On The Solutions of Wave Equation in Three Dimensions using D'alembert Formula

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Abstract *In this paper, we derive explicit formulas, which can be used to solve Cauchy problems of wave equation in three and two dimension spaces, using d'alembert formula. Moreover, we apply those formulas to solve two examples.*

Keywords: Wave equation, Cauchy problem, D'alembert formula.

I. INTROGUCTION

Let us consider the following Cauchy problem for the wave equation:

$$\begin{cases} \dfrac{\partial^2 u(t,x)}{\partial t^2} = \lambda^2 \Delta u(t,x), & t > 0,\ x \in \mathbb{R}^3, \\ u(0,x) = f(x),\ \ u_t(0,x) = g(x), & x \in \mathbb{R}^3, \end{cases} \quad (1)$$

where

$$\Delta u = \frac{\partial^2 u(t,x)}{\partial x_1^2} + \frac{\partial^2 u(t,x)}{\partial x_2^2} + \frac{\partial^2 u(t,x)}{\partial x_3^2} \quad \text{and}$$

$$x = (x_1, x_2, x_3),\ \lambda > 0 \ .$$

This problem has been studied my some authors, for instance, see [1],[2],[3] and [4].

We are interested in finding an explicit representation of a solution of (1), following the way which has been used in [2].

The idea for solving this problem is to reduce the problem to a one-dimensional space and to solve the one-dimensional problem by using d'Alembert formula. Then will apply this formula to find the solution of wave equation in two-dimensional space.

II. POISSON FORMULA

In this section, we aim to find a formula, which will be the solution of problem (1)

Definition (1). (Spherical Mean)[1].

Let $w(x_1, x_2, x_3) \in C^2(\mathbb{R}^3)$ be a smooth function.

Let $x \in \mathbb{R}^3$ be fixed and $S(x,r) = \{y \in \mathbb{R}^3 : \|y - x\| = r\}$ be a sphere centred at x with radius r. The spherical mean is the integral average of w on a sphere $S(x,r)$.

To be more precise,

$$M_w(x,r) = \frac{1}{4\pi r^2} \iint_{S(x,r)} w(x + rn)\, d_r\sigma, \qquad (2)$$

Where $4\pi r^2$ is the area of $S(x,r)$, n is the outward unit normal vector of $S(x,r)$ at $y = x + rn$, and $d_r\sigma$ is the area element of $S(x,r)$.

Note that if a point y is on $S(x,r)$, it can always be denoted by $y = x + rn$,

where $n = \frac{y-x}{\|y-x\|} = \frac{y-x}{r}$, and $y = (y_1, y_2, y_3) = (rn_1, rn_2, rn_3)$.

If we parameteize the sphere $S(x,r)$ by spherical coordinates, we have

$n = (n_1, n_2, n_3), n_1 = \sin\theta \cos\phi, n_2 = \sin\theta \sin\phi,$

$n_3 = \cos\theta;\ \theta \in [0,\pi], \phi \in [0, 2\pi).$

We transform $d_r\sigma$ to $d\theta d\phi$:

$$d_r\sigma = \left\| \frac{\partial rn}{\partial \theta} \times \frac{\partial rn}{\partial \phi} \right\| d\theta d\phi = r^2 \sin\theta d\theta d\phi = r^2\, d_1\sigma,$$

where $d_1\sigma = \sin\theta d\theta d\phi$.

Therefore, we have

$$M_w(x,r) = \frac{1}{4\pi r^2} \iint_{S(x,r)} w(x + rn) r^2 d_1\sigma$$

$$= \frac{1}{4\pi} \iint_{S(0,1)} w(x + rn) 2 d_1\sigma$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} w(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3) \sin\theta d\theta d\phi. \quad (3)$$

The relation between the spherical mean and the original function is the following:

$\lim_{r \to 0_+} M_w(x,r) = w(x).$

(4)

To see this, use mean value theorem in (2.48), i.e.,

$$M_w(x,r) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} w(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3) \sin\theta d\theta d\phi$$

$$= w(x + rn(\theta_0(r), \phi_0(r))) \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\phi$$

$$= w(x + rn(\theta_0(r), \phi_0(r))) + \frac{1}{2}(-\cos\theta)\Big|_0^{\pi}$$

$$= w(x + rn(\theta_0(r), \phi_0(r))),$$

Where $n(\theta_0(r), \phi_0(r))$ is a unit vector depending on r.

Passing to the limit $r \to 0$, we obtain (4).

Our aim is to use the special mean to construct a solution of Cauchy problem (1).

We have the following result.

Lemma (1). Suppose $w(x) \in C^2(\mathbb{R}^3)$ and M_w is given by (3). Then $M_w(x,r)$, as a function of x and r, satisfies

$$\Delta_x M_w(x,r) = \frac{\partial^2 M_w(x,r)}{\partial r^2} + \frac{2}{r} M_w(x,r), \qquad (5)$$

where Δ_x means that derivatives are to be taken with respect to the x −variables.

Proof: We first observe that

$$\frac{\partial M_w(x,r)}{\partial x_1} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\partial}{\partial x_1} w(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3) \sin\theta d\theta d\phi$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\partial}{\partial y_1} w(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3) \sin\theta d\theta d\phi.$$

Notice here that w is a function of y, i.e., $w(y_1, y_2, y_3)$, while y is a function of x and rn, i.e., $(y_1, y_2, y_3) = (x_1 + rn_1, x_2 + rn_2, x_3 + rn_3)$.
Hence by the chain rule we get the formula above.
Repeating this argument, we find

$$\Delta_x M_w = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \Delta_x w(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3)\, \sin\theta d\theta d\phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \Delta_y w(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3)\, \sin\theta d\theta d\phi.$$

On the other hand, we have, still by the chain rule,

$$\frac{\partial M_w(x,r)}{\partial r} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\partial}{\partial y_i} w(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3) n_i \sin\theta d\theta d\phi$$

$$= \frac{1}{4\pi r^2}\iint_{S(x,r)} \sum_{i=1}^{3} \frac{\partial}{\partial y_i} w(x + rn) n_i d_r\sigma$$

$$= \frac{1}{4\pi r^2}\iiint_{B(x,r)} \Delta_y w(y) dy_1 dy_2 dy_3,$$

Where $B(x,r)$ is a ball centred at x with radius r, Here we used Gauss' formula for a vector-valued function $F = (F_1, F_2, F_3)$ on a domain Ω in $\mathbb{R}^3$ with boundary $\partial\Omega$:

$$\iiint_\Omega \left(\frac{\partial F_1}{\partial y_1} + \frac{\partial F_2}{\partial y_2} + \frac{\partial F_3}{\partial y_3}\right) dy_1 dy_2 dy_3 = \iint_{\partial\Omega} F.nd\sigma.$$

Thus, we have

$$\frac{\partial M_w(x,r)}{\partial r} = \frac{1}{4\pi r^2}\iiint_{B(x,r)} \Delta_y w(y) dy_1 dy_2 dy_3$$

$$= \frac{1}{4\pi r^2}\int_0^r\int_0^{2\pi}\int_0^{\pi} \Delta_y w(y) \rho^2 \sin\theta d\theta d\phi d\rho. \qquad (6)$$

Form this, we have, by the method of differentiating integrals with parameters,

$$\frac{\partial^2 M_w(x,r)}{\partial r^2} = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} \Delta_y w(x + rn) r^2 \sin\theta d\theta d\phi$$

$$- \frac{2}{4\pi r^2}\int_0^r\int_0^{2\pi}\int_0^{\pi} \Delta_y w \rho^2 \sin\theta d\theta d\phi d\rho$$

$$= \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} \Delta_y w(x + rn) d_r\sigma - \frac{2}{4\pi r^2}\int_0^r\int_0^{2\pi}\int_0^{\pi} \Delta_y w \rho^2 \sin\theta d\theta d\phi d\rho$$

$$= \Delta_x M_w - \frac{2}{r}\frac{\partial M_w(x,r)}{\partial r}$$

The proof is complete. ■

Notice that (5) can be rewritten as

$$\Delta_x M_w = \frac{1}{r}\frac{\partial^2 (rM_w(x,r))}{\partial r^2}.$$

Now we are able to solve problem (1). In fact, suppose u is a solution of (1), and let $M_u(x,r,t)$ be its spherical mean. We establish a partial differential equation with variables r and t and leave x as a parameter, cf. (5). For this we extend M_u, M_f and M_g to $r < 0$ by using formula (3). This gives an even extensions, e.g., $M_u(x,r,t) = M_u(x,-r,t)$, where M_f and M_g are spherical means of the initial data f and g, respectively, and the function is C^2 since we can differentiate under the integral sign.
For the second derivative we have

$$\frac{\partial^2 M_w(x,r)}{\partial r^2} = \lim_{h\to 0} \frac{\frac{\partial M_u(x,r+h,t)}{\partial r} - \frac{\partial M_u(x,r,t)}{\partial r}}{h}$$

$$= \lim_{h\to 0} \frac{-\frac{\partial M_u(x,-(r+h),t)}{\partial r} + \frac{\partial M_u(x,-r,t)}{\partial r}}{h}$$

$$= \frac{\partial^2 M_w(x,-r,t)}{\partial r^2}$$

Hence, the second derivative is defined and continuous at $r = 0$.
Since u is a solution of problem (1), we have

$$\lambda^2 \frac{1}{r}\frac{\partial^2 (rM_w(x,r))}{\partial r^2} = \lambda^2 \Delta_x M_u(x,r,t)$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \Delta_y \lambda^2 u(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3)\, \sin\theta d\theta d\phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\partial^2}{\partial t^2} u(x_1 + rn_1, x_2 + rn_2, x_3 + rn_3, t)\, \sin\theta d\theta d\phi$$

$$= \frac{\partial^2}{\partial t^2} M_u(x,r,t).$$

Therefore $M_u(x,r,t)$ satisfies

$$\frac{\partial^2}{\partial t^2}(rM_u(x,r,t)) = \lambda^2 \frac{\partial^2 (rM_u(x,r,t))}{\partial r^2} \qquad (7)$$

and

$$rM_u(x,r,t)\,|_{t=0} = rM_f(x,r), \qquad (8)$$

$$\frac{\partial}{\partial t}(rM_u(x,r,t))\,|_{t=0} = rM_g(x,r), \qquad (9)$$

Thus we reduce (1) to (7) – (9), which is much simpler and has only one space variable r.
We can easily solve $rM_u(x,r,t)$ from (7) – (9) by using d'Alembert's formula.
Then we pass to the limit $r \to 0$ to recover u.
By d'Alembert we have

$$rM_u(x,r,t) = \frac{1}{2}\left[(r + \lambda t)M_f(x, r + \lambda t) + (r - \lambda t)M_f(x, r - \lambda t)\right]$$

$$+ \frac{1}{2\lambda}\int_{r-\lambda t}^{r+\lambda t} \mathcal{T} M_g(x,\mathcal{T}) d\mathcal{T}$$

$$M_u(x,r,t) = \frac{1}{2r}\left[(r + \lambda t)M_f(x, r + \lambda t) + (r - \lambda t)M_f(x, r - \lambda t)\right]$$

$$+ \frac{1}{2\lambda r}\int_{r-\lambda t}^{r+\lambda t} \mathcal{T} M_g(x,\mathcal{T}) d\mathcal{T}$$

Passing to the limit $r \to 0$, we have

$$u(t,x) = \lim_{r\to 0} M_u(t,x,r)$$

$$= \underbrace{\lim_{r\to 0}\frac{1}{2r}\left[(r + \lambda t)M_f(x, r + \lambda t) + (r - \lambda t)M_f(x, r - \lambda t)\right]}_{=I_1}$$

$$+ \underbrace{\lim_{r\to 0}\frac{1}{2\lambda r}\int_{r-\lambda t}^{r+\lambda t} \mathcal{T} M_g(x,\mathcal{T}) d\mathcal{T}}_{=I_2}$$

We study I_1 and I_2 separately. Since M_f is even,

$$(r + \lambda t)M_f(x, r + \lambda t) + (r - \lambda t)M_f(x, r - \lambda t) =$$

$$(r + \lambda t)[M_f(x, r + \lambda t) - M_f(x, \lambda t - r)]$$

$$+ (r - \lambda t)M_f(x, r - \lambda t) + (r + \lambda t)M_f(x, \lambda t - r)$$

$$= (r + \lambda t)\left[M_f(x, r + \lambda t) - M_f(x, \lambda t - r)\right]$$

$$+ 2rM_f(x, r - \lambda t).$$

Consequently, we have for M_f

$$I_1 = \lim_{r\to 0}\frac{1}{2r}\left[(r + \lambda t)M_f(x, r + \lambda t) + (r - \lambda t)M_f(x, r - \lambda t)\right]$$

$$= \lim_{r\to 0}\frac{(r + \lambda t)}{2r}\left[M_f(x, r + \lambda t) - M_f(x, \lambda t - r)\right]$$

$$+ \lim_{r\to 0}\frac{2r}{2r} M_f(x, r - \lambda t)$$

$$= \lambda t \frac{\partial}{\partial r} M_f(x, \lambda t) + M_f(x, \lambda t) = \frac{\partial}{\partial t}\left[tM_f(x, \lambda t)\right].$$

Now we turn to I_2. Since $M_g(x,r)$ is an even function of r, $rM_g(x,r)$ is odd. Therefore

$$\int_{r-\lambda t}^{\lambda t - r} \mathcal{T} M_g(x,\mathcal{T}) d\mathcal{T} = 0.$$

Hence, we infer

$$\int_{r-\lambda t}^{r+\lambda t} \mathcal{T} M_g(x,\mathcal{T}) d\mathcal{T} = \int_{r-\lambda t}^{\lambda t - r} \mathcal{T} M_g(x,\mathcal{T}) d\mathcal{T} + \int_{\lambda t - r}^{\lambda t + r} \mathcal{T} M_g(x,\mathcal{T}) d\mathcal{T}$$

$$= \int_{\varepsilon t - r}^{\varepsilon t + r} \mathcal{T} M_g(x,\mathcal{T}) d\mathcal{T}.$$

Thus, the second term I_2 results in
$I_2 = \lim_{r\to 0}\frac{1}{2\lambda r}\int_{\lambda t-r}^{r+\lambda t} \mathcal{T} M_g(x,\mathcal{T})d\mathcal{T} = tM_g(x,\lambda t)$ (mean value theorem).
Therefore

$$u(t,x) = \frac{\partial}{\partial t}\left[tM_f(x,\lambda t)\right] + tM_g(x,\lambda t). \qquad (10)$$

From the definition of spherical means, we infer

$$u(t,x) = \frac{\partial}{\partial t}\frac{1}{4\pi\lambda^2 t}\iint_{S(x,\lambda t)} f(y)d_{\lambda t}\sigma + \frac{1}{4\pi\lambda^2 t}\iint_{S(x,\lambda t)} g(y)d_{\lambda t}\sigma$$

(11)

The relation (11) is called the Poisson formula. If we use spherical coordinates, we have

$$u(t,x) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f\begin{pmatrix} x_1+\lambda t\sin\theta\cos\phi, x_2+ \\ \lambda t\sin\theta\sin\phi, x_3+\lambda t\cos\theta\end{pmatrix}\sin\theta d\theta d\phi\right)$$

$$+\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} g(x_1+\lambda t\sin\theta\cos\phi, x_2+\lambda t\sin\theta\sin\phi, x_3+ \lambda t\cos\theta)\sin\theta d\theta d\phi$$

Example(1) : Solve the following Cauchy problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} = \lambda^2\Delta u(t,x), & t>0 \quad , \quad x\in\mathbb{R}^3, \\ u(0,x) = x_1+x_2+x_3, \quad u_t(0,x) = x_1^2+x_2^2+x_3^2, & x\in\mathbb{R}^3. \end{cases}$$

Solution: Using the initial values $f(x) = x_1+x_2+x_3$, $g(x) = x_1^2+x_2^2+x_3^2$ in Poisson's formula (11), we have

$$u(t,x) = \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\begin{pmatrix}(x_1+\lambda t\sin\theta\cos\phi + x_2+\lambda t\sin\theta\sin\phi \\ +x_3+\lambda t\cos\theta)\sin\theta d\theta d\phi\end{pmatrix}\right)$$

$$+\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\begin{pmatrix}(x_1+\lambda t\sin\theta\cos\phi)^2+(x_2+\lambda t\sin\theta\sin\phi)^2 \\ +(x_3+\lambda t\cos\theta)^2\end{pmatrix}\right]\sin\theta d\theta d\phi$$

So, we calculate the integrals

$$\int_0^{\pi}(x_1+\lambda t\sin\theta\cos\phi + x_2+\lambda t\sin\theta\sin\phi + x_3+\lambda t\cos\theta)\sin\theta d\theta$$

$$\int_0^{\pi}(x_1\sin\theta+\lambda t\sin^2\theta\cos\phi + x_2\sin\theta+\lambda t\sin^2\theta\sin\phi + x_3\sin\theta+\lambda t\cos\theta\sin\theta)\,d\theta$$

$$= (-x_1\cos\theta+\lambda t\cos\phi\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right) - x_2\cos\theta + \lambda t\sin\phi\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right) - x_3\cos\theta+\frac{1}{2}\lambda t\sin^2\theta)\Big|_0^{\pi}$$

$$= -x_1(-1)+ct\cos\phi\left(\frac{1}{2}\pi-0\right) - x_2(-1)+ct\sin\phi\left(\frac{1}{2}\pi-0\right) - x_3(-1)+\frac{1}{2}ct(0) - (-x_1-x_2-x_3+0)$$

$$= x_1+\frac{1}{2}\pi ct\cos\phi + x_2+\frac{1}{2}\pi ct\sin\phi + x_3 + x_1+x_2+x_3$$

$$= 2x_1+2x_2+2x_3+\frac{1}{2}\pi ct(\cos\phi+\sin\phi)$$

$$\int_0^{2\pi}\left(2x_1+2x_2+2x_3+\frac{1}{2}\pi ct(\cos\phi+\sin\phi)\right)d\phi$$

$$= [2x_1\phi+2x_2\phi+2x_3\phi+\frac{1}{2}\pi ct(\sin\phi-\cos\phi)]\Big|_0^{2\pi}$$

$$= 4x_1\pi+4x_2\pi+4x_3\pi+\frac{1}{2}\pi ct(-1)+\frac{1}{2}\pi ct$$

$$= 4\pi(x_1+x_2+x_3)$$

$$\Rightarrow \frac{t}{4\pi}(4\pi(x_1+x_2+x_3) = t(x_1+x_2+x_3)$$

$$\Rightarrow \frac{\partial}{\partial t}(t(x_1+x_2+x_3)) = x_1+x_2+x_3$$

$$+\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\begin{pmatrix}(x_1+\lambda t\sin\theta\cos\phi)^2+(x_2+\lambda t\sin\theta\sin\phi)^2 \\ +(x_3+\lambda t\cos\theta)^2\end{pmatrix}\right]\sin\theta d\theta d\phi$$

$$\int_0^{2\pi}\int_0^{\pi}[x_1^2+2x_1\lambda t\sin\theta\cos\phi+\lambda^2t^2\sin^2\theta\cos^2\phi+x_2^2+ 2x_2\lambda t\sin\theta\sin\phi+\lambda^2t^2\sin^2\theta\sin^2\phi+x_3^2+ 2x_3\lambda t\cos\theta+\lambda c^2t^2\cos^2\theta]\sin\theta d\theta d\phi$$

$$\int_0^{2\pi}\int_0^{\pi}[x_1^2\sin\theta+2x_1\lambda t\sin^2\theta\cos\phi+\lambda^2t^2\sin^2\theta\cos^2\phi+x_2^2\sin\theta+ 2x_2\lambda t\sin^2\theta\sin\phi+\lambda^2t^2\sin^2\theta\sin^2\phi+x_3^2\sin\theta+2x_3\lambda t\sin\theta\cos\theta +\lambda^2t^2\sin\theta\cos^2\theta]d\theta d\phi$$

$$\int_0^{2\pi}[-x_1^2\cos\theta+2x_1\lambda t\cos\phi\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]+ c^2t^2\cos^2\phi\left[\frac{-1}{3}\sin^2\theta\cos\theta-\frac{2}{3}\cos\theta\right]+(-x_2^2\cos\theta) +2x_2\lambda t\sin\phi\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]+c^2t^2\sin^2\phi\left[\frac{-1}{3}\sin^2\theta\cos\theta-\frac{2}{3}\cos\theta\right] +(-x_3^2\cos\theta)+x_3\lambda t\sin^2\theta+\lambda^2t^2\left[\frac{1}{3}\sin^2\theta\cos\theta-\frac{1}{3}\cos\theta\right]\Big|_0^{\pi}]\,d\phi$$

$$\int_0^{2\pi}[x_1^2+x_1\pi\lambda t\cos\phi+\frac{2}{3}\lambda^2t^2\cos^2\phi+x_2^2+x_2\pi\lambda t\sin\phi +\frac{2}{3}\lambda^2t^2\sin^2\phi+x_3^2+\frac{1}{3}\lambda^2t^2+x_1^2+\frac{2}{3}\lambda^2t^2\cos^2\phi +x_2^2+\frac{2}{3}\lambda^2t^2\sin^2\phi+x_3^2+\frac{1}{3}\lambda^2t^2]\,d\phi$$

$$\int_0^{2\pi}[2x_1^2+x_1\pi\lambda t\cos\phi+\frac{4}{3}\lambda^2t^2\cos^2\phi+2x_2^2 +x_2\pi\lambda t\sin\phi+\frac{4}{3}\lambda^2t^2\sin^2\phi+2x_3^2+\frac{2}{3}\lambda^2t^2]\,d\phi$$

$$= [2x_1^2\phi+x_1\pi ct\sin\phi+\frac{4}{3}\lambda^2t^2[\frac{1}{2}\phi+\frac{1}{4}\sin 2\phi]+2x_2^2\phi -x_2\pi\lambda t\cos\phi+\frac{4}{3}\lambda^2t^2[\frac{1}{2}\phi-\frac{1}{4}\sin 2\phi]+2x_3^2\phi+\frac{2}{3}\lambda^2t^2\phi]\Big|_0^{2\pi}$$

$$= 4x_1^2\pi+\frac{4}{3}\lambda^2t^2\pi+4x_2^2\pi-x_2\pi\lambda t+\frac{4}{3}\lambda^2t^2\pi+4x_3^2\pi+\frac{4}{3}\lambda^2t^2\pi+x_3\pi\lambda t$$

$$= 4\pi(x_1^2+x_2^2+x_3^2)+4\pi\lambda^2t^2$$

$$= 4\pi[(x_1^2+x_2^2+x_3^2)+\lambda^2t^2]$$

$$\Rightarrow \frac{t}{4\pi}(4\pi[(x_1^2+x_2^2+x_3^2)+\lambda^2t^2]$$

$$\Rightarrow t(x_1^2+x_2^2+x_3^2)+\lambda^2t^2$$

$$u(t,x) = x_1+x_2+x_3+t(x_1^2+x_2^2+x_3^2)+\lambda^2t^2$$

III. SOULTION OF WAVE EQUATION IN TWO SPACE DIMENTIONS

In this section, we solve the following Cauchy problem in two space dimension:

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} = \lambda^2\Delta u(t,x), & t>0, x\in\mathbb{R}^2, \\ u(0,x) = f(x_1,x_2), \quad u_t(0,x) = g(x_1,x_2), & x\in\mathbb{R}^2, \end{cases} \qquad (12)$$

where
$\Delta u = \frac{\partial^2 u(t,x)}{\partial x_1^2}+\frac{\partial^2 u(t,x)}{\partial x_2^2}$.

Our plan is to use Poisson's formula for $n=3$ in order to solve the problem for $n=2$. Hence, the idea is to consider problem (12) as a problem in three dimensional space with both initial data f and g depending only on two variables (x_1,x_2). Then we hope that Poisson's formula provides a solution independent of the third variable. In fact, consider the Cauchy problem as a problem in $\mathbb{R}^3$ with initial data $f(x_1,x_2)$ and $g(x_1,x_2)$.
By Poisson's formula, the solution is

$$u(t,x_1,x_2,x_3) = \frac{\partial}{\partial t}\frac{1}{4\pi\lambda^2 t}\iint_{S(x,\lambda t)} f(y_1,y_2)\,d_{\lambda t}\sigma + \frac{1}{4\pi\lambda^2 t}\iint_{S(x,\lambda t)} g(y_1,y_2)\,d_{\lambda t}\sigma$$

(13)

Recall that f and g are independent of the third variable. We further have

$$u(t,x_1,x_2,x_3) = \frac{\partial}{\partial t}\frac{1}{4\pi\lambda^2 t}2\iint_{S_+(x,\lambda t)} f(y_1,y_2)\, d_{\lambda t}\sigma + \frac{1}{4\pi\lambda^2 t}2\iint_{S_+(x,\lambda t)} g(y_1,y_2)\, d_{\lambda t}\sigma$$

(14)

where $S_+(x,\lambda t)$ is the upper hemisphere

$$S_+(x,\lambda t) = \{y = (y_1,y_2,y_3) : \|y-x\| = \lambda t,\ y_3 - x_3 \geq 0\}.$$

In fact,

$$\iint_{S(x,\lambda t)} f(y_1,y_2)\, d_{\lambda t}\sigma = (\lambda t)^2\int_0^{2\pi}\int_0^{\pi} f(x_1+\lambda t\sin\theta\cos\phi, x_2+\lambda t\sin\theta\sin\phi)\sin\theta d\theta d\phi = I.$$

The change of variables

$\theta := \alpha + \pi/2, \sin\theta = \sin(\alpha+\pi/2) = \cos\alpha$, implies

$$I = (\lambda t)^2\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x_1+\lambda t\cos\alpha\cos\phi, x_2+\lambda t\cos\alpha\sin\phi)\cos\alpha d\alpha d\phi$$

$$= 2(\lambda t)^2\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{0} f(x_1+\lambda t\cos\alpha\cos\phi, x_2+\lambda t\cos\alpha\sin\phi)\cos\alpha d\alpha d\phi$$

($\cos\alpha$ is even)

$$+ct\sin\theta\sin\phi)\sin\theta d\theta d\phi = 2\iint_{S_+(x,\lambda t)} f(y_1,y_2)\, d_{\lambda t}\sigma.$$

Parameterizing $S_+(x,\lambda t)$ by

$$y_3 = x_3 + \sqrt{(\lambda t)^2 - (y_1-x_1)^2 - (y_2-x_2)^2} := F(y_1,y_2),$$

where

$(y_1,y_2) \in D_{\lambda t} = \{(y_1,y_2), \sqrt{(y_1-x_1)^2+(y_2-x_2)^2} \leq \lambda t\}$,

we have, for some given function $h(y_1,y_2)$,

$$\iint_{S_+(x,\lambda t)} h(y_1,y_2)d_{\lambda t}\sigma = \iint_{D_{\lambda t}} h(y_1,y_2)\sqrt{1+F_{y_1}^2(y_1,y_2)+F_{y_2}^2(y_1,y_2)}dy_1dy_2,$$

where

$$F_{y_1}(y_1,y_2) = \frac{-(y_1-x_1)}{\sqrt{(\lambda t)^2-(y_1-x_1)^2-(y_2-x_2)^2}},$$

$$F_{y_2}(y_1,y_2) = \frac{-(y_2-x_2)}{\sqrt{(\lambda t)^2-(y_1-x_1)^2-(y_2-x_2)^2}},$$

and as a consequence

$$\sqrt{1+F_{y_1}^2(y_1,y_2)+F_{y_2}^2(y_1,y_2)} = \frac{\lambda t}{\sqrt{(\lambda t)^2-(y_1-x_1)^2-(y_2-x_2)^2}}.$$

This gives

$$\iint_{S_+(x,\lambda t)} h(y_1,y_2)d_{\lambda t}\sigma = \iint_{D_{\lambda t}} h(y_1,y_2)\frac{\lambda t}{\sqrt{(\lambda t)^2-(y_1-x_1)^2-(y_2-x_2)^2}}dy_1dy_2$$

(15)

Replacing h by f and g substituting this into Poisson's formula, we obtain

$$u(t,x_1,x_2,x_3) = \frac{\partial}{\partial t}\frac{1}{4\pi\lambda^2 t}\iint_{D_{\lambda t}} f(y_1,y_2)\frac{\lambda t}{\sqrt{(\lambda t)^2-(y_1-x_1)^2-(y_2-x_2)^2}}dy_1dy_2 + 2\frac{1}{4\pi\lambda^2 t}\iint_{D_{\lambda t}} g(y_1,y_2)\frac{\lambda t}{\sqrt{(\lambda t)^2-(y_1-x_1)^2-(y_2-x_2)^2}}dy_1dy_2$$

Notices that the right hand side of (15) is independent of x_3.

Therefore, $u_{x_3} = 0$ and $u = u(x_1,x_2)$ given by (14) is a solution of (12).

If we use polar coordinates

$y_1 = x_1 + \rho\cos\theta, y_2 = x_2 + \rho\sin\theta$, we can rewrite (14) as

$$u(t,x_1,x_2) = \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\int_0^{\lambda t}\int_0^{2\pi} f(x_1+\rho\cos\theta, x_2 + \rho\sin\theta)\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}d\theta d\rho + \frac{1}{2\pi\lambda}\int_0^{\lambda t}\int_0^{2\pi} g(x_1+\rho\cos\theta, x_2+\rho\sin\theta)\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}d\theta d\rho$$

(16)

Both, (14) and (16) are called Poisson's formula for the two dimensional Cauchy problem of the wave equation.

Examle (2): Solve the following Cauchy problem in $\mathbb{R}^2$:

$$\begin{cases} u_{tt} = \lambda^2(u_{x_1x_1}+u_{x_2x_2}), & t>0, (x,y)\in\mathbb{R}^2 \\ u(0,x) = x_1(x_1+x_2), \quad u_t(0,x) = 0, & x\in\mathbb{R}^2 \end{cases}$$

Solution: We apply Poisson formula (15) to $f = x_1(x_1+x_2), g = 0$ the solution is

$$u(t,x_1,x_2) = \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\int_0^{\lambda t}\int_0^{2\pi} f(x_1+\rho\cos\theta)\begin{pmatrix} x_1+\rho\cos\theta \\ +x_2+\rho\sin\theta\end{pmatrix}\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}d\theta d\rho + \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\int_0^{ct}\frac{[2\pi x_1(x_1+x_2)+\pi\rho^2]\rho}{\sqrt{(\lambda t)^2-\rho^2}}d\rho$$

$$u(t,x_1,x_2) = \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\int_0^{ct}\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}\int_0^{2\pi}(x_1+\rho\cos\theta)\begin{pmatrix} x_1 \\ +\rho\cos\theta+x_2+\rho\sin\theta\end{pmatrix}d\theta d\rho$$

$$= \frac{\partial}{\partial t}\frac{1}{2\pi c}\int_0^{ct}\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}\int_0^{2\pi}[x_1^2+2x_1\rho\cos\theta+x_1x_2 + x_1\rho\sin\theta+\rho^2\cos^2\theta+x_2\rho\cos\theta+\rho^2\cos\theta\sin\theta]\,d\theta d\rho$$

$$= \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\int_0^{ct}\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}[x_1^2\theta+2x_1\rho\sin\theta+x_1x_2\theta-x_1\rho\cos\theta + \rho^2(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta)+x_2\rho\sin\theta+\rho^2(\frac{1}{2}\sin^2\theta)]\Big|_0^{2\pi}\,d\rho$$

$$= \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\int_0^{ct}\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}[2\pi x_1^2+2\pi x_1x_2-x_1p+\pi p^2+x_1\rho]\,d\rho$$

$$= \frac{\partial}{\partial t}\frac{1}{2\pi c}\int_0^{ct}\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}[2\pi x_1^2+2\pi x_1x_2+\pi p^2]\,d\rho$$

$$= \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\int_0^{ct}\frac{\rho}{\sqrt{(\lambda t)^2-\rho^2}}[2\pi x_1(x_1+x_2)+\pi p^2]\,d\rho$$

Using the substitution $u = \sqrt{(\lambda t)^2-\rho^2}$ we obtain

$\frac{du}{d\rho} = \frac{-\rho}{\sqrt{(ct)^2-\rho^2}}$ and thus

$u = \sqrt{(\lambda t)^2-\rho^2} = ((\lambda t)^2-\rho^2)^{\frac{1}{2}}$

$\boxed{\rightarrow u^2 = (\lambda t)^2-\rho^2 \rightarrow \rho^2 = (\lambda t)^2-u^2}$

$$\frac{du}{dp} = \frac{1}{2}((\lambda t)^2-\rho^2)^{-\frac{1}{2}}\cdot(-2p)$$

$$\frac{du}{dp} = \frac{-\rho}{\sqrt{(\lambda t)^2-\rho^2}}$$

$$\sqrt{(\lambda t)^2-\rho^2}\,du = -p\,dp$$

$$du = \frac{-\rho\,dp}{\sqrt{(\lambda t)^2-\rho^2}}$$

So, $\quad u(t,x_1,x_2) = \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\int_{\lambda t}^{\sigma}(-1)[2\pi x_1(x_1+x_2)+\pi(\lambda t)^2-\pi u^2]du$

$$= \frac{\partial}{\partial t}\frac{1}{2\pi\lambda}\left[2\pi x_1(x_1+x_2)u+\pi(\lambda t)^2u-\frac{\pi u^3}{3}\right]\Big|_0^{\lambda t}$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2\lambda}\left(2x_1(x_1+x_2)\lambda t+(\lambda t)^2-\frac{(\lambda t)^2}{3}\right)\right]$$

$$= \frac{\partial}{\partial t}\left[\left(x_1(x_1+x_2)t+\frac{\lambda^2t^2}{2}-\frac{\lambda^2t^2}{6}\right)\right]$$

$$= x_1(x_1 + x_2) + \frac{3\lambda^2 t^2}{2} - \frac{3\lambda^2 t^2}{6}$$
$$= x_1(x_1 + x_2) + \lambda^2 t^2 \left[\frac{3}{2} - \frac{1}{2}\right]$$
$$= x_1(x_1 + x_2) + \frac{\lambda t^2}{2}$$

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