conversely, suppose that there exists a minimal e-closed set G in X, such that $Int(G) \subset B \subset G$. Hence $X \setminus G \subset X \setminus B \subset X \setminus Int(G) = Cl(X \setminus G)$. So, there exists a maximal e-open set $U = X \setminus G$ such that $U \subset X \setminus B \subset Cl(U)$, which implies that $X \setminus B$ is e-semi-maximal open in X. It follows that B is e-semi-minimal closed.

Theorem 4.6. If G is e-semi-minimal closed in **(X,τ)** and if $Int(G) \subset F \subset G$, then F is also e-semi - minimal closed in X.

Proof. Let G be a e-semi-minimal closed set of X. Then there exists a minimal e-closed set H in X, such that $Int(H) \subset G \subset H$. Hence, $Int(H) \subset Int(G) \subset F \subset G \subset H$. It follows $Int(H) \subset F \subset H$. Therefore, F is a e-semi-minimal closed set in X.

Theorem 4.7. Let (Y,τ_Y) be an open subspace of **(X,τ)** and $A \subset Y$ and A is a e-semi-maximal open set of X, then A is also a e-semi-maximal open set of Y.

Proof. Since A is a e-semi-maximal open set of X, there exists a maximal e-open set U such that $U \subset A \subset Cl(U)$. Hence, U is a subset of Y. Since U is maximal e-open in X, $Y \cap U = U$ is maximal e-open in Y and $U = Y \cap U \subset Y \cap A \subset Y \cap Cl(U) \Rightarrow U \subset A \subset Cl_Y(U)$. Hence A is e-semi-maximal open in Y.

Theorem 4.8. If G_λ is a e-semi-maximal open set of topological spaces $(X_\lambda,\tau_{X1\times X2})$ (λ = 1, 2), then $G_1\times G_2$ is a e-semi-maximal open set in the Product Space $(X_1\times X_2, \tau_{X1\times X2})$.

Proof. Let G_λ be a e-semi-maximal open set of topological spaces X_λ (λ = 1, 2), then there exists a maximal e-open set U_λ such that $U_\lambda \subset G_\lambda \subset Cl_{X\lambda}(U_\lambda)$, for each λ. Therefore, $U_1\times U_2 \subset G_1\times G_2 \subset Cl_{X1}(U_1) \times Cl_{X2}(U_2) = Cl_{X1\times X2}(U_1 \times U_2)$. Hence $G_1\times G_2$ is e-semi-maximal open in $(X_1\times X_2, \tau_{X1\times X2})$.

5. Conclusions

In this work, the concept of maximal e-open sets, minimal e-closed sets, e-semi maximal open and e-semi minimal closed sets which are fundamental results for further research on topological spaces are introduced and aimed in investigating the properties of these new notions of open sets with example, counter examples and some of their fundamental results are also established. Hope that the findings in this paper will help researcher enhance and promote the further study on topological spaces to carry out a general framework for their applications in separation axioms, connectedness, compactness etc. and also in practical life.

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