On Tri Complex Representation Methods of a Quaternion Matrix Trace Inequality

K. Gunasekaran[1], M.Rahamathunisha[2]

[(1,2)] *Ramanujan Research Centre, PG and Research Department of Mathematics, Government Arts College (Autonomous), Kumabakonam - 612002*

Abstract :

In this note, the matrix trace inequality for positive semi definite quaternion matrices A and B,$tr(A * B)^m \leq \{tr(A)^{2m} * tr\ (B)^{2m}\}^{1/2}$ is established, where m is an integer.

The above inequality for improves the result given by yang (J. Math Anal Appl. 250 (2000), 372 – 374).

Key Words : Product of quaternion matrix, Positive semi definite, Positive definite, Hermitian matrix.

1.Introduction :

Recently, Yang [5] proved two matrix trace inequalities for positive semi definite matrices $A \in C^{n\times n}$ and $B \in C^{n\times n}$,

$$0 \leq tr(AB)^{2n} \leq (trA)^2(trA^2)^{n-1}\ (trB^2)^n,$$

$$0 \leq tr(AB)^{2n+1} \leq (trA)(trB)\ (trA^2)^n\ (trB^2)^n,$$

$$for\ n = 1,2, \ldots$$

The purpose of this note is improve the above inequalities; our main results are the following inequalities.

2.Definitions :

Definition 2.1: Positive Semi definite.

A matrix A is said to positive semi definite if there exist $X \neq 0$ such that $A\,X \geq 0$.

Definition 2.2: Positive definite.

A matrix A is positive definite if there exists $X \neq 0$ such that $AX > 0$

Note:

A Positive definite matrix is positive semi definite, but the converse need not true.

3.Lemmas:

Let $A \in H^{n\times n}$ with eigenvalues $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ and singular values

$\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)$ respectively. Let $B \epsilon H^{n\times n}$ with eigenvalues $\lambda_1(B), \lambda_2(B), \ldots, \lambda_n(B)$ and singular values $\sigma_1(B), \sigma_2(B), \ldots, \sigma_n(B)$ respectively. They are arranged in such a way that,

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \ldots \geq |\lambda_n(A)| \text{ and}$$

$$|\sigma_1(A)| \geq |\sigma_2(A)| \geq \ldots \geq |\sigma_n(A)|.$$

Similarly,

$|\lambda_1(B)| \geq |\lambda_2(B)| \geq \ldots \geq |\lambda_n(B)|$ and

$|\sigma_1(B)| \geq |\sigma_2(B)| \geq \ldots \geq |\sigma_n(B)|$.

Lemma 3.1: [1,2,3] If A, $B \in H^{n\times n}$, then

$$\prod_{i=1}^{k} \sigma_i(A*B) \leq \prod_{i=1}^{k} \sigma_i(A)*\sigma_i(B) \qquad (1 \leq k < n) \qquad (1)$$

Proof : For any quaternion matrix $A, B \in H^{n\times n}$, then A and B uniquely represented as

A = $A_0 + A_1 j + A_2 k$, $B = B_0 + B_1 j + B_2 k$ and

$$A*B \quad = A_0B_0 + A_1B_1 j + A_2B_2 k$$

$$\sigma_i(A*B) = \sigma_i(A_0B_0 + A_1B_1 j + A_2B_2 k$$

$$= \sigma_i(A_0B_0) + \sigma_i(A_1B_1 j) + \sigma_i(A_2B_2 k)$$

$$\leq \sigma_i(A_0)\sigma_i(B_0) + \sigma_i(A_1)\sigma_i(B_1 j) + \sigma_i(A_2)\sigma_i(B_2 K)$$

$$[\because \sigma_i(A*B) \leq \sigma_i(A)*\sigma_i(B)]$$

$$= (\sigma_i A)*(\sigma_i B)$$

$$\sigma_i(A*B) \leq \sigma_i(A)*\sigma_i(B)$$

$$\prod_{i=1}^{k} \sigma_i(A*B) \leq \prod_{i=1}^{k} \sigma_i\,(A)*\sigma_i(B)$$

The proof is complete.

Lemma 3.2: [3] If $A \in H^{n\times n}$, then

$$\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \sigma_i(A) \qquad (1 \leq k \leq n) \qquad (2)$$

Proof: Let A$\in H^{n\times n}$ with eigenvalues $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ and singular values $\sigma_1(A), \sigma_2(A), \ldots, \sigma_n(A)$ respectively.

For any quaternion matrix A$\in H^{n\times n}$, A can be uniquely represented as A=$A_0 + A_1 j + A_2$k

$$\lambda_i(A) \;\; = \lambda_i(A_0 + A_i j + A_2 k)$$

$$|\lambda_i(A)| = |\lambda_i(A_0 + A_1 j + A_2 k)|$$

$$|\lambda_i(A)| \leq |\lambda_i(A)| + |\lambda_i(A_1 j)| + |\lambda_i(A_2 k)|$$

$$\prod_{i=1}^{k} |\lambda_i(A)| \leq \prod_{i=1}^{k} |\lambda_i(A)| + \prod_{i=1}^{k} |\lambda_i(A_1 j)| + \prod_{i=1}^{k} |\lambda_i(A_2 k)|$$

$$\sigma_i(A) \;\; = \sigma_i(A_0 + A_1 j + A_2 k) = \sigma_i(A_0) + \sigma_i(A_1 j) + \sigma_i(A_2 k)$$

$$\prod_{i=1}^{k}|\sigma_i(A)| \le \prod_{i=1}^{k}|\sigma_i(A_0)|) + \prod_{i=1}^{k}|\sigma_i(A_1 j)| + \prod_{i=1}^{k}|\sigma_i(A_2 k)|)$$

$$\prod_{i=1}^{k}|\lambda_i(A)| \le \prod_{i=1}^{k}\sigma_i(A)$$

The proof is complete.

Lemma 3.3: [1,2,4] Let $x_1, x_{2,\dots,}, x_n$ and $y_1, y_2, \dots, x_n$ *be real numbers*.

$If\ x_1 \ge x_2 \ge \dots \ge x_{n,}\ y_1 \ge y_2 \ge \cdots \ge y_n,\ and$

$$\sum_{i=1}^{k} x_i \le \sum_{i=1}^{k} y_i \ \ for\ k = 1,2,\dots,n, then\ for\ every\ convex\ and\ increasing\ function\ f,$$

$$\sum_{i=1}^{k} f(x_i) \le \sum_{i=1}^{k} f(y_i),\ for\ k = 1,2,\dots,n.$$

Lemma 3.4: Let a_{ij} (i=1, … , n, j=1, … , m) be non negative real numbers and

Let $\alpha_1, \dots, \alpha_m$ be positive numbers such that ${}^1/_{\alpha_1} + \dots + {}^1/_{\alpha_m} = 1$. Then

$$\sum_{i=1}^{n} a_{il\dots} a_{im} \le (\sum_{i=1}^{n} a_{il}^{\alpha_1})^{1/\alpha_1} \dots (\sum_{i=1}^{n} a_{im}^{\alpha_m})^{1/\alpha_m}$$

4. Theorems and Corollary :

Theorem 4.1 : If $A_1, A_2, \dots, A_m \in H^{n\times n}$, then

$$\prod_{i=1}^{k}\sigma_i\left(\prod_{i=1}^{m}A_j\right) \le (\prod_{i=1}^{k}\prod_{j=1}^{m}\sigma_i(A_j)), \qquad 1 \le k \le n$$

$$\boldsymbol{Proof}: \prod_{i=1}^{k}\sigma_i(A_1 \dots A_m) = \prod_{i=1}^{k}\lambda_i((A_1 \dots A_m)^*(A_1 \dots A_m)) = \prod_{i=1}^{k}\lambda_i(A_m^* \dots A_1^* A_1 A_2 \dots A_m)$$

$$= \lambda_1\big((A_m^* \dots A_1^*)(A_1 A_2 \dots A_m)\big)\lambda_2(A_m^* \dots A_1^*)(A_1 A_2 \dots A_m) \dots \lambda_k((A_m^* \dots A_1^*)(A_1 A_2 \dots A_m))$$

$$Now, \prod_{i=1}^{k}\prod_{j=1}^{m}\sigma_i(A_j) = \prod_{i=1}^{k}[\lambda_i(A_1^*A_1)\lambda_i(A_2^*A_2)\dots\lambda_i(A_m^*A_m)]$$

$$= \lambda_1(A_1^*A_1)\lambda_1(A_2^*A_2)\dots\lambda_1(A_m^*A_m)$$

$$\lambda_2(A_1^*A_1)\lambda_2(A_2^*A_2)\dots\lambda_2(A_m^*A_m)\dots.\lambda_k(A_1^*A_1)\dots\lambda_k(A_m^*A_m)$$

So,

$$\prod_{i=1}^{k}\sigma_i(\prod_{i=1}^{m}A_j) \le \prod_{i=1}^{k}\prod_{j=1}^{m}\sigma_i(A_j)$$

The Proof is complete.

Corollary :4.2: If $A_1, A_2, \dots A_m \in H^{n\times n}$ are positive semi definite matrices, then

$$\prod_{i=1}^{k}\left|\lambda_i(\prod_{j=1}^{m}A_j)\right| \le \prod_{i=1}^{k}\prod_{j=1}^{m}\sigma_i(A_j), \qquad 1\le k\le n$$

Proof:

$$\prod_{i=1}^{k}\left|\lambda_i(\prod_{j=1}^{m}A_j)\right| = \prod_{i=1}^{k}|\lambda_i(A_1A_2\dots A_m)| = \prod|\lambda_1(A_1A_2\dots A_m)|$$

$$\prod|\lambda_2(A_1A_2\dots A_m)|\dots\prod|\lambda_k(A_1A_2\dots A_m)|$$

$$\le \prod\sigma_1(A_1\dots A_m)\dots\prod\sigma_k\,(A_1\dots A_m) \qquad [Lemma\ 3.2]$$

$$= \prod_{i=1}^{k}\sigma_i(A_m)\sigma_i(A_{m-1})\dots\sigma_i(A_1) = \prod_{i=1}^{k}\prod_{j=1}^{m}\sigma_i(A_j)$$

$$\ge \prod_{i=1}^{k}\left|\lambda_i(\prod_{j=1}^{m}A_j)\right|$$

$$\therefore \prod_{i=1}^{k}\left|\lambda_i(\prod_{j=1}^{m}A_j)\right| \le \prod_{i=1}^{k}\prod_{j=1}^{m}\sigma_i(A_j)$$

The Proof is complete.

Theorem 4.3: If $A_1, A_2, \dots A_m \in H^{n\times n}$, then

$$\sum_{i=1}^{k}\sigma_i(\prod_{j=1}^{m}A_j) \le \sum_{i=1}^{k}\prod_{j=1}^{m}\sigma_i(A_j), \qquad 1\le k\le n.$$

Proof: For any $k: 1\le k\le n$, without loss of generality, we can suppose that

$$\sigma_r(\prod_{j=1}^{m}A_j) > 0, \sigma_{r+1}(\prod_{j=1}^{m}A_j) = 0, \qquad 1\le r\le k$$

From Theorem 4.1, we have

$$\prod_{j=1}^{m}\sigma_r(A_j) > 0$$

Let

$$y_i = l_n \prod_{j=1}^{m} \sigma_i(A_{j)}, \qquad x_i = l_n \sigma_i(\prod_{j=1}^{m} A_j), \qquad 1 \le i \le r$$

Thus, $x_1 \ge x_2 \ge \cdots \ge x_n$, $\quad y_1 \ge y_2 \ge \cdots \ge y_n$, and

$$\sum_{i=1}^{l} x_i \le \sum_{i=1}^{l} y_i \,, \qquad 1 \le l \le r$$

$By\ Lemma\ 3.3, for\ f = e^x$

$$\sum_{i=1}^{r} \sigma_i(\prod_{j=1}^{m} A_j) \le \sum_{i=1}^{r} \prod_{j=1}^{m} \sigma_i(A_j), \qquad 1 \le r \le k$$

Therefore, from the supposition we obtain

$$\sum_{i=1}^{k} \sigma_i(\prod_{j=1}^{m} A_j) \le \sum_{i=1}^{k} \prod_{j=1}^{m} \sigma_i(A_j), \qquad 1 \le k \le n$$

The Proof is complete.

Theorem 4.4 : Let $A \in H^{n\times n}\ and\ B \in H^{n\times n}\ be\ positive\ semi\ definite\ matrices;$

$then\ we\ have\ \ tr\ (A * B)^m \le \{tr(A)^{2m} * tr(B)^{2m)^{1/2}}\}$, Where m is an integer.

Proof : From the inequality,

$$tr(A * B)^m = \sum_{i=1}^{n} \lambda_i (A * B)^m \le \sum_{i=1}^{n} |\lambda_i((A * B)^m| \le \sum_{i=1}^{n} \sigma_i(A * B)^m \qquad (3)$$

and theorem 4.3 we have,

$$tr(A * B)^m \le \sum_{i=1}^{n} \sigma_i^m(A) * \sigma_i^m(B) \qquad (4)$$

On the other hand, from Lemma 3.4 and positive semi definite of A and B, we have

$$\sum_{i=1}^{n} \sigma_i^m(A) * \sigma_i^m(B) \le \{(\sum_{i=1}^{n} \sigma_i^{2m}(A)) * \sum_{i=1}^{n} \sigma_i^{2m}(B))\}^{1/2} \qquad (5)$$

Note that $\sigma_i^{2m}(A) = \lambda_i^m(A^2)$,

$\sigma_i^{2m}(B) = \lambda_i^m(B^2)$, $\qquad$ i = 1,2, … , n $\qquad$ (6)

Combining inequalities (3) – (5) and

Equation (6), we have

$tr(A * B)^m \le \{tr(A)^{2m} * tr(B)^{2m}\}^{1/2}$, $\qquad$ for m = 1,2, …

The proof is complete.

Using inequality $tr(A * B) \leq tr(A) * tr(B)$ for positive semidefinite matrices of the same order A and B as well as theorem 4.4, it is easy to verify the following corollary, which is the main result in [5].

Corollary 4.5: If $A \in H^{n\times n}$ *and* $B \in H^{n\times n}$, then

$0\leq tr(A * B)^{2n} \leq (trA)^2 * (trA^2)^{n-1}(trB^2)^n$

$0\leq tr(A * B)^{2n+1} \leq (trA) * (trB) * (trA^2)^n * (trB^2)^n$ For n = 1,2, …

Remark : It is easy to show that theorem 4.4 in the paper proper improves the result in [1] by a simple example. [2]

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