A New Class Integral Representation Involving Exton's Triple Hypergeometric Function X_1

Radha Mathur[1] Mukesh M Joshi[2] Rachana Mathur[3]

[1]Department of Mathematics, Govt. Engineering College, Bikaner, India

[2]Department of Mathematics, Govt. College of Engineering & Technology, Bikaner, India

[3]Department of Mathematics, Govt. P.G. Dungar College, Bikaner, India

***Abstract*— The aim of this paper is to obtain eulerian kind generalized single integral which include Exton's triple hypergeometric function X_1. The results are established with the help of generalized Watson's theorem. Special cases have also been obtained.**

***Keywords*—Hypergeometric function, Exton's triple hypergeometric function, generalized Watson's theorem, Eulerian integral representations.**

I. Hypergeometric function Introduction and Preliminaries

In the theory of hypergeometric function of triple variables, a remarkable twenty distinct triple hypergeometric functions have been introduced by Exton [1], which he denoted by $X_i, i = 1$ to 20. He also investigated their twenty Laplace integral representations whose Kernel include the confluent hypergeometric functions and Humbert hypergeometric functions. In this paper, we have established generalized single Euler type integral which involves Exton's function X_1 and hypergeometric function ${}_2F_1$ in its integrand. We recall here the definition of Exton's function X_1.

$$X_1(a, b; c, d; x, y, z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{2m+2n+p}(b)_p x^m y^n z^p}{(c)_m (d)_{n+p}\, m!\, n!\, p!} \tag{1.1}$$

with the condition of convergence given by Srivastava and Karlsson [2].

$$\sqrt{r} + \sqrt{s} < \frac{1}{2} \wedge t < \frac{1}{2}(1 - 2\sqrt{r}) + \frac{1}{2}\sqrt{(1 - 2\sqrt{r})^2 - 4s} \tag{1.2}$$

where,

$|x| = r, |y| = s, |z| = t$

$(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \epsilon c$) by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)}$$

$$= \begin{cases} 1 \quad , & n = 0 \\ \lambda(\lambda + 1) \ldots (\lambda + n - 1), & n \epsilon N \end{cases} \tag{1.3}$$

II. Results Required

$$\int_0^1 x^{c-1}(1 - x)^{c+j-1}\, {}_2F_1\left(a, b; \frac{1}{2}(a + b + 1 + i); x\right) dx$$

$$= \frac{\Gamma(c)\Gamma(c + j)}{\Gamma(2c + j)}\, {}_3F_2\left(\begin{matrix} a \;,\; b \;,\; c \\ \frac{1}{2}(a + b + i + 1), 2c + j \end{matrix}; 1\right)$$

Provided that $\mathrm{Re}(c) > 0, Re(c + j) > 0$ and $\mathrm{Re}(2c - a - b + i + j) > 0, i, j = 0, 1, 2, 3 \ldots\ldots$ (2.1)

Lavoie et al [3] obtained the following contiguous extension of generalized Watson's theorem on the sum of a ${}_3F_2$.

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+j \end{matrix};1\right] $$

$$ = A_{i,j}\frac{2^{a+b+i-2}\,\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{i}{2}+\frac{1}{2}\right)\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}-\frac{|i+j|}{2}-\frac{j}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(a)\Gamma(b)} $$

$$ \times\left\{B_{i,j}\frac{\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1-(-1)^{ij}\right)\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{(-1)^j}{4}\left(1-(-1)^{ij}\right)\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]\right)}\right. $$

$$ \left.+\,C_{i,j}\frac{\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1-(-1)^{ij}\right)\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}\left(1-(-1)^{ij}\right)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)}\right\} \qquad (2.2) $$

for $i, j = 0, \pm 1, \pm 2$ and $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients of $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are given in the tables [3].

Legendre's duplication formula

$$ \sqrt{n}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) \qquad (2.3) $$

III. Main Result

$$ \int_0^1 x^{c-1}(1-x)^{c+j-1}\,{}_2F_1\left(a,b;\frac{1}{2}(a+b+1+i);x\right)X_1(\alpha,\beta;\gamma,\delta;x(1-x),x(1-x),x(1-x))dx $$

$$ = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha)_{2m+2m+p}(\beta)_p}{(\gamma)_m(\delta)_{n+p}\ \ m!\,n!\,p!}\ \mathrm{x}\ \frac{\Gamma(c+m+n+p)\,\Gamma(c+m+n+p+j)}{\Gamma(2(c+m+n+p)+j)} $$

$$ \times A_{i,j}\frac{2^{a+b+i-2}\,\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{i}{2}+\frac{1}{2}\right)\Gamma\left(c+m+n+p+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\Gamma\left(c+m+n+p-\frac{a}{2}-\frac{b}{2}-\frac{|i+j|}{2}-\frac{j}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma(a)\Gamma(b)} $$

$$ \times\left\{B_{i,j}\frac{\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1-(-1)^{ij}\right)\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c+m+n+p-\frac{a}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{(-1)^j}{4}\left(1-(-1)^{ij}\right)\right)\Gamma\left(c+m+n+p-\frac{b}{2}+\frac{1}{2}+\left[\frac{j}{2}\right]\right)}\right. $$

$$\left. + C_{i,j} \frac{\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1-(-1)^{i\,j}\right)\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(c+m+n+p-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}\left(1-(-1)^{i\,j}\right)\right)\Gamma\left(c+m+n+p-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)}\right\} \quad (3.1)$$

IV. Proof of the Main Result

In order to prove the result (3.1), express integral on the left hand side as I, we have

$$I = \int_0^1 x^{c-1}(1-x)^{c+j-1}\ {}_2F_1\left(a, b; \frac{(a+b+1+i)}{2}; x\right) X_1(\alpha, \beta; \gamma, \delta; x(1-x), x(1-x), x(1-x))dx \quad (4.1)$$

Expressing the Exton's function X_1 in series form (1.1) and then changing the order of integration and summation in the left hand side of (4.1), we have

$$I = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha)_{2m+2m+p}(\beta)_p}{(\gamma)_m(\delta)_{n+p}\ m!\,n!\,p!}\int_0^1 x^{c+m+n+p-1}(1-x)^{c+m+n+p+j-1}$$

$$\times\ {}_2F_1\left(a, b; \frac{1}{2}(a+b+1+i); x\right)dx \quad (4.2)$$

on using the result (2.1) in (4.2), we obtain

$$I = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha)_{2m+2m+p}(\beta)_p}{(\gamma)_m(\delta)_{n+p}\ m!\,n!\,p!}\ \frac{\Gamma(c+m+n+p)\,\Gamma(c+m+n+p+j)}{\Gamma(2(c+m+n+p)+j)}$$

$$\times\ {}_3F_2\left[\begin{matrix} a, & b, & c+m+n+p \\ \frac{1}{2}(a+b+i+1), & 2(c+m+n+p)+j & \end{matrix}; 1\right] \quad (4.3)$$

on applying the result (2.2) in (4.3), we obtain the desired result (3.1).

V. Special cases

(i) if we take $i = j = 0$ in (3.1), we have

$$\int_0^1 x^{c-1}(1-x)^c\ {}_2F_1\left(a, b; \frac{1}{2}(a+b+1); x\right) X_1(\alpha, \beta; \gamma, \delta; x(1-x), x(1-x), x(1-x))dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\alpha)_{2m+2m+p}(\beta)_p}{(\gamma)_m(\delta)_{n+p}\ m!\,n!\,p!}\ \frac{\Gamma(c+m+n+p)\,\Gamma(c+m+n+p)}{\Gamma 2(c+m+n+p)}$$

$$\times A_{o,o}\frac{2^{a+b-2}\,\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c+m+n+p+\frac{1}{2}\right)\Gamma\left(c+m+n+p-\frac{a}{2}-\frac{b}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\ \Gamma(a)\Gamma\ (b)}$$

$$\times\left\{B_{o,o}\frac{\Gamma\left(\frac{a}{2}\right)\ \Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c+m+n+p-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c+m+n+p-\frac{b}{2}+\frac{1}{2}\right)}\right.$$

$$+ C_{o,o} \frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(c+m+n+p-\frac{a}{2}\right)\Gamma\left(c+m+n+p-\frac{b}{2}\right)}\Bigg\} \quad (5.1)$$

on putting the values of coefficients $A_{o,o}$, $B_{o,o}$ and $C_{o,o}$ from table[3] an applying the results (2.3) in (5.1), we obtain

$$\int_0^1 x^{c-1}(1-x)^{c-1}\,{}_2F_1\left(a,b;\frac{1}{2}(a+b+1);x\right)X_1(\alpha,\beta;\gamma,\delta;x(1-x),x(1-x),x(1-x))dx$$

$$= \frac{\pi\,\Gamma(c)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}-\frac{1}{2}\right)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\times 2^{1-2c-2m-2n-2p}}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}+\frac{1}{2}\right)}$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\alpha)_{2m+2m+p}(\beta)_p(c)_{m+n+p}\left(c-\frac{a}{2}-\frac{b}{2}-\frac{1}{2}\right)_{m+n+p}}{(\gamma)_m(\delta)_{n+p}\left(c-\frac{a}{2}+\frac{1}{2}\right)_{m+n+p}\left(c-\frac{b}{2}+\frac{1}{2}\right)_{m+n+p} m!\,n!\,p!}$$

(ii) If we take $i = o, j = 1$ in (3.1), we have

$$\int_0^1 x^{c-1}(1-x)^{c}\,{}_2F_1\left(a,b;\frac{1}{2}(a+b+1);x\right)X_1(\alpha,\beta;\gamma,\delta;x(1-x),x(1-x),x(1-x))dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\alpha)_{2m+2m+p}(\beta)_p\,\Gamma(c+m+n+p)\Gamma(c+m+n+p+1)}{(\gamma)_m(\delta)_{n+p}\; m!\,n!\,p!\;\Gamma(2(c+m+n+p)+1)}$$

$$\times A_{o,1}\frac{2^{a+b-2}\,\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c+m+n+p+\frac{1}{2}\right)\Gamma\left(c+m+n+p-\frac{a}{2}-\frac{b}{2}-\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)}$$

$$\times\Bigg\{B_{o,1}\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c+m+n+p-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c+m+n+p-\frac{b}{2}+\frac{1}{2}\right)}$$

$$+ C_{o,1}\frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\Gamma\left(c+m+n+p-\frac{a}{2}+1\right)\Gamma\left(c+m+n+p-\frac{b}{2}+1\right)}\Bigg\} \quad (5.2)$$

on using the result (2.3) and putting the values of the coefficients $A_{o,1}$, $B_{o,1}$, and $C_{o,1}$ from table[3] in (5.2), we have

$$\int_0^1 x^{c-1}(1-x)^{c}\,{}_2F_1\left(a,b;\frac{1}{2}(a+b+1);x\right)X_1(\alpha,\beta;\gamma,\delta;x(1-x),x(1-x),x(1-x))dx$$

$$= \frac{\Gamma(c)\Gamma\left(\frac{a}{2}+\frac{b}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{a}{2}-\frac{b}{2}-\frac{3}{2}\right)}{\Gamma(a)\Gamma(b)}$$

$$\times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(\alpha)_{2m+2m+p}(\beta)_p(c)_{m+n+p}\left(c-\frac{a}{2}-\frac{b}{2}-\frac{3}{2}\right)_{m+n+p}}{(\gamma)_m(\delta)_{n+p}\; m!\,n!\,p!}$$

$$\times\left\{\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}{\left(c-\frac{a}{2}+\frac{1}{2}\right)_{m+n+p}\left(c-\frac{b}{2}+\frac{1}{2}\right)_{m+n+p}\Gamma\left(c-\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(c-\frac{b}{2}-\frac{1}{2}\right)}\right.$$

$$\left.-\frac{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(\frac{b}{2}+\frac{1}{2}\right)}{\left(c-\frac{a}{2}+1\right)_{m+n+p}\left(c-\frac{b}{2}+1\right)_{m+n+p}\Gamma\left(c-\frac{a}{2}+1\right)\Gamma\left(c-\frac{b}{2}+1\right)}\right\}$$

VI. CONCLUSION

A number of special cases for various values of i, j can derive. Also a variety of integral representations can establish for hypergeometric functions.

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