

# Unsteady Helical Flow of a Generalized Oldroyd-B Fluid with Fractional Derivative

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**Abstract**—This paper considered some unsteady helical flows of a generalized Oldroyd-B fluid between two infinite concentric cylinders and an infinite circular cylinder. The flow is due to the cylinders oscillate around their common axis and accelerating slide in the direction of the same axis with prescribed velocities. Exact solutions of some unsteady helical flows are obtained by using Laplace transform coupled with Hankel transform for fractional calculus. The corresponding solutions for generalized second grade fluid, Maxwell fluid, ordinary Oldroyd-B fluid or Newtonian fluid are obtained as limiting cases of general solutions. Finally, the influence of the fractional parameters  $\alpha$  and  $\beta$  on the fluid motion is underlined by graphical illustrations.

**Keywords**—Helical fluid, Oldroyd-B fluid, Laplace transform, Hankel transform.

## I. INTRODUCTION

The interest for motion problems of non-Newtonian fluids has considerably grown because of the wide range of their applications, such as extrusion of polymer fluids, exotic lubricants, colloidal and suspension solutions, food stuffs, slurry fuels and many others. These fluids have been modeled in a number of diverse manners with their constitutive equations varying greatly in complexity. Models of differential type and rate type, which are used to describe the response of fluids that have slight memory such as dilute polymeric solutions, have received much attention. Among them the Oldroyd-B fluid has obtained a lot of attention for it has been found some success in describing polymeric liquids.

The fractional derivatives [1] are found to be quite flexible for describing the behaviors of viscoelastic fluids. The starting point of the fractional derivative model of a non-Newtonian fluid is usually a classical differential equation which is modified by replacing the time derivative of an integer order by the so-called Riemann-Liouville fractional calculus operators. Tan *et al.* [2] and Xu and Tan [3] examined the velocity field, stress field and vortex sheet of a generalized second-order fluid with fractional anomalous diffusion. Jiang [4] achieved satisfactory result to apply the constitutive equation with fractional derivative to the experimental data of viscoelasticity. Fetecau [5]-[6] considered the motions of a second grade fluid due to the longitudinal and torsional oscillations of a circular cylinder. Wang [7] investigated the unsteady axial Couette flow of fractional second grade fluid and fractional Maxwell fluid between two infinitely long concentric circular cylinders. Fetecau [8]-[11] studied some helical fluids in cylindrical domains. Khan *et al.* [12]-[13] considered the unsteady flow of a non-Newtonian fluid between two infinitely long concentric circular cylinders with fractional derivative model. Tong [14]-[15] investigated the helical flows for Oldroyd-B fluid in concentric cylinders and a circular cylinder. The velocity fields and the associated tangential stresses are determined in forms of series in terms of Bessel functions.

Motivated by the above mentioned works, this paper considers some helical flows of a generalized Oldroyd-B fluid with the fractional derivative. The flow is due to the cylinders oscillate around their common axis and accelerating slide in the direction of the same axis with prescribed velocities. By means of Laplace transform coupled with Hankel transform, the velocity field and shear stress are determined. The similar solutions corresponding to the helical flow within an infinite circular cylinder are obtained.

## II. GOVERNING EQUATIONS

The constitutive equation of an incompressible, generalized Oldroyd-B fluid is written in the form [16]-[17]:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda \frac{D^\alpha \mathbf{S}}{Dt^\alpha} = \mu(1 + \lambda_r \frac{D^\beta}{Dt^\beta})\mathbf{A} \quad (1)$$

where  $\mathbf{T}$  is the Cauchy stress tensor,  $-p\mathbf{I}$  denotes the indeterminate spherical stress,  $\mathbf{S}$  is the extra-stress tensor,  $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$  is the first Rivlin-Ericksen tensor,  $\mathbf{L}$  is the velocity gradient,  $\mu, \lambda, \lambda_r$  are material constants, and

$$\frac{D^\alpha \mathbf{S}}{Dt^\alpha} = D_t^\alpha \mathbf{S} + \mathbf{V} \cdot \nabla \mathbf{S} - \mathbf{L} \mathbf{S} - \mathbf{S} \mathbf{L}^T, \quad \frac{D^\beta \mathbf{A}}{Dt^\beta} = D_t^\beta \mathbf{A} + \mathbf{V} \cdot \nabla \mathbf{A} - \mathbf{L} \mathbf{A} - \mathbf{A} \mathbf{L}^T \quad (2)$$

In the above relations  $\mathbf{V}$  is the velocity,  $\nabla$  is the gradient operator,  $D_t^\alpha$  and  $D_t^\beta$  are based on Riemann- Liouville’s definition is defined as [1]:

$$D_t^p f(t) = \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^p} d\tau, 0 \leq p < 1 \tag{3}$$

where  $\Gamma(\cdot)$  is the Gamma function.

In cylindrical coordinates  $(r, \theta, z)$ , the helical flow velocity is given by [15], [18]

$$\mathbf{V} = \mathbf{V}(r, t) = u(r, t)\mathbf{e}_z + v(r, t)\mathbf{e}_\theta \tag{4}$$

where  $\mathbf{e}_z$  and  $\mathbf{e}_\theta$  are the unit vectors in the  $z$ -and  $\theta$ -directions. Since the velocity field is independent of  $z$  and  $\theta$ , the extra-stress tensor  $\mathbf{S}$  will also be independent of  $z$  and  $\theta$ , and the incompressibility condition is automatically satisfied.

Substituting Eq. (4) into Eq. (1) and taking into account the initial condition, we find that  $S_{rr} = 0$  and

$$(1 + \lambda D_t^\alpha)\tau_1 = \mu(1 + \lambda_r D_t^\beta) \frac{\partial u}{\partial r} \tag{5}$$

$$(1 + \lambda D_t^\alpha)\tau_2 = \mu(1 + \lambda_r D_t^\beta) \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) \tag{6}$$

where  $\tau_1 = S_{rz}$  and  $\tau_2 = S_{r\theta}$  are the shear stress, which are different of zero.

In the absence of body forces and pressure gradient, the equations of motion reduce to

$$\rho \frac{\partial u}{\partial r} = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \tau_1, \quad \rho \frac{\partial v}{\partial r} = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \tau_2 \tag{7}$$

Eliminating  $\tau_1$  and  $\tau_2$  among Eqs. (5)- (7), we obtain the governing equations

$$(1 + \lambda D_t^\alpha) \frac{\partial u(r, t)}{\partial t} = \nu(1 + \lambda_r D_t^\beta) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u(r, t) \tag{8}$$

$$(1 + \lambda D_t^\alpha) \frac{\partial v(r, t)}{\partial t} = \nu(1 + \lambda_r D_t^\beta) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) v(r, t) \tag{9}$$

where  $\nu = \mu / \rho$  is the kinematic viscosity of the fluid.

### III. HELICAL FLOW BETWEEN TWO CONCENTRIC CYLINDERS

We consider unsteady helical flow between two infinite coaxial cylinders of radius  $R_1$  and  $R_2 (> R_1)$ . The fluid is assumed to be at rest at the moment  $t = 0$ , when  $t = 0^+$ , the two cylinders begin to oscillate around their common axis ( $r = 0$ ) with the velocities  $R_1 \sin(\omega t)$  and  $R_2 \sin(\omega t)$  and to slide along the same axis with the velocities  $Ut^a$  and  $Ut^b$ , where  $\omega$  is the angular frequency of velocity,  $U$ ,  $a$  and  $b$  are constants. The associated initial and boundary conditions are:

$$\text{Initial condition: } u(r, 0) = \partial_t u(r, 0) = v(r, 0) = \partial_t v(r, 0) = 0, \quad r \in (R_1, R_2) \tag{10}$$

$$\text{Boundary conditions: } u(R_1, t) = Ut^a, \quad u(R_2, t) = Ut^b, \tag{11}$$

$$v(R_1, t) = R_1 \sin(\omega t), \quad v(R_2, t) = R_2 \sin(\omega t) \tag{12}$$

#### Velocity field

Applying Laplace transform principle of sequential fractional derivatives to Eqs. (8) and (9), and using initial conditions Eq.(10), we obtain

$$p(1 + \lambda p^\alpha) \bar{u}(r, p) = \nu(1 + \lambda_r p^\beta) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \bar{u}(r, p) \tag{13}$$

$$\bar{u}(R_1, p) = U \frac{\Gamma(a+1)}{p^{a+1}}, \quad \bar{u}(R_2, p) = U \frac{\Gamma(b+1)}{p^{b+1}} \tag{14}$$

$$p(1 + \lambda p^\alpha) \bar{v}(r, p) = \nu(1 + \lambda_r p^\beta) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \bar{v}(r, p) \tag{15}$$

$$\bar{v}(R_1, p) = R_1 \frac{\omega}{p^2 + \omega^2}, \quad \bar{v}(R_2, t) = R_2 \frac{\omega}{p^2 + \omega^2} \tag{16}$$

In order to obtain an analytical solution of Eq.(13), the Hankel transform [19] method with respect to  $r$  is used, and is defined as follows

$$\tilde{u}(s_{1n}, p) = \int_{R_1}^{R_2} r \bar{u}(r, p) \varphi_1(s_{1n} r) dr \tag{17}$$

The inverse Hankel transform is

$$\bar{u}(r, p) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{s_{1n}^2 J_0^2(s_{1n} R_1) \tilde{u}(s_{1n}, p) \varphi_1(s_{1n} r)}{J_0^2(s_{1n} R_1) - J_0^2(s_{1n} R_2)} \tag{18}$$

where  $\varphi_1(s_{1n} r) = Y_0(R_2 s_{1n}) J_0(s_{1n} r) - J_0(R_2 s_{1n}) Y_0(s_{1n} r)$ ,  $s_{1n}$  is the positive root of  $\varphi_1(s_{1n} R_1) = 0$ . Applying the above transform to Eqs. (13) and (14), we obtain

$$\tilde{u}(r, p) = \frac{2\nu(1 + \lambda_r p^\beta) U[J_0(s_{1n} R_2) \frac{\Gamma(a+1)}{p^{a+1}} - J_0(s_{1n} R_1) \frac{\Gamma(b+1)}{p^{b+1}}]}{\pi J_0(s_{1n} R_1) [p(1 + \lambda p^\alpha) + \nu s_{1n}^2 (1 + \lambda_r p^\beta)]} \tag{19}$$

Substituting Eq. (19) into Eq. (18), we have

$$\bar{u}(r, p) = \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n} R_1) \varphi_1(s_{1n} r)}{J_0^2(s_{1n} R_1) - J_0^2(s_{1n} R_2)} U[J_0(s_{1n} R_2) \frac{\Gamma(a+1)}{p^{a+1}} - J_0(s_{1n} R_1) \frac{\Gamma(b+1)}{p^{b+1}}] \frac{s_{1n}^2 \nu (1 + \lambda_r p^\beta)}{[p(1 + \lambda p^\alpha) + \nu s_{1n}^2 (1 + \lambda_r p^\beta)]} \tag{20}$$

In order to avoid the burdensome calculations of residues and contour integrals, we apply the discrete inverse Laplace transform method. We rewrite the last factor from Eq. (20) in series form, and apply the discrete inverse Laplace transform, and then we have

$$u(r, t) = \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n} R_1) \varphi_1(s_{1n} r)}{J_0^2(s_{1n} R_1) - J_0^2(s_{1n} R_2)} U[J_0(s_{1n} R_2) t^a - J_0(s_{1n} R_1) t^b] - \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n} R_1) \varphi_1(s_{1n} r)}{J_0^2(s_{1n} R_1) - J_0^2(s_{1n} R_2)} U \int_0^t [J_0(s_{1n} R_2) (t-s)^a - J_0(s_{1n} R_1) (t-s)^b] G(s_{1n}, s) ds \tag{21}$$

where

$$G(s_{1n}, s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} \left( \frac{\nu s_{1n}^2}{\lambda} \right)^m \sum_{k=0}^m \binom{m}{k} \lambda_r^k s^{m(\alpha+1) - \beta k - 1} E_{\alpha, \alpha+m-\beta k}^{(m-1)}(-\lambda^{-1} s^\alpha) \tag{22}$$

In order to obtain an analytical solution of Eq.(15), the Hankel transform with respect to  $r$  is used, and is defined as follows

$$\tilde{v}(s_{2n}, p) = \int_{R_1}^{R_2} r \bar{v}(r, p) \varphi_2(s_{2n} r) dr \tag{23}$$

The inverse Hankel transform is

$$\bar{v}(r, p) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{s_{2n}^2 J_0^2(s_{2n} R_1) \tilde{v}(s_{2n}, p) \varphi_2(s_{2n} r)}{J_0^2(s_{2n} R_1) - J_0^2(s_{2n} R_2)} \tag{24}$$

where  $\varphi_2(s_{2n} r) = Y_1(R_2 s_{2n}) J_1(s_{2n} r) - J_1(R_2 s_{2n}) Y_1(s_{2n} r)$ ,  $s_{2n}$  is the positive root of  $\varphi_2(s_{2n} R_1) = 0$ . Applying the above transform to Eqs. (15) and (16), we obtain

$$\tilde{v}(s_{2n}, p) = \frac{2}{\pi} \frac{\nu(1 + \lambda_r p^\beta) \omega}{p^2 + \omega^2} \frac{J_1(s_{2n} R_2) R_1 - J_1(s_{2n} R_1) R_2}{J_1(s_{2n} R_1) [p(1 + \lambda p^\alpha) + \nu s_{2n}^2 (1 + \lambda_r p^\beta)]} \tag{25}$$

Substituting Eq. (25) into Eq. (24), we have

$$\bar{v}(r, p) = \pi \sum_{n=1}^{\infty} \frac{J_1(s_{2n} R_1) \varphi_2(s_{2n} r)}{J_1^2(s_{2n} R_1) - J_1^2(s_{2n} R_2)} \frac{\omega}{p^2 + \omega^2} [J_1(s_{2n} R_2) R_1 - J_1(s_{2n} R_1) R_2] \frac{\nu s_{2n}^2 (1 + \lambda_r p^\beta)}{[p(1 + \lambda p^\alpha) + \nu s_{2n}^2 (1 + \lambda_r p^\beta)]} \tag{26}$$

And following the same way as before we get

$$v(r, p) = \pi \sum_{n=1}^{\infty} \frac{J_1(s_{2n} R_1) \varphi_2(s_{2n} r)}{J_1^2(s_{2n} R_1) - J_1^2(s_{2n} R_2)} [J_1(s_{2n} R_2) R_1 - J_1(s_{2n} R_1) R_2] [\sin(\omega t) - \int_0^t \sin(\omega(t-s)) G(s_{2n}, s) ds] \tag{27}$$

**Shear stress**

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Apply the Laplace transform to Eqs. (5) and (6), we find that

$$(1 + \lambda p^\alpha) \bar{\tau}_1(r, p) = \mu(1 + \lambda_r p^\beta) \frac{\partial \bar{u}(r, p)}{\partial r} \tag{28}$$

$$(1 + \lambda p^\alpha) \bar{\tau}_2(r, p) = \mu(1 + \lambda_r p^\beta) \left( \frac{\partial}{\partial r} - \frac{1}{r} \right) \bar{v}(r, p) \tag{29}$$

Substituting Eqs. (20) and (26) into Eqs. (28)-(29), we have

$$\begin{aligned} \bar{\tau}_1(r, p) = & \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n} R_1) \varphi_3(s_{1n} r)}{J_0^2(s_{1n} R_2) - J_0^2(s_{1n} R_1)} U [J_0(s_{1n} R_2) \frac{\Gamma(a+1)}{p^{a+1}} - J_0(s_{1n} R_1) \frac{\Gamma(b+1)}{p^{b+1}}] \\ & \times \frac{\mu(1 + \lambda_r p^\beta)}{1 + \lambda p^\alpha} s_{1n} \frac{s_{1n}^2 \nu(1 + \lambda_r p^\beta)}{[p(1 + \lambda p^\alpha) + \nu s_{1n}^2(1 + \lambda_r p^\beta)]} \end{aligned} \tag{30}$$

$$\begin{aligned} \bar{\tau}_2(r, p) = & \pi \left[ \sum_{n=1}^{\infty} \frac{J_1(s_{2n} R_1) \varphi_3(s_{2n} r)}{J_1^2(s_{2n} R_2) - J_1^2(s_{2n} R_1)} - \sum_{n=1}^{\infty} \frac{J_1(s_{2n} R_1) \varphi_2(s_{2n} r)}{r s_{2n} [J_1^2(s_{2n} R_1) - J_1^2(s_{2n} R_2)]} \right] \frac{\omega}{p^2 + \omega^2} \\ & \times [J_1(s_{2n} R_2) R_1 - J_1(s_{2n} R_1) R_2] \frac{\mu(1 + \lambda_r p^\beta)}{1 + \lambda p^\alpha} s_{2n} \frac{\nu s_{2n}^2(1 + \lambda_r p^\beta)}{[p(1 + \lambda p^\alpha) + \nu s_{2n}^2(1 + \lambda_r p^\beta)]} \end{aligned} \tag{31}$$

And following the same way as the velocity field, we obtain

$$\tau_1(r, t) = \rho \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n} R_1) \varphi_3(s_{1n} r)}{s_{1n} [J_0^2(s_{1n} R_2) - J_0^2(s_{1n} R_1)]} U \int_0^t [J_0(s_{1n} R_2)(t-s)^a - J_0(s_{1n} R_1)(t-s)^b] G_1(s_{1n}, s) ds \tag{32}$$

where

$$G_1(s_{1n}, s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \left( \frac{\nu s_{1n}^2}{\lambda} \right)^{m+2} \sum_{k=0}^{m+2} \binom{m+2}{k} \lambda_r^k s^{\alpha(m+2)+m-\beta k} E_{\alpha, \alpha+m+1-\beta k}^{(m+1)} (-\lambda^{-1} s^\alpha) \tag{33}$$

And

$$\begin{aligned} \tau_2(r, t) = & \pi \left[ \sum_{n=1}^{\infty} \frac{J_1(s_{2n} R_1) \varphi_3(s_{2n} r)}{J_1^2(s_{2n} R_2) - J_1^2(s_{2n} R_1)} - \sum_{n=1}^{\infty} \frac{J_1(s_{2n} R_1) \varphi_2(s_{2n} r)}{r s_{2n} [J_1^2(s_{2n} R_1) - J_1^2(s_{2n} R_2)]} \right] \\ & \times [J_1(s_{2n} R_2) R_1 - J_1(s_{2n} R_1) R_2] \int_0^t \sin(\omega(t-s)) G_1(s_{2n}, s) ds \end{aligned} \tag{34}$$

**Limiting cases**

(1) Making the limit of Eqs. (21)-(22), (27), (32)-(34), when  $\alpha \neq 0$  and  $\lambda \rightarrow 0$ , we attain to the similar solution for a generalized second grade fluid

$$\begin{aligned} u(r, t) = & \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n} R_1) \varphi_1(s_{1n} r)}{J_0^2(s_{1n} R_1) - J_0^2(s_{1n} R_2)} U [J_0(s_{1n} R_2) t^a - J_0(s_{1n} R_1) t^b] - \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n} R_1) \varphi_1(s_{1n} r)}{J_0^2(s_{1n} R_1) - J_0^2(s_{1n} R_2)} \\ & \times U \int_0^t [J_0(s_{1n} R_2)(t-s)^a - J_0(s_{1n} R_1)(t-s)^b] \sum_{m=0}^{\infty} (-1)^m (\nu s_{1n}^2)^m \sum_{k=0}^m \binom{m}{k} \lambda_r^k \frac{s^{m-\beta k-1}}{\Gamma(m-\beta k)} ds \end{aligned} \tag{35}$$

$$\begin{aligned} v(r, t) = & \pi \sum_{n=1}^{\infty} \frac{J_1(s_{2n} R_1) \varphi_2(s_{2n} r)}{J_1^2(s_{2n} R_1) - J_1^2(s_{2n} R_2)} [J_1(s_{2n} R_2) R_1 - J_1(s_{2n} R_1) R_2] [\sin(\omega t) \\ & - \int_0^t \sin(\omega(t-s)) \sum_{m=0}^{\infty} (-1)^m (\nu s_{2n}^2)^m \sum_{k=0}^m \binom{m}{k} \lambda_r^k \frac{s^{m-\beta k-1}}{\Gamma(m-\beta k)} ds] \end{aligned} \tag{36}$$

$$\tau_1(r, t) = \rho\pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_3(s_{1n}r)}{s_{1n}[J_0^2(s_{1n}R_2) - J_0^2(s_{1n}R_1)]} U \int_0^t [J_0(s_{1n}R_2)(t-s)^a - J_0(s_{1n}R_1)(t-s)^b] \times \sum_{m=0}^{\infty} (-1)^m (vs_{1n}^2)^m \sum_{k=0}^{m+2} \binom{m+2}{k} \lambda_r^k \frac{s^{m-\beta k}}{\Gamma(m-\beta k+1)} ds \tag{37}$$

$$\tau_2(r, t) = \rho\pi \left[ \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_3(s_{2n}r)}{s_{1n}[J_1^2(s_{2n}R_2) - J_1^2(s_{2n}R_1)]} - \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_2(s_{2n}r)}{rs_{2n}[J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)]} \right] \times [J_1(s_{2n}R_2)R_1 - J_1(s_{2n}R_1)R_2] \int_0^t \sin(\omega(t-s)) \sum_{m=0}^{\infty} (-1)^m (vs_{2n}^2)^m \sum_{k=0}^{m+2} \binom{m+2}{k} \lambda_r^k \frac{s^{m-\beta k}}{\Gamma(m-\beta k+1)} ds \tag{38}$$

(2) When  $\beta \neq 0$  and  $\lambda_r \rightarrow 0$  in Eqs. (21)-(22), (27), (32)-(34), we can obtain the solutions for a generalized Maxwell fluid.

$$u(r, t) = \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_1(s_{1n}r)}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} U [J_0(s_{1n}R_2)t^a - J_0(s_{1n}R_1)t^b] - \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_1(s_{1n}r)}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} \times U \int_0^t [J_0(s_{1n}R_2)(t-s)^a - J_0(s_{1n}R_1)(t-s)^b] \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} \left( \frac{vs_{1n}^2}{\lambda} \right)^m s^{m(\alpha+1)-1} E_{\alpha, \alpha+m}^{(m-1)}(-\lambda^{-1}s^\alpha) ds \tag{39}$$

$$v(r, t) = \pi \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_2(s_{2n}r)}{J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)} [J_1(s_{2n}R_2)R_1 - J_1(s_{2n}R_1)R_2] [\sin(\omega t) - \int_0^t \sin(\omega(t-s)) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} \left( \frac{vs_{2n}^2}{\lambda} \right)^m s^{m(\alpha+1)-1} E_{\alpha, \alpha+m}^{(m-1)}(-\lambda^{-1}s^\alpha) ds] \tag{40}$$

$$\tau_1(r, t) = \rho\pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_3(s_{1n}r)}{s_{1n}[J_0^2(s_{1n}R_2) - J_0^2(s_{1n}R_1)]} U \int_0^t [J_0(s_{1n}R_2)(t-s)^a - J_0(s_{1n}R_1)(t-s)^b] \times \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \left( \frac{vs_{1n}^2}{\lambda} \right)^{m+2} s^{\alpha(m+2)+m} E_{\alpha, \alpha+m+1}^{(m+1)}(-\lambda^{-1}s^\alpha) ds \tag{41}$$

$$\tau_2(r, t) = \pi \left[ \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_3(s_{2n}r)}{J_1^2(s_{2n}R_2) - J_1^2(s_{2n}R_1)} - \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_2(s_{2n}r)}{rs_{2n}[J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)]} \right] [J_1(s_{2n}R_2)R_1 - J_1(s_{2n}R_1)R_2] \times \int_0^t \sin(\omega(t-s)) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \left( \frac{vs_{2n}^2}{\lambda} \right)^{m+2} s^{\alpha(m+2)+m} E_{\alpha, \alpha+m+1}^{(m+1)}(-\lambda^{-1}s^\alpha) ds \tag{42}$$

(3) In the special case when  $\alpha = \beta = 1$ , Eqs.(21)-(22), (27), (32)-(34) can be simplified as

$$u(r, t) = \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_1(s_{1n}r)}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} U [J_0(s_{1n}R_2)t^a - J_0(s_{1n}R_1)t^b] - \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_1(s_{1n}r)}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} \times U \int_0^t [J_0(s_{1n}R_2)(t-s)^a - J_0(s_{1n}R_1)(t-s)^b] \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} \left( \frac{vs_{1n}^2}{\lambda} \right)^m \sum_{k=0}^m \binom{m}{k} \lambda_r^k s^{2m-k-1} E_{1, 1+m-k}^{(m-1)}(-\lambda^{-1}s) ds \tag{43}$$

$$v(r, p) = \pi \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_2(s_{2n}r)}{J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)} [J_1(s_{2n}R_2)R_1 - J_1(s_{2n}R_1)R_2] [\sin(\omega t) - \int_0^t \sin(\omega(t-s))] \times \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} \left( \frac{vs_{2n}^2}{\lambda} \right)^m \sum_{k=0}^m \binom{m}{k} \lambda_r^k s^{2m-k-1} E_{1, 1+m-k}^{(m-1)}(-\lambda^{-1}s) ds \tag{44}$$

$$\tau_1(r, t) = \rho\pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_3(s_{1n}r)}{s_{1n}[J_0^2(s_{1n}R_2) - J_0^2(s_{1n}R_1)]} U \int_0^t [J_0(s_{1n}R_2)(t-s)^a - J_0(s_{1n}R_1)(t-s)^b] \times \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \left( \frac{vs_{1n}^2}{\lambda} \right)^{m+2} \sum_{k=0}^{m+2} \binom{m+2}{k} \lambda_r^k s^{2m+2-k} E_{1, 2+m-k}^{(m+1)}(-\lambda^{-1}s) ds \tag{45}$$

$$\tau_2(r, t) = \pi \left[ \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_3(s_{2n}r)}{J_1^2(s_{2n}R_2) - J_1^2(s_{2n}R_1)} - \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_2(s_{2n}r)}{rs_{2n}[J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)]} \right] [J_1(s_{2n}R_2)R_1 - J_1(s_{2n}R_1)R_2] \tag{46}$$

$$\times \int_0^t \sin(\omega(t-s)) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} \left( \frac{vs_{2n}^2}{\lambda} \right)^{m+2} \sum_{k=0}^{m+2} \binom{m+2}{k} \lambda_r^k s^{2m+2-k} E_{1,2+m-k}^{(m+1)}(-\lambda^{-1}s) ds$$

which correspond to the similar solution for an ordinary Oldroyd-B fluid.

(4) Setting  $\beta = 0, \theta \rightarrow 0$  in Eqs.(35)-(38) or setting  $\alpha \neq 0, \lambda \rightarrow 0$  in Eqs.(39)-(42), we obtain the solutions of Newtonian fluid

$$u(r, t) = \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_1(s_{1n}r)}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} U [J_0(s_{1n}R_2)t^a - J_0(s_{1n}R_1)t^b] - \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_1(s_{1n}r)}{J_0^2(s_{1n}R_1) - J_0^2(s_{1n}R_2)} \tag{47}$$

$$\times U \int_0^t [J_0(s_{1n}R_2)(t-s)^a - J_0(s_{1n}R_1)(t-s)^b] \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} (vs_{1n}^2)^m s^{m-1} ds$$

$$v(r, p) = \pi \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_2(s_{2n}r)}{J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)} [J_1(s_{2n}R_2)R_1 - J_1(s_{2n}R_1)R_2] \tag{48}$$

$$\times [\sin(\omega t) - \int_0^t \sin(\omega(t-s)) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!} (vs_{2n}^2)^m s^{m-1} ds]$$

$$\tau_1(r, t) = \rho \pi \sum_{n=1}^{\infty} \frac{J_0(s_{1n}R_1)\varphi_3(s_{1n}r)}{s_{1n}[J_0^2(s_{1n}R_2) - J_0^2(s_{1n}R_1)]} U \int_0^t [J_0(s_{1n}R_2)(t-s)^a - J_0(s_{1n}R_1)(t-s)^b] \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (vs_{1n}^2)^{m+2} s^m ds \tag{49}$$

$$\tau_2(r, t) = \rho \pi \left[ \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_3(s_{2n}r)}{s_{2n}[J_1^2(s_{2n}R_2) - J_1^2(s_{2n}R_1)]} - \sum_{n=1}^{\infty} \frac{J_1(s_{2n}R_1)\varphi_2(s_{2n}r)}{rs_{2n}[J_1^2(s_{2n}R_1) - J_1^2(s_{2n}R_2)]} \right] \tag{50}$$

$$\times [J_1(s_{2n}R_2)R_1 - J_1(s_{2n}R_1)R_2] \int_0^t \sin(\omega(t-s)) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} (vs_{2n}^2)^{m+2} s^m ds$$

#### IV. HELICAL FLOW THROUGH A CIRCULAR CYLINDER

Taking the limit of Eq.(17) when  $R_1 \rightarrow 0$  and  $R_2 \rightarrow R$ , we find the Hankel transform

$$\tilde{u} = \int_0^R r \bar{u}(r, p) J_0(s_{3n}r) dr \tag{51}$$

And the inverse Hankel transform is

$$\bar{u}(r, p) = \frac{2}{R^2} \sum_{n=1}^{\infty} \frac{J_0(s_{3n}r)\tilde{u}(r, p)}{J_1^2(s_{3n}R)} \tag{52}$$

The boundary conditions must be changes by

$$|u(0, t)| < \infty, u(R, t) = Ut^b, |v(0, t)| < \infty, v(R, t) = R \sin(\omega t). \tag{53}$$

And the velocity  $u(r, t)$  takes the form

$$u(r, t) = \frac{2}{R} \sum_{n=1}^{\infty} \frac{J_0(s_{3n}r)}{s_{3n}J_1(s_{3n}R)} Ut^b - \frac{2}{R} \sum_{n=1}^{\infty} \frac{J_0(s_{3n}r)}{s_{3n}J_1(s_{3n}R)} U \int_0^t (t-s)^b G(s_{3n}, s) ds \tag{54}$$

where  $s_{3n}$  is the positive root of  $J_0(s_{3n}R) = 0$ , and

$$v(r, t) = \frac{2}{R} \sum_{n=1}^{\infty} \frac{J_0(s_{4n}r)}{J_1^2(s_{4n}R)} \frac{[Rs_{4n}J_0(Rs_{4n}) - J_1(Rs_{4n})]}{s_{4n}^2} [\sin(\omega t) - \int_0^t \sin(\omega(t-s))G(s_{4n}, s)] ds \tag{55}$$

where  $s_{4n}$  is the positive root of  $J_0(s_{4n}R) = 0$ . The associated shear stresses are

$$\tau_1(r, t) = -\frac{2}{Rs_{3n}} \sum_{n=1}^{\infty} \frac{J_1(s_{3n}r)}{J_1(s_{3n}R)} \int_0^t U(t-s)^b G_1(s_{3n}, s) ds \tag{56}$$

$$\tau_2(r, t) = -\frac{2}{R} \frac{\mu(1 + \lambda_r p^\beta)}{1 + \lambda p^\alpha} \left[ \sum_{n=1}^{\infty} \frac{J_1(s_{4n}r)}{J_1^2(s_{4n}R)} - \sum_{n=1}^{\infty} \frac{J_0(s_{4n}r)}{rs_{4n}J_1^2(s_{4n}R)} \right] \frac{[Rs_{4n}J_0(Rs_{4n}) - J_1(Rs_{4n})]}{s_{4n}^2} \int_0^t \sin(\omega(t-s))G_2(s_{4n}, s) ds \tag{57}$$

V. RESULTS AND DISCUSSION

In this paper, we considered some unsteady helical flows of a generalized Oldroyd-B fluid between two infinite concentric cylinders and an infinite circular cylinder. The fractional calculus approach is used in the constitutive relationship model of a viscoelastic fluid. With the help of integral transforms, the solutions are obtained in terms of Bessel function and Mittag-Leffler function. The similar solutions for generalized second grade, Maxwell fluid, ordinary Oldroyd-B fluid or Newtonian fluid are also given by the limiting cases.

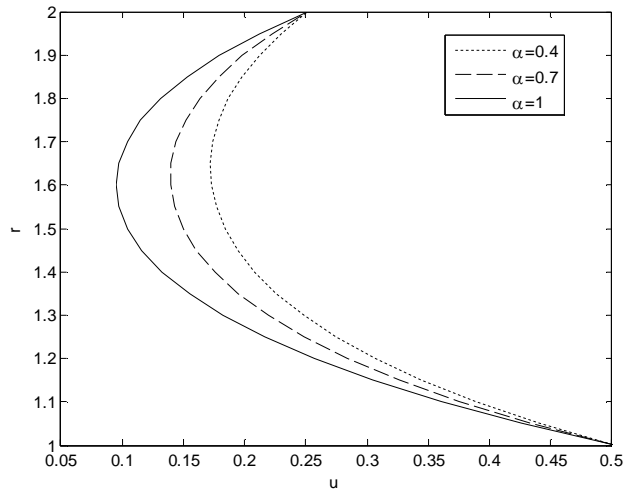


Fig.1 Velocity profiles  $u(r,t)$  for different values of  $\alpha$  when  $\beta = 0.8$ .

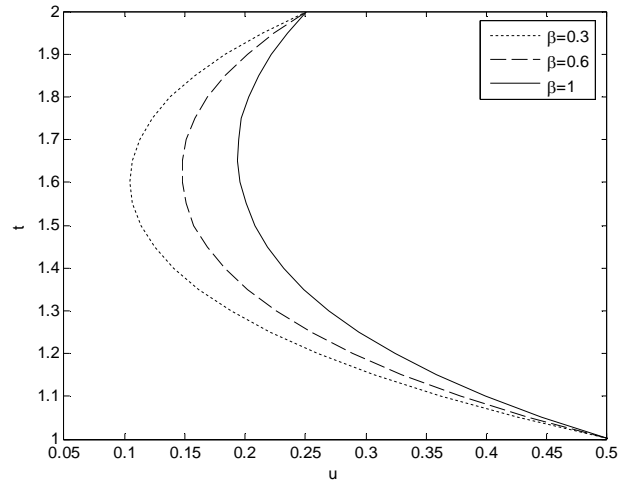


Fig.2 Velocity profiles  $u(r,t)$  for different values of  $\beta$  when  $\alpha = 0.4$ .

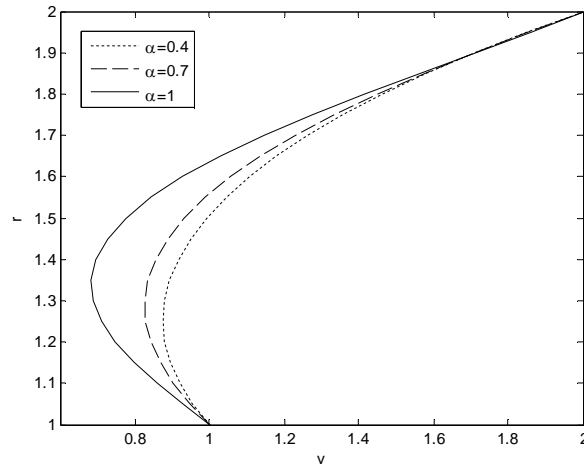


Fig.3 Velocity profiles  $v(r, t)$  for different values of  $\alpha$  when  $\beta = 0.8$ .

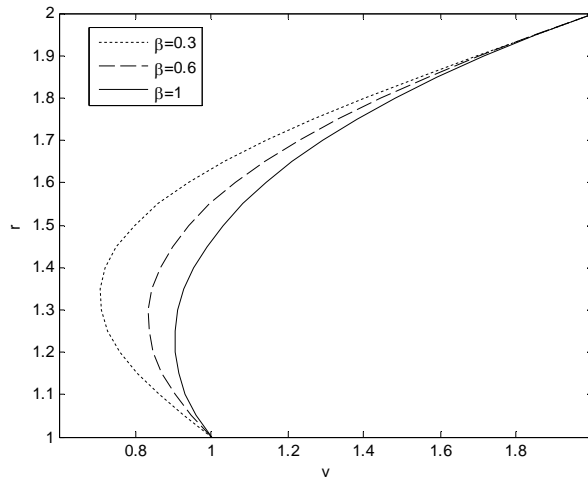


Fig.4 Velocity profiles  $v(r, t)$  for different values of  $\beta$  when  $\alpha = 0.4$ .

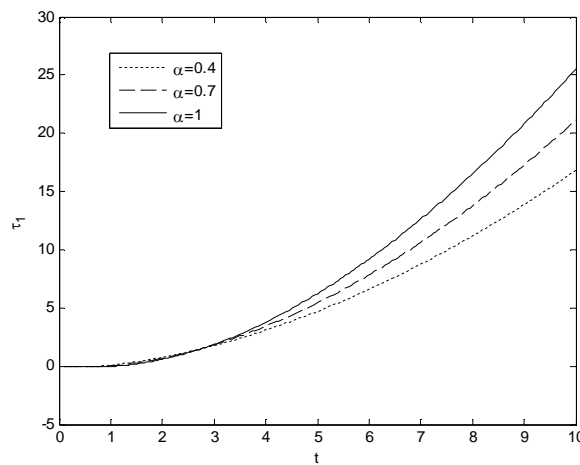


Fig.5 Shear stress  $\tau_1(r, t)$  for different values of  $\alpha$  when  $\beta = 0.8$ .



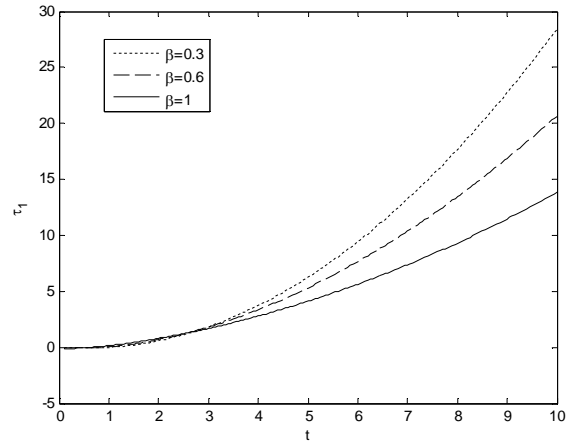


Fig.6 Shear stress  $\tau_1(r,t)$  for different values of  $\beta$  when  $\alpha = 0.4$ .

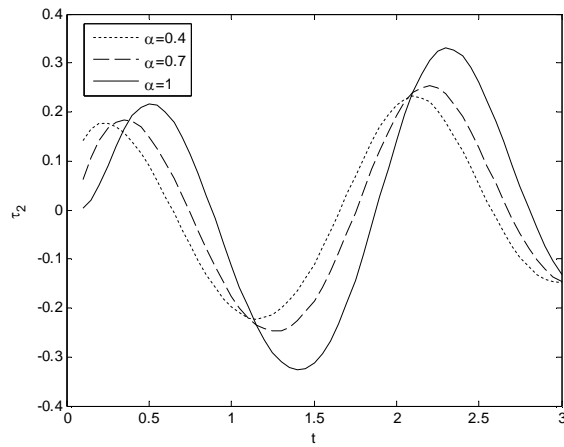


Fig.7 Shear stress  $\tau_2(r,t)$  for different values of  $\alpha$  when  $\beta = 0.8$ .

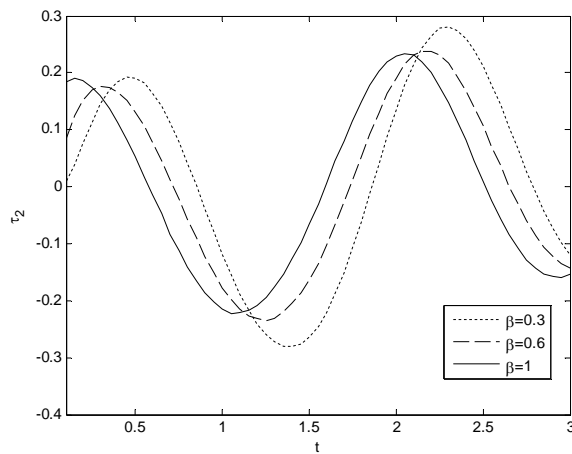


Fig.8 Shear stress  $\tau_2(r,t)$  for different values of  $\beta$  when  $\alpha = 0.4$ .

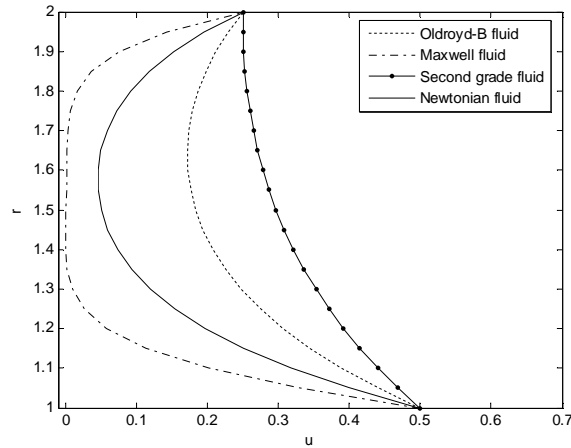


Fig.9 Velocity profiles  $u(r,t)$  of four kinds fluids.

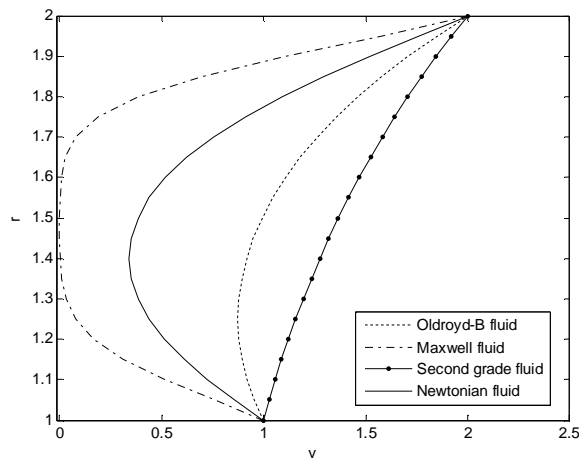


Fig.10 Velocity profiles  $v(r,t)$  of four kinds fluids.

In the following, we analyze the characteristics of the velocity field and the shear stress by using the exact solutions. In the following figures, we take  $U = 1, R_1 = 1, R_2 = 2, a = 1, b = 2, \lambda = 2, \lambda_r = 5, \nu = 0.165$ . Figs. 1-4 demonstrate the velocity changes with the fractional parameters  $\alpha$  and  $\beta$  are plotted for  $t = 0.5$ . It can be seen that the smaller the values of  $\alpha$ , the more slowly the velocity decays for the flow. However, one can see that an increase in material parameter  $\beta$  has quite the opposite effect to that of  $\alpha$ . Figs. 5-8 show the shear stress changes with the fractional parameters  $\alpha$  and  $\beta$ , are plotted for  $\omega = \pi, r = 1.5$ . From Figs. 7-8, we can see the smaller the values of  $\alpha$ , the more steady of the shear stress. And the parameter  $\beta$  has quite opposite effect. Thus, it can be speculated that the parameters in the generalized Oldroyd-B fluid model have strong effects on velocity and shear stress. Figs. 9-10 demonstrate the velocity of generalized oldroyd-B fluid, generalized Maxwell fluid, generalized second grade fluid and Newtonian fluid in  $z$ -direction and  $\theta$ -direction. We can see the velocity of generalized second grade is fastest, and the generalized Maxwell fluid is most slowly.

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