The Analysis Solutions for Two-Dimensional Fractional Diffusion Equations with Variable Coefficients

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***Abstract*—This paper deals with a fractional diffusion equation with variable coefficients developed by a non-local method with temporal and spatial correlations. The time-fractional derivative is described in the Caputo sense while the space-fractional derivatives are described in the Riemann-Liouville sense. The variational iteration method is used to derive the solutions. Two examples are given to demonstrate the validity of the method.**

***Keywords*—Variational iteration method, Fractional differential equation, Caputo derivative, Diffusion equation.**

I. INTRODUCTION

Recently, considerable attention has been paid to the problem of how to predict the behavior of fractional differential equations. Fractional diffusion equations are used to model problems in many fields such as engineering, physical, chemistry, and so on. The fluid-dynamic traffic model with fractional derivative [1]-[2] can be used to model the anomalous diffusion of particle in porous medium, and can eliminate the deficiency arising from the assumption of continuum traffic flow [3]-[4]. Based on the experimental data of anomalous diffusion of particle, the differential equations with fractional order have been proved to be effective to model many physical phenomena.

Most nonlinear fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques are used extensively. The numerical solutions are often obtained by difference scheme [4]-[6]. There are several methods for approximate solutions, such as the variational iteration method [7], the adomian decomposition method [7]-[8], the homotopy perturbation method and so on [9].

Motivated by the above mentioned works, we investigate the applications of fractional diffusion equation. A special emphasis is given on the formulation of fractional diffusion equation, which provides approximation analytic solution. Fractional diffusion equations with variable coefficients are developed by a non-local method with temporal and spatial correlations based on the classical local 2-nd order advection-diffusion developed. The time-space fractional advection diffusion equation (TSFADE) with various coefficients on a finite domain is obtained from the standard diffusion equation by replacing the first-order time derivative by the Caputo fractional derivative of order $0 < \alpha \le 1$, and the second-order space derivatives by the Riemann-Liouville fractional derivatives of order $1 < \beta,\ \ \gamma \le 2$. The variational iteration method is employed to derive the solutions.

The principles of the variational method and its applicability for various kinds of differential equations are given in [7]. He [2] applied the variational iteration method to fractional differential equations, firstly. Recently Odibat and Momani implemented the variational iteration method to solve time-fractional partial differential equations [7]. In this paper, we apply the variational iteration method to solve time-space fractional derivatives.

II. NON-LOCAL METHOD AND CONTROL EQUATION

The classical advection-diffusion equation based on the assumption of continuum traffic flow, but actual medium has a large number of fractal structures which can induce anomalous diffusion. In order to eliminate the deficiency, Chang *et al*. [10] extend the classical Fick's law for standard diffusion to a general fractional Fick's law, and develop a non-local method with temporal and spatial correlations to introduce a fractional order advection-diffusion equation.

In this paper,we consider the following two-dimensional fractional diffusion equation

$$\frac{\partial^{\alpha}u(x,y,t)}{\partial t^{\alpha}} = f(x,y)\frac{\partial^{\beta}u(x,y,t)}{\partial x^{\beta}} + g(x,y)\frac{\partial^{\gamma}u(x,y,t)}{\partial y^{\gamma}} + q(x,y,t)$$

$$0 < x < a\,, 0 < y < b\,, 0 < \alpha \le 1, 1 < \beta,\ \ \gamma \le 2. \tag{1}$$

Where $u(x, y, t)$ is the solute concentration, $f(x, y) > 0$ and $g(x, y) > 0$ represent the diffusion coefficients, and the forcing function $q(x, y, t)$ can be used to represent sources and sinks.

The initial condition:

$$u(x, y, 0) = \psi(x, y). \tag{2}$$

The boundary conditions:

$$u(0, y, t) = f_1(y, t)\,,\quad u(a, y, t) = f_2(y, t)\,,\quad u(x, 0, t) = g_1(x, t)\,,\quad u(x, b, t) = g_2(x, t). \tag{3}$$

The time fractional derivative ${}_0D_t^{\alpha} = \dfrac{\partial^{\alpha} c(x,t)}{\partial t^{\alpha}}$ is Caputo fractional derivative [5]:

$$\frac{\partial^{\alpha} c(x, y, t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial c(x, y, \tau)}{\partial \tau} d\tau \tag{4}$$

While the space fractional derivatives $\dfrac{\partial^{\gamma} u(x, y, t)}{\partial y^{\gamma}}$ and $\dfrac{\partial^{\beta} u(x, y, t)}{\partial x^{\beta}}$ are the Riemann-Liouville fractional derivatives [1]:

$$\frac{\partial^{\beta} u(x, y, t)}{\partial x^{\beta}} = D_x^{\beta} u(x, y, t) = \frac{1}{\Gamma(m-\beta)} \frac{d^m}{d\xi^m} \int_0^t (x-\xi)^{m-\beta-1} u(\xi, y, t) d\xi. \tag{5}$$

$$\frac{\partial^{\gamma} u(x, y, t)}{\partial y^{\gamma}} = D_y^{\gamma} u(x, y, t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{d\eta^n} \int_0^t (y-\eta)^{n-\gamma-1} u(x, \eta, t) d\eta. \tag{6}$$

III. VARIATION ITERATION METHOD

Variational iteration method was first proposed by the mathematician He [2]. This method has been employed to solve a large variety of linear and nonlinear problems with approximations converging rapidly to accurate solutions. According to the variational iteration method, the correction functional for Eq. (1) can be approximately expressed as follows:

$$u_{k+1}(x, y, t) = u_k(x, y, t) + \int_0^t \lambda(\xi) \left(\frac{\partial^m}{\partial \xi^m} u_k(x, y, \xi) - f(x, y) \frac{\partial^{\beta}}{\partial x^{\beta}} \tilde{u}_k(x, y, \xi) - g(x, y) \frac{\partial^{\gamma}}{\partial y^{\gamma}} \tilde{u}_k(x, y, \xi) - q(x, y, \xi) \right) d\xi \tag{7}$$

Where λ is a general Lagrange multiplier, which can be identified optimally by variational theory;here u_n is the n th-order approximate, and $\dfrac{\partial^{\beta} \tilde{u}_k}{\partial x^{\beta}}$, $\dfrac{\partial^{\gamma} \tilde{u}_k}{\partial y^{\gamma}}$ are considered as restricted variations, i.e., $\delta \tilde{u}_n = 0$. Making the above functional stationary,

$$\delta u_{k+1}(x, y, t) = \delta u_k(x, y, t) + \delta \int_0^t \lambda(\xi) \left(\frac{\partial^m}{\partial \xi^m} u_k(x, y, \xi) - q(x, y, \xi) \right) d\xi \tag{8}$$

For $m = 1$, we obtain for Eq. (8) the following stationary conditions

$$1 + \lambda(t)|_{\xi = t} = 0, \tag{9}$$

$$\lambda'(\xi) = 0. \tag{10}$$

Therefore, the general Lagrange multiplier can be identified:

$$\lambda(\xi) = -1, \tag{11}$$

and we obtain the following iteration formula:

$$u_{k+1}(x, y, t) = u_k(x, y, t) - \int_0^t \left(\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} u_k(x, y, \xi) - f(x, y) \frac{\partial^{\beta}}{\partial x^{\beta}} \tilde{u}_k(x, y, \xi) - g(x, y) \frac{\partial^{\gamma}}{\partial y^{\gamma}} \tilde{u}_k(x, y, \xi) - q(x, y, \xi) \right) d\xi\,. \tag{12}$$

Here, we begin with the initial approximation

$$u_0(x, y, t) = \psi(x, y). \tag{13}$$

Consequently, the exact solution is obtained as

$$u(x, y, t) = \lim_{k \to \infty} u_k(x, y, t). \tag{14}$$

The convergence of the variation iteration method has been investigated in [11]. He obtained some results about the speed of convergence of this method.

IV. APPLICATION AND RESULTS

In this section, we present two examples to demonstrate that the variation iteration method is effective and the results are in good agreement with the exact solutions.

Example 1

We consider the time-space fractional diffusion equation with variable coefficients on a finite domain:

$$\frac{\partial^{0.8}u(x,y,t)}{\partial t^{0.8}} = f(x,y)\frac{\partial^{1.8}u(x,y,t)}{\partial x^{1.8}} + g(x,y)\frac{\partial^{1.5}u(x,y,t)}{\partial y^{1.5}} + q(x,y,t),$$

$$0 < x < 1, 0 < y < 1, 0 < \alpha \le 1, 0 \le t \le T_{end}. \tag{15}$$

Diffusion coefficients are

$$f(x,y) = x^{1.8}/\Gamma(3.8), \qquad g(x,y) = \Gamma(2.5)y^{1.5}/6. \tag{16}$$

The source term is

$$q(x,y,t) = -2(t^2 - t^{1.2}/\Gamma(2.2) + 1)x^{2.8}y^3. \tag{17}$$

The initial condition:

$$u(x,y,0) = x^{2.8}y^3. \tag{18}$$

Boundary conditions:

$$u(0,y,t) = u(x,0,t) = 0, \quad u(1,y,t) = y^3(1+t^2), \quad u(x,1,t) = x^{2.8}(1+t^2). \tag{19}$$

when $\alpha = 0.8$, equation（15）has exact solution

$$u(x,y,t) = x^{2.8}y^3(1+t^2). \tag{20}$$

To solve the problem by the variational iteration method, we use formula (12) to construct the iteration formula for Eq. (15) as follows:

$$u_{k+1}(x,y,t) = u_k(x,y,t) - \int_0^t \left(\frac{\partial^{0.8}}{\partial \xi^{0.8}} u_k(x,y,\xi) - f(x,y)\frac{\partial^{1.8}}{\partial x^{1.8}}\tilde{u}_k(x,y,\xi) - g(x,y)\frac{\partial^{1.5}}{\partial y^{1.5}}\tilde{u}_k(x,y,\xi) - q(x,y,\xi) \right) d\xi \tag{21}$$

Using the above variational iteration formula with $u_0(x,y,t) = x^{2.8}y^3$, we can obtain the following approximations

$$u_0(x,y,t) = x^{2.8}y^3, \tag{22}$$

$$u_1(x,y,t) = x^{2.8}y^3(1 + \frac{2}{\Gamma(3.2)}t^{2.2}), \tag{23}$$

$$u_2(x,y,t) = x^{2.8}y^3(1 + \frac{4}{\Gamma(3.2)}t^{2.2} - \frac{2}{\Gamma(3.4)}t^{2.4} - \frac{2}{3}t^3 + \frac{4}{\Gamma(4.2)}t^{3.2}), \tag{24}$$

$$u_3(x,y,t) = x^{2.8}y^3(1 + \frac{6}{\Gamma(3.2)}t^{2.2} - \frac{6}{\Gamma(3.4)}t^{2.4} + \frac{2}{\Gamma(3.6)}t^{2.6} - \frac{4}{3}t^3 + \frac{16}{\Gamma(4.2)}t^{3.2} - \frac{8}{\Gamma(4.4)}t^{3.4} - \frac{1}{3}t^4 + \frac{8}{\Gamma(5.2)}t^{4.2}) \tag{25}$$

$$u_4(x,y,t) = x^{2.8}y^3(1 + \frac{8}{\Gamma(3.2)}t^{2.2} - \frac{12}{\Gamma(3.4)}t^{2.4} + \frac{8}{\Gamma(3.6)}t^{2.6} - \frac{2}{\Gamma(3.8)}t^{2.8} - 2t^3$$

$$+ \frac{36}{\Gamma(4.2)}t^{3.2} - \frac{36}{\Gamma(4.4)}t^{3.4} + \frac{12}{\Gamma(4.6)}t^{3.6} - t^4 + \frac{48}{\Gamma(5.2)}t^{4.2} - \frac{24}{\Gamma(5.4)}t^{4.4} - \frac{2}{15}t^5 + \frac{16}{\Gamma(6.2)}t^{5.2}), \tag{26}$$

$$\vdots$$

in this manner the rest of components can be obtained using the Mathematica package.

Fig.1 shows the comparison of the approximate solution and the exact solution when $x = y = 0.5$. It is obvious that the approximate results are in good agreement with the exact ones.

Fig.2 indicates the approximation of $u(x,y,t)$ with the variation of k.

It is seen that the efficiency of this approaches can be dramatically enhanced by computing further terms when the variational iteration method is used

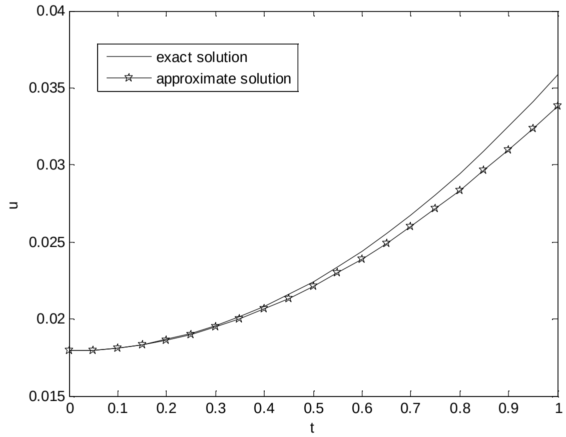

Fig.1. When $x = y = 0.5$, comparing the exact solution with the approximate solution.

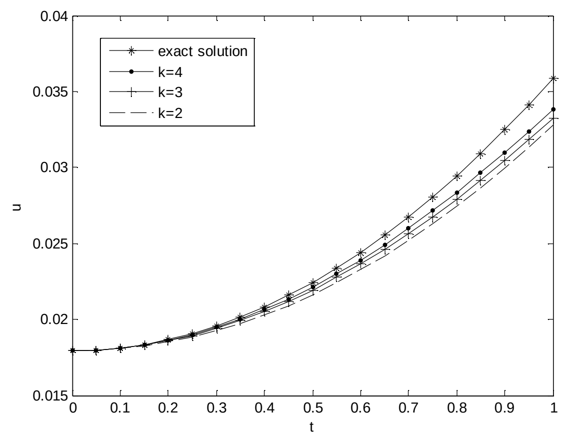

Fig.2. Comparing with the exact solution, the characteristic of $u(x, y, t)$ with the variation of k.

Example 2

In this example we consider the following two-dimensional fractional heat-like problem

$$D_t^{\alpha} u = \frac{1}{2}(x^2 u_{xx} + y^2 u_{yy}) + x^2 + y^2, \qquad 0 < x, y < 1,\ 0 < \alpha \le 1,\ t > 0 \tag{27}$$

Subject to the following boundary conditions:

$$u(0, y, t) = y^2(2e^t - 1), \qquad u(x, 0, t) = x^2(2e^t - 1),$$
$$u(1, y, t) = (1 + y^2)(2e^t - 1), \qquad u(x, 1, t) = (1 + x^2)(2e^t - 1). \tag{28}$$

And the initial condition:

$$u(x, y, 0) = x^2 + y^2. \tag{29}$$

When $\alpha = 1$, which is easily seen to have exact solution

$$u(x, y, t) = -(x^2 + y^2)(1 - 2e^t). \tag{30}$$

To solve the problem using the variational iteration method, we use formula (12) to construct the iteration formula for Eq. (27) as follows:

$$u_{k+1}(x,y,t)=u_k(x,y,t)-\int_0^t\left(\frac{\partial^\alpha}{\partial\xi^\alpha}u_k(x,y,\xi)-\frac{1}{2}x^2\frac{\partial^2}{\partial x^2}\tilde{u}_k(x,y,\xi)-\frac{1}{2}y^2\frac{\partial^2}{\partial y^2}\tilde{u}_k(x,y,\xi)-x^2-y^2\right)d\xi \quad (31)$$

Using the above variational iteration formula with $u_0(x,y,t)=x^2+y^2$, we can obtain the following approximations

$$u_0(x,y,t)=x^2+y^2, \quad (32)$$

$$u_1(x,y,t)=(x^2+y^2)(1+2t), \quad (33)$$

$$u_2(x,y,t)=(x^2+y^2)[1+4t+t^2-\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}], \quad (34)$$

$$u_3(x,y,t)=(x^2+y^2)[1+6t+3t^2+\frac{1}{3}t^3-\frac{6}{\Gamma(3-\alpha)}t^{2-\alpha}-\frac{4}{\Gamma(4-\alpha)}t^{3-\alpha}+\frac{2}{\Gamma(4-2\alpha)}t^{3-2\alpha}], \quad (35)$$

$$u_4(x,y,t)=(x^2+y^2)[1+8t+6t^2+\frac{4}{3}t^3+\frac{1}{12}t^4-\frac{12}{\Gamma(3-\alpha)}t^{2-\alpha}-\frac{16}{\Gamma(4-\alpha)}t^{3-\alpha}$$
$$-\frac{6}{\Gamma(5-\alpha)}t^{4-\alpha}+\frac{8}{\Gamma(4-2\alpha)}t^{3-2\alpha}+\frac{6}{\Gamma(5-2\alpha)}t^{4-2\alpha}-\frac{2}{\Gamma(5-3\alpha)}t^{4-3\alpha}], \quad (36)$$

$\vdots$

in the same manner the rest of components can be obtained using the Mathematica package.

We can see that the solution for the standard heat-like equation ($\alpha=1$) is given by

$$u(x,y,t)=(x^2+y^2)(1+2t+t^2+\frac{1}{3}t^3+\frac{1}{12}t^4), \quad (37)$$

and the solution in a closed form

$$u(x,y,t)=-(x^2+y^2)(1-2e^t), \quad (38)$$

is readily obtained.

Fig.3 shows the comparison of the approximate solution and the exact solution when $x^2+y^2=1$. It is obvious that the approximate results are in good agreement with the exact ones.

Fig.4 indicates the approximation of $u(x,y,t)$ with the variation of k.

It is seen that the efficiency of this approaches can be dramatically enhanced by computing further terms when the variational iteration method is used.

Fig.5 indicates the characteristics of $u(x,y,t)$ with different values of α.

It is seen that the approximate solution of Eq. (27) obtained for different values of α using the variational iteration method. For the case of $\alpha=1$, we know that the exact solution and the approximate solution by variational iteration method are

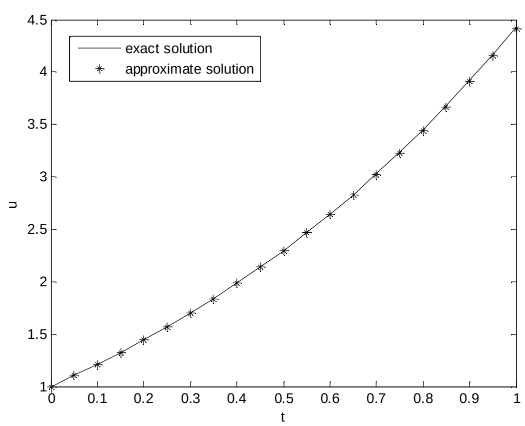

Fig.3. When $x^2+y^2=1$, comparing of the exact solution and the approximate solution.

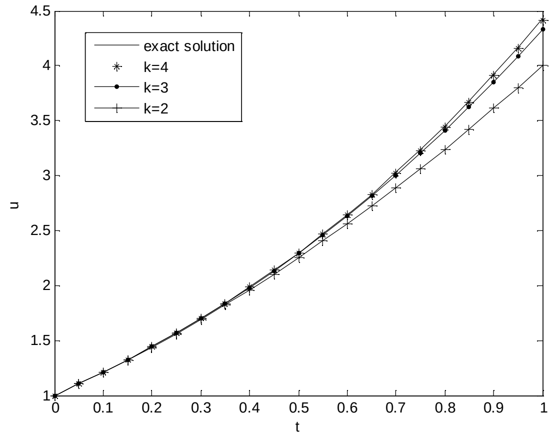

Fig.4. Comparing with the exact solution, the characteristics of $u(x, y, t)$ with the variation of k.

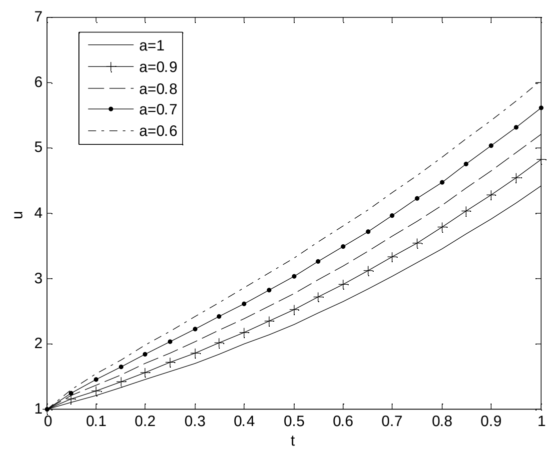

Fig.5. The characteristics of $u(x, y, t)$ with the variation of α .

efficient. We also can seen when the values of α is more closed to 1, the characteristics of $u(x, y, t)$ are more closed to the exact solution.

V. CONCLUSIONS

In this paper, the time and space fractional differential equation has been studied using the variational iteration method. This works shows that the variational iteration method is a very efficient and powerful tool in solving the nonlinear fractional differential equations. The results also reveal that the technique introduced here is very effective and convenient in solving partial differential equations of fractional order.

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