Contra θgs-Continuous Functions

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***Abstract*—The aim of this paper is to introduce and study of a new generalization of contra continuity called contra θgs-continuous functions utilizing θgs-closed set.**

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I. INTRODUCTION

In 1996, Dontchev [4] introduced the notion of contra continuity and strong S-closedness in topological spaces. A new weaker form of this class of functions called contra semi continuous function is introduced and investigated by Dontchev and Noiri [5]. Recently in [8] the notion of θ-generalized semi closed (briefly, θgs-closed) set was introduced. The aim of this paper is to introduce and study of a new generalization of contra continuity called contra θgs-continuous functions utilizing θgs-closed set.

II. PRELIMINARIES

Throughout this paper (X,τ) and (Y,σ) (or simply X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of space X, then Cl(A) and Int(A) denote the closure of A and the interior of A in X respectively.
The following definitions are useful in the sequel:

Definition 2.1: A subset A of space X is called
(i) a semi-open set [7] if $A \subseteq Cl(Int(A))$
(ii) a semi-closed set [1] if $Int(Cl(A)) \subseteq A$

Definition 2.2 [3]**:** A point $x \in X$ is called a semi-θ-cluster point of A if $A \cap sCl(U) \neq \emptyset$ for each semi-open set U containing x. The set of all semi-θ-cluster point of A is called semi- θ-closure of A and is denoted by $sCl_\theta(A)$. A subset A is called semi- θ-closed if $sCl_\theta(A) = A$. The complement of semi-θ-closed set is semi-θ-open set.

Definition 2.3 [8]**:** A subset A of a topological space X is called θ-generalized-semi closed (briefly, θgs-closed) if $sCl_\theta(A) \subset U$, whenever $A \subset U$ and U is open in X. The complement of θgs-closed set is θ-generalized-semi open (briefly, θgs-open). We denote the family of θgs-closed sets of X by $\theta GSC(X, \tau)$ and θgs-open sets by $\theta GSO(X, \tau)$.

Definition 2.4 [8]**:** The intersection of all θgs-closed sets containing A is called θgs-closure of A and is denoted by θgsCl(A). A set A is θgs-closed set if and only if θgsCl(A) = A.

Definition 2.5**:** A topological space X is called
(i) T_b-space [2] if every gs-closed set of X is closed set.
(ii)$T_{\theta gs}$-space [12] if every θgs-closed set in it is closed set.

Definition 2.6 [11]**:** A topological space X is called θgs-T_2 if for each pair of distinct points x and y of X, there exist disjoint θgs-open sets, one containing x and other containing y.

Definition 2.7 [9]**:** A function f: X → Y is called
(i) θ-generalized semi-continuous (in briefly,θ-gs-continuous), if f^{-1} (F) is θ-gs-closed in X for every closed set F of Y.
(ii) θ-generalized semi-irresolute (in briefly, θ-gs-irresolute), if f^{-1}(F) is θ-gs-closed in X for every θ-gs-closed set F of Y.

Definition 2.8 [10]: A function f: X →Y is said to be θgs-open (resp., θgs-closed) if f (V) is θgs-open (resp., θgs-closed) in Y for every open set (resp., closed) V in X.

Definition 2.9 [15]: (i) A topological space X is called Ultra Hausdroff space, if for every pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X containing x and y respectively.
(ii) A topological space X is said to be ultra normal if each pair of disjoint closed sets can be separated by disjoint clopen sets.

Definition 2.10 [13]: A topological space X is said to be θgs-normal if each pair of disjoint closed sets can be separated by disjoint θgs-open sets.

Definition 2.11 [14]: A topological space X is said to be θgs-connected if X cannot be written as disjoint union of two non-empty θgs-open sets.

III. CONTRA θgs-CONTINOUS FUNCTIONS

In this section, the notion of a new class of function called contra θgs-continuous functions is introduced and obtain some of their characterizations and properties. Also, the relationships with some other related functions are discussed.

Definition 3.1: A function f: X→Y is said to be contra θgs-continuous if f^{-1}(V) is θgs-closed set in X for each open set V of Y.

Remark 3.2: From the following example, it is clear that both contra θgs-continuous and θgs-continuous both are independent notions of each other.

Example 3.3: Let X=Y= {a, b, c, d} and τ = {X, ϕ, {b}, {d}, {a, b, d}, {b, c, d}}, σ = {Y, ϕ, {b, d}, {a}} be topologies on X and Y respectively. We have θgs-closed sets in X are {X, ϕ, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {c, d}, {a, b, c}, {a, c, d}}.
(i) Define a function f: X→Y by f(a) = c, f(b) = b, f(c) = a and f(d) = d. Then f is θgs-continuous function but not contra θgs-continuous, because for open set {b, d} in Y, f^{-1}({b, d}) = {b, d} is not θgs-closed set in X.
(ii) Define a function g: X→Y by g(a) = b, g(b) = c, g(c) = d and g(d) = a. Then g is contra θgs-continuous function but not θgs-continuous, because for closed set {a, c} in Y, g^{-1}({a, c}) = {b, d} is not θgs-closed set in X.

Theorem 3.4: If f: X→Y is contra continuous, then it is contra θgs -continuous.
Proof: Let V be an open set in Y. Since f is contra continuous, f^{-1}(V) is closed in X. Since every closed set is θgs-closed, f^{-1}(V) is θgs -closed in X. Therefore f is contra θgs -continuous.

Remark 3.5: Converse of the above theorem need not be true in general as seen from the following example.

Example 3.6: Let X= {a, b, c}, Y= {a, b, c, d} and τ = {X, ϕ, {b}, {c}, {b, c}}, σ = {Y, ϕ, {a, b}} be topologies on X and Y respectively. We have θgs-closed sets in X are {X, ϕ, {a}, {b}, {c}, {a, b}, {a, c}}. Define a function f: X→Y by f(a) = c, f(b) = d, f(c) = b. Then f is contra θgs-continuous function but not contra-continuous, because for open set {a, b} in Y, f^{-1}({a, b}) = {c, d} is not closed set in X.

Lemma 3.7 [6]: The following properties hold for a subsets A and B of a space X

(i) x $\in$ ker(A) if and only if $A \cap F \neq \emptyset$ for any closed set F of X containing x.

(ii) A $\subset$ ker(A) and A = ker(A) if A is open in X.

(iii) If A $\subset$ B, then ker(A) $\subset$ ker(B).

Theorem 3.8: If f: X→Y is a function, then the following are equivalent.

(i) f is contra θgs-continuous.

(ii) For every closed set F of Y, f^{-1}(F) is θgs-open set of X.

(iii) For each x $\in$ X and each closed set F of Y containing f(x), there exists θgs-open set U containing x such that f(U) $\subset$ F.

(iv) For each x $\in$ X and each open set V of Y containing f(x), there exists θgs-closed set K not containing x such that f^{-1}(V) $\subset$ K.

(v) f (θgsCl(A)) $\subset$ ker(A) for every subset A of X.

(vi) θgsCl(f^{-1}(B)) $\subset$ f^{-1}(ker(B)) for every subset B of Y.

Proof: (i) $\Rightarrow$ (ii) Let F be a closed set in Y, then Y- F is an open set in Y. By (i), f^{-1}(Y-F) = X- f^{-1}(F) is θgs-closed set in X. This implies f^{-1}(F) is θgs-open set in X. Therefore, (ii) holds.

(ii) $\Rightarrow$ (i) Let G be an open set of Y. Then Y-G is a closed set in Y. By (ii), f^{-1}(Y-G) is θgs-open set in X. This implies X- f^{-1}(G) is θgs-open set in X, which implies f^{-1}(G) is θgs-closed set in X. Therefore, (i) holds.

(ii) $\Rightarrow$ (iii) Let F be a closed set in Y containing f(x), then x$\in$ f^{-1}(F). By (ii), f^{-1}(F) is θgs-open containing x. Let U= f^{-1}(F), implies f(U) = f(f^{-1}(F)) $\subset$ F. Therefore (iii) holds.

(iii) $\Rightarrow$ (ii) Let F be a closed set in Y containing f(x), then x$\in$ f^{-1}(F). From (iii), there exists θgs-open U_x containing x such that f(U_x) $\subset$ F. That is $U_x \subset f^{-1}$(F). Thus f^{-1}(F) = $\cup$\{U_x: x$\in$ f^{-1}(F)\} is a f^{-1}(F) is θgs-open set of X.

(iii) $\Rightarrow$ (iv) Let V be an open set in Y not containing f(x). Then Y-V is closed set in Y containing f(x). From (iii), there exists a θgs-open set U in X containing x such that f(U) $\subset$ Y-V. This implies U$\subset$ f^{-1}(Y-V) = X- f^{-1}(V). Hence, f^{-1}(V) $\subset$ X- U, Set K= X- U, is θgs-closed set not containing x in X.

(iv) $\Rightarrow$ (iii) Let F be a closed set in Y containing f(x). Then Y-F is an open set in Y not containing f(x). From (iv), there exists θgs-closed set K in X not containing x such that f^{-1}(Y-F) $\subset$ K. This implies X- f^{-1}(F) $\subset$ K. Hence, X- K $\subset$ f^{-1}(F), that is f(X- K) $\subset$ F and X- K is θgs-open set containing x in X.

(ii) ⇒ (v) Let A be any subset of X. Suppose $y \notin ker(f(A))$. Then by lemma 3.7, there exists a closed set F in Y containing y such that $f(A) \cap F = \emptyset$. Thus we have, $A \cap f^{-1}(F) = \emptyset$. Therefore $A \subset X - f^{-1}(F)$. By (ii), $f^{-1}(F)$ is θgs-open set in X and hence $X - f^{-1}(F)$ is θgs-closed set in X. Therefore, $\theta gsCl(X - f^{-1}(F)) = X - f^{-1}(F)$. Now $A \subset X - f^{-1}(F)$, implies $\theta gsCl(A) \subset \theta gsCl(X - f^{-1}(F)) = X - f^{-1}(F)$. Therefore $\theta gsCl(A) \cap f^{-1}(F) = \emptyset$, implies $f(\theta gsCl(A)) \cap F = \emptyset$ and $y \notin \theta gsCl(A)$. Hence $f(\theta gsCl(A)) \subset ker(A)$ for every subset A of X.

(v) ⇒ (vi) Let $B \subset Y$, then $f^{-1}(B) \subset X$. By (iv) and Lemma 3.7, we have, $f(\theta gsCl(f^{-1}(B))) \subset ker(f(f^{-1}(B))) \subset Ker(B)$. Thus, $\theta gsCl(f^{-1}(B)) \subset f^{-1}(ker(B))$ for every subset B of Y.

(vi) ⇒(i) Let V be any open subset of Y. Then by (vi) and lemma 3.7, $\theta gsCl(f^{-1}(V) \subset f^{-1}(ker(V)) = f^{-1}(V)$ and $\theta gsCl(f^{-1}(V)) = f^{-1}(V)$. Therefore, $f^{-1}(V)$ is θgs-closed set in X. This shows that f is contra θgs-continuous.

Remark 3.9: If a function f: X→Y is contra θgs–continuous and X is $T_{\theta gs}$-space, then f is contra continuous.

Definition 3.10: A space X is called *locally θgs-indiscrete* if every θgs-open set is closed set in X.

Theorem 3.11: If a function f: X→Y is contra θgs–continuous and X is locally θgs-indiscrete space, then f is continuous.

Proof: Let U be an open set in Y. Since f is contra θgs-continuous and X is locally θgs-indiscrete space, implies $f^{-1}(U)$ is an open set in X. Therefore f is continuous.

***Theorem 3.12*:** (i) If a function f: X→Y is θgs-coninuous and X is locally θgs-indiscrete space, then f is contra continuous.
(ii) If a function f: X→Y is contra θgs-continuous and X is T_b-space then f is contra continuous

Proof: (i) Let V be an open set in Y. Since f is θgs-continuous, $f^{-1}(V)$ is θgs-open set in X and X is locally θgs-indiscrete space, implies $f^{-1}(U)$ is a closed set in X. Therefore f is contra continuous.

(ii) Let V be an open set in Y. Since f is contra θgs-continuous, $f^{-1}(V)$ is θgs-closed set in X and every θgs-closed set is gs-closed set and X is T_b-space, implies $f^{-1}(V)$ is closed set in X. Therefore f is contra continuous.

Theorem 3.13: If f: X→Y is a contra θgs-continuous from a θgs-connected space X onto any space Y, then Y is not a discrete space.

Proof: Let f: X→Y is a contra θgs-continuous and X is θgs-connected space. Suppose Y is a discrete space. Let A be a proper non empty open and closed subset of Y. Then $f^{-1}(A)$ is a proper non empty θgs-open and θgs-closed subset of X, which is contradiction to the fact that X is θgs-connected space. Therefore, Y is not a discrete space.

Theorem 3.14: If f: X→Y is a contra θgs-continuous surjection and X is θgs-connected space, then Y is connected.

Proof: Let f: X→Y is a contra θgs-continuous and X is θgs-connected space. Suppose Y is a not connected space. Then there exist disjoint open sets U and V such that $Y = U \cup V$. Therefore U and V are clopen in Y. Since f is contra θgs-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are θgs-open sets in X. Further f is surjective, $f^{-1}(U)$ and $f^{-1}(V)$ are non empty disjoint and $X= f^{-1}(U) \cup f^{-1}(V)$. This shows that X is not θgs-connected space. This is contradiction. Therefore, Y is connected

Theorem 3.15: Let X be θgs-connected space and Y T_1, if f: X→Y is contra θgs-continuous, then f is constant.

Proof: Let f: X→Y is contra θgs-continuous, X be θgs-connected space and Y is T_1. Since Y is T_1 space, $\Delta = \{f^{-1}(V): y\in Y\}$ is a disjoint θgs-open partition of X. If $|\Delta| \geq 2$, then X is union of two non-empty θgs-open sets. This is contradiction to the fact that X is θgs-connected. Therefore $|\Delta| = 1$ and hence f is constant.

Theorem 3.16: If f: X→Y is contra θgs-continuous injective function from space X into a Hausdroff space Y, then X is θgs-T_2.

Proof: Let x and y be any two distinct points in X. Since f is injective $f(x) \neq f(y)$ and Y is Hausdroff space, implies there exist disjoint clopen sets U and V of Y containing f(x) and f(y) respectively. Then $x \in f^{-1}(U)$ and $y \in f^{-1}(V)$, where $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint θgs-open sets in X. Therefore X is θgs-T_2.

Theorem 3.17: If f: X→Y is contra θgs-continuous closed injection and Y is ultra normal, then X is θgs-normal.

Proof: Let E and F be disjoint closed subsets of X. Since f is closed and injective f(E) and f(F) are disjoint closed sets in Y. Since Y is ultra normal there exist disjoint clopen sets U and V in Y such that $f(E) \subset U$ and $f(F) \subset V$. This implies $E\subset f^{-1}(U)$ and $F \subset f^{-1}(V)$. Since f is contra θgs-continuous injection, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint θgs-open sets in X. This shows X is θgs-normal.

***Remark 3.18*:** If f: X→Y is contra θgs-continuous and g: Y→Z is continuous, then g∘f: X→Z is contra θgs-continuous.

***Theorem 3.19*:** Let f: X→Y is contra θgs-continuous and g: Y→Z is θgs-continuous. If Y is $T_{\theta gs}$-space, then g∘f: X→Z is contra θgs-continuous.

Proof: Let V be any open set in Z. Since g is θgs-continuous $g^{-1}(V)$ is θgs-open in Y and Y is $T_{\theta gs}$-space implies $g^{-1}(V)$ open in Y. Since f is contra θgs-continuous $f^{-1}(g^{-1}(V)) = (g\circ f)^{-1}(V)$ is θgs-closed sets in X. Therefore, g∘f is contra θgs-continuous.

Definition3.20: If f: X→Y is said to be strongly θgs-open (or strongly θgs-closed) if image of every θgs-open (resp. θgs-closed) set of X is θgs-open (resp. θgs-closed) set in Y.

***Theorem 3.21*:** If f: X→Y is surjective θgs-open (or θgs-closed) and g: Y→Z is a function such that g∘f: X→Z is contra θgs-continuous, then g is contra θgs-continuous

Proof: Let V be any closed (resp. open) set in Z. Since g∘f is contra θgs-continuous, implies $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V)$ is θgs-open (resp. θgs-closed). Since f is surjective and strongly θgs-open (or strongly θgs-closed), implies $f(f^{-1}(g^{-1}(V))= g^{-1}(V)$ is θgs-open (or θgs-closed). Therefore g is contra θgs-continuous.

Definition3.22: A space X is said to be

(i) θgs-closed compact if every closed cover of X has a finite subcover.
(ii) Countably θgs-closed compact if every countable cover of X by θgs-closed sets has a finite subcover.

Theorem 3.23**:** Let f: X→Y be a contra θgs-continuous surjection. Then the following properties hold.
(i)If X is θgs-closed compact, then Y is compact.
(ii)If X is countably θgs-closed compact, then Y is countably compact.

Proof: (i) Let $\{V_\alpha : \alpha \in I\}$ be an open cover of Y. Since f is contra θgs-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is θgs-closed cover of X. Since X is θgs-closed compact, there exists a finite subset I_o of I such that $X = \cup \{f^{-1}(V_\alpha) : \alpha \in I_o\}$. Since f is surjective, $Y = \cup \{f^{-1}(V_\alpha) : \alpha \in I_o\}$, which is finite subcover of Y. Therefore Y is compact.

(ii) Let $\{V_\alpha : \alpha \in I\}$ be countable open cover of Y. Since f is contra θgs-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable θgs-closed cover of X. Since X is countably θgs-closed compact, there exists a finite subset I_o of I such that $X = \cup \{f^{-1}(V_\alpha) : \alpha \in I_o\}$. Since f is surjective, $Y = \cup \{f^{-1}(V_\alpha) : \alpha \in I_o\}$, is finite subcover of Y. Therefore Y is countably compact.

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