# n _ Multi-Series of the Generalized Difference Equations to Circular Functions

G.Britto Antony Xavier[1], S.Sathya [2]
and S.U.Vasantha Kumar [3]
[1,2,3] Department of Mathematics,
Sacred Heart College,
Tirupattur - 635601, Vellore District
Tamil Nadu, S.India.

**ABSTRACT**

We investigate the numerical-complete relation to certain type of higher order generalized difference equation to find the value of n _ multi-series to circular functions in the field of finite difference methods. We also give an example to illustrate the n _ multi-series.

**Key words:** Complete solution, Circular function, Generalized difference operator, Numerical solution.

**AMS Subject classification:** 39A70, 47B39, 39A10.

## 1. INTRODUCTION

The Fractional Calculus is currently a very important research field in several different areas: physics (including classical and quantum mechanics and thermodynamics), chemistry, biology, economics and control theory ([9], [10], [11]). In 1989, K.S.Miller and Ross [12] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The main definition of fractional difference equation (as done in [12]) is the $\nu$ fractional sum of $f(t)$ by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma_{(\nu)}} \sum_{s=a}^{t-\nu} \frac{\Gamma_{(t-s)}}{\Gamma_{(t-s-(\nu-1))}} f(s), \tag{1}$$

where $\nu > 0$. On the other hand when $\nu = m$ is a positive integer, if we replace $f(t)$ by $u(k)$ and $\Delta$ by $\Delta_\ell$, defined by $\Delta_\ell u(k) = u(k+\ell) - u(k)$, (1) becomes

$$\mathrm{u}_{n(\ell)}(k) = \Delta_\ell^{-n} u(k) = \sum_{r=n}^{\left[\frac{k}{\ell}\right]} \frac{(r-1)^{(n-1)}}{(n-1)!} u(k - r\ell). \tag{2}$$

Let $\ell_i > 0$, $u(k)$ be real valued function on $[0,\infty)$, $u(k) = 0$ for all $k \in (-\infty, 0]$, $[k/\ell_i]$ be the integer part of $k/\ell_i$, $\ell_i(k) = k - [k/\ell_i]\ell_i$ for $i = 1,2,\cdots,n$ and $\ell_0(k) = k$.

Then for $n \geq 2$, (2) induces n_multi – series $\mathrm{u}_{\ell_{[1,n]}}(k) = \sum_{r_n=1}^{\left[\frac{k}{\ell_n}\right]} \mathrm{u}_{\ell_{[1,n-1]}}(k - r_n \ell_n)$, (3)

where $\mathrm{u}_{\ell_{[1,1]}}(k) = \sum_{r_1=1}^{\left[\frac{k}{\ell_1}\right]} u(k - r_1 \ell_1)$ (1 _ series with respect to $\ell_1$),

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$$\underset{\ell_{[1,2]}}{\mathrm{u}}(k)=\sum_{r_2=1}^{\left[\frac{k}{\ell_2}\right]}\underset{\ell_{[1,1]}}{\mathrm{u}}(k-r_2\ell_2) \text{ (2\_ multi-series with respect to } \ell_1,\ell_2),$$

… … … …

Substituting $\underset{\ell_{[1,1]}}{\mathrm{u}}$, $\underset{\ell_{[1,2]}}{\mathrm{u}}$, $\cdots$, $\underset{\ell_{[1,n-1]}}{\mathrm{u}}$ in (3), we get

$$\underset{\ell_{[1,n]}}{\mathrm{u}}(k)=\sum_{r_n=1}^{\left[\frac{k}{\ell_n}\right]}\sum_{r_{n-1}=1}^{\left[\frac{k-r_n\ell_n}{\ell_{n-1}}\right]}\sum_{r_{n-2}=1}^{\left[\frac{k-r_n\ell_n-r_{n-1}\ell_{n-1}}{\ell_{n-2}}\right]}\cdots\sum_{r_1=1}^{\left[\frac{k-\sum_{i=2}^{n}r_i\ell_i}{\ell_1}\right]}u(k-\sum_{i=1}^{n}r_i\ell_i), \tag{4}$$

which is a numerical solution of the generalized difference equation

$$\underset{\ell_{[1,n]}}{\Delta}v(k)\equiv\Delta_{\ell_1}(\Delta_{\ell_2}\cdots\Delta_{\ell_n}(v(k))\cdots)=u(k), k\geq 0. \tag{5}$$

We denote R.H.S of (4) as $\sum_{\ell_{[1,n]}}u(\tilde{k})$, which depends on $\ell_1$, $\ell_2$, $\cdots$, $\ell_n$, $k$ and $u(k)$. By this notation, (3) and (4)can be expressed as $\underset{\ell_{[1,n]}}{\mathrm{u}}(k)=\sum_{\ell_{[1,n]}}u(\tilde{k})$.

$$\text{Inparticular, when } n=1,\ \underset{\ell_{[1,1]}}{\mathrm{u}}(k)=\underset{\ell_{[1,1]}}{\Delta^{-1}}u(k)\big|_{\ell_1(k)}^{k}=\sum_{\ell_{[1,1]}}u(\tilde{k}). \tag{6}$$

**Remark 1.1** $$\sum_{\ell_{[m,n]}}\underset{\ell_{[1,m-1]}}{\mathrm{u}}(\ell_{m-1}(\tilde{k}))=\sum_{r_n=1}^{\left[\frac{k}{\ell_n}\right]}\sum_{r_{n-1}=1}^{\left[\frac{k-r_n\ell_n}{\ell_{n-1}}\right]}\sum_{r_{n-2}=1}^{\left[\frac{k-r_n\ell_n-r_{n-1}\ell_{n-1}}{\ell_{n-2}}\right]}\cdots\sum_{r_m=1}^{\left[\frac{k-\sum_{i=m+1}^{n}r_i\ell_i}{\ell_m}\right]}\underset{\ell_{[1,m-1]}}{\mathrm{u}}(\ell_{m-1}(k-\sum_{i=m}^{n}r_i\ell_i)).$$

where $\ell_{m-1}(k-\sum_{i=m}^{n}r_i\ell_i)=(k-\sum_{i=m}^{n}r_i\ell_i)-\left[\frac{k-\sum_{i=m}^{n}r_i\ell_i}{\ell_{m-1}}\right]\ell_{m-1}$,

When $\ell_1=\ell_2=\cdots\ell_n=\ell$, the above $n\_$ multi-series $\sum_{\ell_{[1,n]}}u(\tilde{k})$ becomes $\underset{n(\ell)}{\mathrm{u}}(k)$ given in (2). We find that, by expanding the terms, $\underset{\ell_{[1,n]}}{\mathrm{u}}(k)$ is independent of the order of the parameters $\ell_1$, $\ell_2$, $\cdots$, $\ell_n$. There are direct formula to find the $n\_$ series when $u(k)=k^m$, $k_\ell^{(m)}$, $a^k$, $k^m a^k$ *etc* and $\ell_1=\ell_2=\cdots=\ell_n=\ell$ [?, ?].

There is no direct formula to find the value of $n\_$ multi-series in the existing literature. We find that the $n\_$ multi-series $\sum_{\ell_{[1,n]}}u(\tilde{k})$ is the numerical solution of the generalized difference equation

$$\underset{\ell_{[1,n]}}{\Delta}v(k)\equiv\sum_{r=0}^{n}(-1)^{n-r}\{\sum_{A\in r(L_n)}v(k+\sum_{\ell\in A}\ell)\}=u(k), \tag{7}$$

where the operator $\underset{\ell_{[1,n]}}{\Delta}$ is given in (5) and $r(L_n)$ is the set of all subsets of the size t from the set $L_n = \{\ell_1, \ell_2, \cdots, \ell_n\}$. The complete solution of equation (7) is denoted by $\underset{\ell_{[1,n]}}{u}(k)$. Hence, in this paper we obtain numerical-complete relation of the equation (7) and arrive the $n$_ multi-series to the circular functions.

## 2. Preliminaries

In this section, we present some notations, basic definitions and preliminary results. Let $J_n = \{1,2,\ldots,n\}$, $0(J_n) = \{\phi\}$, $\phi$ is empty set, $1(J_n) = \{\{1\},\{2\}, \cdots, \{n\}\}$, $2(J_n) = \{\{1,2\},\{1,3\}, \cdots, \{1,n\},\{2,3\}, \cdots, \{2,n\}, \cdots,\{n-2,n-1\}\}$. In general, $t(J_n)$ = set of all subsets of size $t$ in ascending order from the set $J_n$, $\wp(J_n) = \bigcup_{t=0}^{n} t(J_n)$ is power set of $J_n$.

Let $\sum_{t=1}^{n} f(t) = 0$ for all integers $n < 1$, $\prod_{i=2}^{t} f(i) = 1$ if $t \le 1$ and for $1 \le p < q \le n$, $\underset{\ell_{[p,q]}}{\Delta^{-1}} u(k)$ $= \Delta^{-1}_{\ell_p}(\Delta^{-1}_{\ell_{p+1}} \cdots \Delta^{-1}_{\ell_q}(u(k))\cdots)$, $\underset{\ell_{[1,i]}}{u}(k) = \Delta^{-1}_{\ell_i} \underset{\ell_{[1,i-1]}}{u}(k)\Big|^{k}_{\ell_{i-1}(k)} = \Delta^{-1}_{\ell_i} \underset{\ell_{[1,i-1]}}{u}(k) - \Delta^{-1}_{\ell_i} \underset{\ell_{[1,i-1]}}{u}(\ell_{i-1}(k))$ for $i = 2,\cdots,n$, $\underset{\ell_{[1,1]}}{u}(k) = \Delta^{-1}_{\ell_1} u(k)$ and $\underset{\ell_{[1,0]}}{u}(k) = u(k)$.

Now we consider following lemma on circular functions.

**Lemma 2.1** *[1] Let p and q be any real numbers. Then,*

$$\Delta^{-1}_{\ell} \sin pk = \frac{\sin p(k-\ell) - \sin pk}{2(1-\cos p\ell)} + c_j. \tag{8}$$

$$\text{and } \Delta^{-1}_{\ell} \cos qk = \frac{\cos q(k-\ell) - \cos qk}{2(1-\cos q\ell)} + c_j. \tag{9}$$

**Remark 2.2** *(i) Hereafter, we take* $P = p(n_1 - 2r_1) + q(n_2 - 2r_2)$ *and* $\overline{P} = p(n_1 - 2r_1) - q(n_2 - 2r_2)$ *and hence P and* $\overline{P}$ *are varying with respect to* $n_1$, $n_2$, $r_1$, $r_2$, $p$ *and* $q$,
$n^{(r)} = n(n-1)(n-2)\cdots(n-(r-1))$.

(ii) $P\ell_i, \overline{P}\ell_i, \left(\frac{P+\overline{P}}{2}\right)\ell_i, \left(\frac{P-\overline{P}}{2}\right)\ell_i$ are not multiple of $2\pi$, for $i = 1,2,\cdots,n$.

**Corollary 2.3** *[1] (i) If* $n_1$ *and* $n_2$ *are odd positive integers, then*

$$\sin^{n_1} pk \cos^{n_2} qk = \frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-1}{2}} \sum_{r_2=0}^{\frac{n_2-1}{2}} (-1)^{r_1} \frac{n_1^{(r_1)}}{r_1!} \frac{n_2^{(r_2)}}{r_2!} \left\{\sin Pk + \sin \overline{P}k\right\}. \tag{10}$$

(ii) If $n_1$ and $n_2$ are even positive integers, then

$$\sin^{n_1} pk \cos^{n_2} qk = \frac{1}{2^{n_1+n_2-1}}\left\{\left(\sum_{r_1=0}^{\frac{n_1-2}{2}}(-1)^{\frac{n_1}{2}+r_1}\frac{n_1^{(r_1)}}{r_1!}\left(\sum_{r_2=0}^{\frac{n_2-2}{2}}\frac{n_2^{(r_2)}}{r_2!}\left(\cos Pk+\cos\overline{P}k\right)\right.\right.\right.$$

$$\left.\left.\left.+\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\cos\left(\frac{P-\overline{P}}{2}\right)k\right)+\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\cos\left(\frac{P+\overline{P}}{2}\right)k\right)+\frac{1}{2}\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\right\}. \quad (11)$$

**Lemma 2.4** *[3] If* $s_r^n$ *and* $S_r^n$ *are the Stirling numbers of the first and second kinds, and* $k_\ell^{(n)} = k(k-\ell)(k-2\ell)\cdots(k-(n-1)\ell)$*, then*

$$k_\ell^{(n)} = \sum_{r=1}^{n} s_r^n \ell^{n-r} k^r,\ \ k^n = \sum_{r=1}^{n} S_r^n \ell^{n-r} k_\ell^{(r)} \text{ and } \Delta_\ell^{-1} k_\ell^{(\nu)} = \frac{k_\ell^{(\nu+1)}}{(\nu+1)\ell}. \quad (12)$$

## 3. Main Results

Here we introduce Stirling numbers of third kind and express the polynomial factorial $k_{\ell_a}^{(n)}$ in terms of $k_{\ell_b}^{(r)}$, $r = 1,2,\cdots n$ and Stirling numbers of third kind. Also we derive the Bernoulli's multi-series and $n\_$ multi-series to circular functions.

**Definition 3.1** *Let* $1 \le p \le n$*, The Stirling number of third kind for the positive reals* $\ell_a$ *and* $\ell_b$ *is defined by*

$$S_{p\_\ell_a}^{n\_\ell_b} = \sum_{t=p}^{n} s_t^n S_p^t \ell_a^{n-t} \ell_b^{t-p}. \quad (13)$$

**Lemma 3.2** *The expression of* $k_{\ell_a}^{(n)}$ *in terms of* $k_{\ell_b}^{(p)}$ *is given by*

$$k_{\ell_a}^{(n)} = \sum_{p=1}^{n} S_{p\_\ell_a}^{n\_\ell_b} k_{\ell_b}^{(p)}. \quad (14)$$

*Proof.* The proof follows from (13) and first, second terms of (12).

**Theorem 3.3** *Let* $p_1 = 1$, $\ell_1, \ell_2, \cdots \ell_n$ *be a set of positive reals and* $\underset{\ell_{[1,n]}}{\Delta^{-1}} = \Delta_{\ell_1}^{-1}\Delta_{\ell_2}^{-1}\cdots\Delta_{\ell_n}^{-1}$. Then

$$\Delta_{\ell_1}^{-1} k^{(0)} = \frac{k_{\ell_1}^{(1)}}{\ell_1} \text{ and } \underset{\ell_{[1,n]}}{\Delta^{-1}} k^{(0)} = \left[\prod_{r=2}^{n-1}\sum_{p_r=1}^{1+p_{r-1}} \frac{S_{p_r\_\ell_r}^{1+p_{r-1}\_\ell_{r+1}}}{(1+p_{r-1})\ell_r}\right]\frac{k_{\ell_n}^{(1+p_{n-1})}}{\ell_1(1+p_{n-1})\ell_n}. \quad (15)$$

*Proof*. Since $1 = k_{\ell_1}^{(0)} = k^{(0)}$ and $k_{\ell_1}^{(1)} = k_{\ell_2}^{(1)}$ from (12), we get $\Delta_{\ell_1}^{-1} k_{\ell_1}^{(0)} = \dfrac{k_{\ell_1}^{(1)}}{\ell_1}$. Again taking $\Delta_{\ell_2}^{-1}$, we get $\Delta_{\ell_{[1,2]}}^{-1} k^{(0)} = \dfrac{k_{\ell_2}^{(2)}}{2\ell_1\ell_2}$. Again taking $\Delta_{\ell_3}^{-1}$ on both sides of the above and applying (14), we obtain $\Delta_{\ell_{[1,3]}}^{-1} k^{(0)} = \dfrac{1}{2\ell_1\ell_2}\Delta_{\ell_3}^{-1} k_{\ell_2}^{(2)} = \dfrac{1}{2\ell_1\ell_2}\Delta_{\ell_3}^{-1}\sum_{p_2=1}^{2} S_{p_2-\ell_2}^{2-\ell_3} k_{\ell_3}^{(p_2)} = \sum_{p_2=1}^{2} S_{p_2-\ell_2}^{2-\ell_3} \dfrac{k_{\ell_3}^{(1+p_2)}}{2\ell_1\ell_2\ell_3(1+p_2)}$.

Now the proof is completed by taking $\Delta_{\ell_i}^{-1}$ and applying third relation of (12) for $i = 4,5,\cdots,n$ respectively.

The following theorem gives the complete solution of the equation (7).

**Theorem 3.4** *Consider the functions* $\underset{\ell_{[1,i]}}{u}(k)$, $\ell_i(k)$ *for* $i = 1,2,\cdots,n$, *given in the notations and above. Assume that for each* $i$, $1 \le i \le n$, $\Delta_{\ell_{[1,i]}}^{-1} u(k)$ *be any closed form solution of the difference equation* $\Delta_{\ell_{[1,i]}} v(k) = u(k)$. *Then, for* $k \ge \max_{1\le i\le n} \ell_i$,

$$\underset{\ell_{[1,n]}}{u}(k)\Big|_{\ell_n(k)}^{k} = \Delta_{\ell_{[1,n]}}^{-1} u(k) + \sum_{t=1}^{n}(-1)^t \sum_{\{m_s\}_{s=1}^{t}\in t(J_n)} \Delta_{\ell_{[1,m_1]}}^{-1} u(\ell_{m_1}(k))$$

$$\times\prod_{i=1}^{t} \Delta_{\ell_{[1+m_i,m_{i+1}]}}^{-1} (\ell_{m_{i+1}}(k))^{(0)} \Delta_{\ell_{[1+m_t,n]}}^{-1} k^{(0)} \qquad (16)$$

is the complete solution of equation (7).

*Proof*. Since $1 = k^{(0)}$, applying the limit from $\ell_1(k)$ to $k$ for $\Delta_{\ell_1}^{-1} u(k)$, we have

$$\Delta_{\ell_1}^{-1} u(k)\Big|_{\ell_1(k)}^{k} = \Delta_{\ell_1}^{-1} u(k) - \Delta_{\ell_1}^{-1} u(\ell_1(k)) k^{(0)},$$

which is a complete form solution of equation (7) for $n = 1$.

Taking $\Delta_{\ell_2}^{-1}$ on both sides and applying the limits from $\ell_2(k)$ to $k$ and keeping $\Delta_{\ell_1}^{-1} u(\ell_1(k))$ as a constant, we obtain

$$\Delta_{\ell_2}^{-1}(\Delta_{\ell_1}^{-1} u(k)\Big|_{\ell_1(k)}^{k})\Big|_{\ell_2(k)}^{k} = \Delta_{\ell_{[1,2]}}^{-1} u(k)\Big|_{\ell_2(k)}^{k} - \Delta_{\ell_1}^{-1} u(\ell_1(k))\Delta_{\ell_2}^{-1} k^{(0)}\Big|_{\ell_2(k)}^{k}$$

which is the complete solution of the equation (7) and it can be expressed as

$$\underset{\ell_{[1,2]}}{u}(k)\Big|_{\ell_2(k)}^{k} = \Delta_{\ell_{[1,2]}}^{-1} u(k) - \Delta_{\ell_1}^{-1} u(\ell_1(k))\Delta_{\ell_2}^{-1} k^{(0)} - \Delta_{\ell_{[1,2]}}^{-1} u(\ell_2(k)) + \Delta_{\ell_1}^{-1} u(\ell_1(k))\Delta_{\ell_2}^{-1}(\ell_2(k))^{(0)}.$$

In the right hand side of the above expression, second term is associated to $\{m_1\} = \{1\} \in 1(J_2)$, third term to $\{m_1\} = \{2\} \in 1(J_2)$ and the fourth term to $\{m_1, m_2\} = \{1,2\} \in 2(J_2)$. Taking $\Delta_{\ell_3}^{-1}$ on $u_2(k)$, applying the limits $\ell_3(k)$ and $k$, and as $\Delta_{\ell_1}^{-1} u(\ell_1(k))$, $\Delta_{\ell_2}^{-1}(\ell_2(k))^{(0)}$ and $\Delta_{\ell_{[1,2]}}^{-1} u(\ell_2(k))$ are constants, we get $\underset{\ell_{[1,3]}}{u}(k)\Big|_{\ell_3(k)}^{k} = \Delta_{\ell_3}^{-1}\underset{\ell_{[1,2]}}{u}(k) - \Delta_{\ell_3}^{-1}\underset{\ell_{[1,2]}}{u}(\ell_3(k))$ and is same as

$$\underset{\ell_{[1,3]}}{u}(k)\Big|_{\ell_3(k)}^{k}=\underset{\ell_{[1,3]}}{\Delta^{-1}}u(k)+\sum_{t=1}^{3}(-1)^t\sum_{\{m_s\}_1^t\in t(J_n)}\underset{\ell_{[1,m_1]}}{\Delta^{-1}}u(\ell_{m_1}(k))\times\prod_{i=1}^{t}\underset{\ell_{[1+m_i,m_{i+1}]}}{\Delta^{-1}}(\ell_{m_{i+1}}(k))^{(0)}\underset{\ell_{[1+m_t,3]}}{\Delta^{-1}}k^{(0)}$$

which is a complete solution of the equation (7) for $n=3$.

As all the lower limit values are constants, the proof is completed by taking $\Delta_{\ell_i}^{-1}$ and applying the limit from $\ell_i(k)$ to $k$ on $\underset{[1,3]}{u}(k)$ successively for $i=4,5,\cdots,n$.

The following theorem gives a numerical solution of the equation (7).

**Theorem 3.5** *Consider the assumptions of Theorem 3.4. Then, for* $k\geq\sum_{i=1}^{n}\ell_i$,

$$v(k)=\sum_{m=1}^{n}\sum_{\ell_{[m,n]}}\underset{\ell_{[1,m-1]}}{u}(\ell_{m-1}(\tilde{k})) \tag{17}$$

is the numerical solution of the difference equation (7).

*Proof*. From equation (6), we have

$$\Delta_{\ell_1}^{-1}u(k)\Big|_{\ell_1(k)}^{k}=\sum_{\ell_{[1,1]}}u(\tilde{k})=\underset{\ell_{[1,1]}}{u}(k)-\underset{\ell_{[1,1]}}{u}(\ell_1(k))=z_1(k),\text{(say)} \tag{18}$$

is a numerical solution of the equation (7) for $n=1$. Again taking $\Delta_{\ell_2}^{-1}$ on $z_1(k)$ and applying equation (6), we get

$$\Delta_{\ell_2}^{-1}z_1(k)\Big|_{\ell_2(k)}^{k}=\sum_{\ell_{[2,2]}}z_1(k)=z_2(k),\text{(say)} \tag{19}$$

which is a numerical solution of the equation (7) for $n=2$.
Replacing $k$ by $k-r_2\ell_2$ in (18), we obtain

$$z_1(k-r_2\ell_2)=\underset{\ell_{[1,1]}}{u}(k-r_2\ell_2)-\underset{\ell_{[1,1]}}{u}(\ell_1(k-r_2\ell_2)). \tag{20}$$

Substituting (20) in (19), we find that

$$z_2(k)=\sum_{\ell_{[2,2]}}\underset{\ell_{[1,1]}}{u}(\tilde{k})-\sum_{\ell_{[2,2]}}\underset{\ell_{[1,1]}}{u}(\ell_1(\tilde{k})) \tag{21}$$

which is the same as

$$z_2(k)=\underset{\ell_{[1,2]}}{u}(k)-\underset{\ell_{[1,2]}}{u}(\ell_2(k))-\sum_{\ell_{[2,2]}}\underset{\ell_{[1,1]}}{u}(\ell_1(\tilde{k})). \tag{22}$$

Applying the numerical solution $z_2(k)=\sum_{\ell_{[1,2]}}u(\tilde{k})$ on (22), we get

$$\underset{\ell_{[1,2]}}{u}(k)\Big|_{\ell_2(k)}^{k}=\sum_{\ell_{[1,2]}}u(\tilde{k})+\sum_{\ell_{[2,2]}}\underset{\ell_{[1,1]}}{u}(\ell_1(\tilde{k})), \tag{23}$$

where the values $\underset{\ell_{[1,1]}}{u}(\ell_1(k-r_2\ell_2))$ can be evaluated by replacing $k$ by $\ell_1(k-r_2\ell_2)$ in the closed form solution $\underset{\ell_{[1,1]}}{u}(k)\Big|_{\ell_1(k)}^{k}$ given in Theorem 3.4 for $n=1$.

Taking $\Delta_{\ell_3}^{-1}$ on $z_2(k)$ and applying equation (7) yield

$$\Delta_{\ell_3}^{-1}z_2(k)\Big|_{\ell_3(k)}^{k}=\sum_{\ell_{[3,3]}}z_2(k)=z_3(k)\text{ (say)}. \tag{24}$$

Replacing $k$ by $k-r_3\ell_3$ in (22), we have

$$z_2(k-r_3\ell_3) = \underset{\ell_{[1,2]}}{u}(k-r_3\ell_3) - \underset{\ell_{[1,2]}}{u}(\ell_2(k-r_3\ell_3)) - \sum_{r_2=1}^{\left[\frac{k-r_3\ell_3}{\ell_2}\right]} \underset{\ell_{[1,1]}}{u}(\ell_1(k-r_3\ell_3-r_2\ell_2)). \qquad (25)$$

Substituting (25) in (24), we obtain

$$z_3(k) = \sum_{\ell_{[3,3]}} \underset{\ell_{[1,2]}}{u}(\widetilde{k}) - \sum_{\ell_{[3,3]}} \underset{\ell_{[1,2]}}{u}(\ell_2(\widetilde{k})) - \sum_{\ell_{[2,3]}} \underset{\ell_{[1,1]}}{u}(\ell_1(\widetilde{k}))$$

which is the same as

$$z_3(k) = \underset{\ell_{[1,3]}}{u}(k) - \underset{\ell_{[1,3]}}{u}(\ell_3(k)) - \sum_{\ell_{[3,3]}} \underset{\ell_{[1,2]}}{u}(\ell_2(\widetilde{k})) - \sum_{\ell_{[2,3]}} \underset{\ell_{[1,1]}}{u}(\ell_1(\widetilde{k})).$$

Since $z_3(k)$ is a solution of the equation (7), taking the numerical solution for it, then for $n=3$, we find that

$$\underset{\ell_{[1,3]}}{u}(k)\Big|_{\ell_3(k)}^{k} = \sum_{\ell_{[1,3]}} u(\widetilde{k}) + \sum_{\ell_{[2,3]}} \underset{\ell_{[1,1]}}{u}(\ell_1(\widetilde{k})) + \sum_{\ell_{[3,3]}} \underset{\ell_{[1,2]}}{u}(\ell_2(\widetilde{k})) \qquad (26)$$

where the values $\underset{\ell_{[1,1]}}{u}(\ell_1(k-r_3\ell_3-r_2\ell_2))$ and $\underset{\ell_{[1,2]}}{u}(\ell_2(k-r_3\ell_3))$ can be evaluated by replacing $k$ by $\ell_1(k-r_3\ell_3-r_2\ell_2)$ and $\ell_2(k-r_3\ell_3)$ in the closed form solutions $\underset{\ell_{[1,n]}}{u}(k)\Big|_{\ell_n(k)}^{k}$ given in Theorem 3.4 for $n=1,2$.

The proof is completed by taking $\Delta_{\ell_i}^{-1}$ on $z_3(k)$ and applying the numerical solution mentioned in (7) successively for $i=4,5,\cdots,n$.

The following theorem is the $n\_$ multi-series of $u(k)$.

**Theorem 3.6** *The numerical-complete relation of the difference equation (7) is given by*

$$\sum_{m=1}^{n} \sum_{\ell_{[m,n]}} \underset{\ell_{[1,m-1]}}{u}(\ell_{m-1}(\widetilde{k})) = \underset{\ell_{[1,n]}}{\Delta^{-1}} u(k) + \sum_{t=1}^{n} \sum_{\{m_s\}_{s=1}^{t} \in t(J_n)} (-1)^t$$
$$\times \underset{\ell_{[1,m_1]}}{\Delta^{-1}} u(\ell_{m_1}(k)) \prod_{i=1}^{t} \underset{\ell_{[1+m_i,m_{i+1}]}}{\Delta^{-1}} (\ell_{m_{i+1}}(k))^{(0)} \underset{\ell_{[1+m_t,n]}}{\Delta^{-1}} k^{(0)}. \qquad (27)$$

*Proof.* The proof follows by equating the numerical solution given in Theorem 3.5 and the complete solution given in Theorem 3.4.

**Theorem 3.7** *If $n_1$ and $n_2$ are odd positive integers, then*

$$\underset{\ell_{[1,n]}}{\Delta^{-1}} \sin^{n_1} pk \cos^{n_2} qk = \frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2+n-1}} \sum_{s_1=0}^{\frac{n_1-1}{2}} \sum_{s_2=0}^{\frac{n_2-1}{2}} \sum_{t=0}^{n} \sum_{A\in t(L_n)} (-1)^{(n-t)+s_1}$$
$$\times \frac{n_1^{(s_1)}}{s_1!} \frac{n_2^{(s_2)}}{s_2!} \left( \frac{\sin P(k-A)}{\prod_{i=1}^{n}\left(1-\cos P\ell_i\right)} + \frac{\sin \overline{P}(k-A)}{\prod_{i=1}^{n}\left(1-\cos \overline{P}\ell_i\right)} \right). \qquad (28)$$

*Proof.* Applying $\Delta_{\ell_1}^{-1}$ to equation (10), we get

$$\Delta_{\ell_1}^{-1}\sin^{n_1}pk\cos^{n_2}qk=\sum_{s_1=0}^{\frac{n_1-1}{2}}\sum_{s_2=0}^{\frac{n_2-1}{2}}\frac{(-1)^{\frac{n_1-1}{2}+(n-t)+s_1}}{2^{n_1+n_2}}\frac{n_1^{(s_1)}}{s_1!}\frac{n_2^{(s_2)}}{s_2!}$$

$$\times\left(\frac{\sin P(k-\ell_1)-\sin P(k)}{(1-\cos P\ell_1)}+\frac{\sin \overline{P}(k-\ell_1)-\sin \overline{P}(k)}{(1-\cos \overline{P}\ell_1)}\right). \qquad (29)$$

Similarly we find $\underset{\ell_{[1,2]}}{\Delta^{-1}}\sin^{n_1}pk\cos^{n_2}qk$ ,

$$\sum_{s_1=0}^{\frac{n_1-1}{2}}\sum_{s_2=0}^{\frac{n_2-1}{2}}\sum_{t=0}^{2}\sum_{A\in t(L_2)}\frac{(-1)^{\frac{n_1-1}{2}+(2-t)+s_1}}{2^{n_1+n_2+1}}\frac{n_1^{(s_1)}}{s_1!}\frac{n_2^{(s_2)}}{s_2!}\left(\frac{\sin P(k-A)}{\prod_{i=1}^{2}(1-\cos P\ell_i)}+\frac{\sin \overline{P}(k-A)}{\prod_{i=1}^{2}(1-\cos \overline{P}\ell_i)}\right).$$

Proceeding like this, we get (28).

**Remark 3.8** $\underset{\ell_{[1,n]}}{\Delta^{-1}}\sin^{n_1}pk$ *and* $\underset{\ell_{[1,n]}}{\Delta^{-1}}\cos^{n_2}pk$ *can be obtained by putting* $n_2=0$ *and* $n_1=0$ *i n (28) respectively.*

**Theorem 3.9** *If* $n_1$ *is an odd and* $n_2$ *is an even positive integers, then*

$$\underset{\ell_{[1,n]}}{\Delta^{-1}}\sin^{n_1}pk\cos^{n_2}qk=\frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2+n-1}}\sum_{s_1=0}^{\frac{n_1-1}{2}}\frac{n_1^{(s_1)}}{s_1!}\sum_{t=0}^{n}\sum_{A\in t(L_n)}(-1)^{(n-t)+s_1}\left\{\sum_{s_2=0}^{\frac{n_2-1}{2}}\frac{n_2^{(s_2)}}{s_2!}\right.$$

$$\left.\times\left(\frac{\sin P(k-A)}{\prod_{i=1}^{n}(1-\cos P\ell_i)}+\frac{\sin \overline{P}(k-A)}{\prod_{i=1}^{n}(1-\cos \overline{P}\ell_i)}\right)+\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\frac{\sin\left(\frac{P+\overline{P}}{2}\right)(k-A)}{\prod_{i=1}^{n}\left(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell_i\right)}\right\}. \qquad (30)$$

**Theorem 3.10** *If* $n_1$ *is an even and* $n_2$ *is an odd positive integers, then*

$$\underset{\ell_{[1,n]}}{\Delta^{-1}}\sin^{n_1}pk\cos^{n_2}qk=\frac{(-1)^{\frac{n_1}{2}}}{2^{n_1+n_2+n-1}}\sum_{s_2=0}^{\frac{n_2-1}{2}}\frac{n_2^{(s_2)}}{s_2!}\left\{\sum_{t=0}^{n}\sum_{A\in t(L_n)}\sum_{s_1=0}^{\frac{n_1-1}{2}}\frac{n_1^{(s_1)}}{s_1!}(-1)^{(n-t)+s_1}\right.$$

$$\left.\times\left(\frac{\cos P(k-A)}{\prod_{i=1}^{n}(1-\cos P\ell_i)}+\frac{\cos \overline{P}(k-A)}{\prod_{i=1}^{n}(1-\cos \overline{P}\ell_i)}\right)+\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\frac{\cos\left(\frac{P-\overline{P}}{2}\right)(k-A)}{\prod_{i=1}^{n}\left(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell_i\right)}\right\}. \qquad (31)$$

**Theorem 3.11** *If* $n_1$ *and* $n_2$ *are even positive integers, then*

$$\underset{\ell_{[1,n]}}{\Delta^{-1}} \sin^{n_1} pk \cos^{n_2} qk = \frac{(-1)^{\frac{n_1}{2}}}{2^{n_1+n_2+n-1}}\left\{\left(\sum_{s_2=0}^{\frac{n_2-2}{2}} \frac{n_2^{(s_2)}}{s_2!}\left(\sum_{t=0}^{n}\sum_{A\in t(L_n)}\sum_{s_1=0}^{\frac{n_1-2}{2}}\frac{n_1^{(s_1)}}{s_1!}(-1)^{(n-t)+s_1}\right.\right.\right.$$

$$\times\left(\frac{\cos P(k-A)}{\prod_{i=1}^{n}(1-\cos P\ell_i)}+\frac{\cos \overline{P}(k-A)}{\prod_{i=1}^{n}(1-\cos \overline{P}\ell_i)}\right)+\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\frac{\cos\left(\frac{P+\overline{P}}{2}\right)(k-A)}{\prod_{i=1}^{n}\left(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell_i\right)}\Bigg)$$

$$+\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\frac{\cos\left(\frac{P-\overline{P}}{2}\right)(k-A)}{\prod_{i=1}^{n}\left(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell_i\right)}\Bigg)+2^{n-1}\frac{n_1^{\left(\frac{n_1}{2}\right)}}{\left(\frac{n_1}{2}\right)!}\frac{n_2^{\left(\frac{n_2}{2}\right)}}{\left(\frac{n_2}{2}\right)!}\left[\prod_{r=2}^{n-1}\sum_{p_r=1}^{1+p_{r-1}}\frac{S_{p_r-\ell_r}^{1+p_{r-1}-\ell_r+1}}{(1+p_{r-1})\ell_r}\right]\frac{k_{\ell_n}^{(1+p_{n-1})}}{\ell_1(1+p_{n-1})\ell_n}\Bigg\}. \tag{32}$$

**Note 3.12** *The n_multiseries to* $\sin^{n_1} pk\cos^{n_2} qk$ *for the four cases can be obtained by applying (28), (30), (31) and (32) to equation (27) respectively.*

**Remark 3.13** *If we take* $\ell_1=\ell_2=\ell_3=\cdots=\ell_n=\ell$, *we get* $\underset{\ell_{[1,n]}}{\Delta^{-1}} u(k)=\Delta_\ell^{-m}u(k)$. *and all n_multi-series becomes m-series denoted in [1].*

The following example illustrates a 2_multi-series for $\sin^{n_1} pk\cos^{n_2} qk$

**Example 3.14** *Consider the case* $n=2$ *in equation (27),* $n_1=4$, $n_2=4$, $k=10.3$, $\ell_1=3.1$, $\ell_2=4.2$, $p=7, q=3$, *then* $\ell_1(k)=1$, $\ell_2(k)=1.9$. *Let* $P=\left(7(2-2s_1)+3(2-2s_2)\right)$ *and* $\overline{P}=\left(7(2-2s_1)-3(2-2s_2)\right)$.

$$\sum_{m=1}^{2}\sum_{\ell_{[m,2]}}\underset{\ell_{[1,m-1]}}{\mathrm{u}}(\ell_{m-1}(\tilde{k})) = \underset{\ell_{[1,2]}}{\Delta^{-1}} u(k)+\sum_{t=1}^{2}\sum_{\{m_s\}_{s=1}^{t}\in t(J_2)}(-1)^t$$

$$\times\underset{\ell_{[1,m_1]}}{\Delta^{-1}} u(\ell_{m_1}(\tilde{k}))\prod_{i=1}^{t}\underset{\ell_{[1+m_i,m_{i+1}]}}{\Delta^{-1}}(\ell_{m_{i+1}}(k))^{(0)}\underset{\ell_{[1+m_t,n]}}{\Delta^{-1}} k^{(0)}. \tag{33}$$

LHS of equation (33) is the sum of the terms

$m=1;\ \sum_{\ell_{[1,2]}} u(\tilde{k})$.

$m=2;\ \sum_{\ell_{[2,2]}}\underset{\ell[1,1]}{\mathrm{u}}(\ell_1(\tilde{k}))=\underset{\ell[1,1]}{\mathrm{u}}(3)+\underset{\ell[1,1]}{\mathrm{u}}(1.9)$.

RHS of equation (33) is the sum of the terms

$$\text{(i) } \Delta_{\ell_2}^{-1}\sin^4 pk\cos^4 qk=\frac{(-1)^{\frac{4}{2}}}{2^9}\left\{\left(\sum_{s_2=0}^{\frac{4-2}{2}}\frac{4^{(s_2)}}{s_2!}\left(\sum_{t=0}^{2}\sum_{A\in t(L_2)}\sum_{s_1=0}^{\frac{4-2}{2}}\frac{4^{(s_1)}}{s_1!}(-1)^{(2-t)+s_1}\right.\right.\right.$$

$$\times\left(\frac{\cos P(10.3-A)}{\prod_{i=1}^{2}(1-\cos P\ell_i)}+\frac{\cos\overline{P}(10.3-A)}{\prod_{i=1}^{2}(1-\cos\overline{P}\ell_i)}\right)+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}$$

$$\left.\left.\times\frac{\cos\left(\frac{P+\overline{P}}{2}\right)(10.3-A)}{\prod_{i=1}^{2}\left(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell_i\right)}\right)+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{\cos\left(\frac{P-\overline{P}}{2}\right)(10.3-A)}{\prod_{i=1}^{2}\left(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell_i\right)}\right)2\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{10.3_{\ell_2}^{(2)}}{2\ell_1\ell_2}\right\}$$

( the terms for $1(J_4)=\{\{1\},\{2\}\}$)

$$\text{(ii) } \Delta_{\ell_1}^{-1}\sin^4 p\ell_1\cos^4 q\ell_1\times\frac{10.3_{\ell_2}^{(1)}}{\ell_2}=-\frac{(-1)^{\frac{4}{2}}}{2^8}\left\{\left(\sum_{s_2=0}^{\frac{4-2}{2}}\frac{4^{(s_2)}}{s_2!}\left(\sum_{t=0}^{1}\sum_{A\in t(L_2)}\sum_{s_1=0}^{\frac{4-2}{2}}\frac{4^{(s_1)}}{s_1!}(-1)^{(1-t)+s_1}\right.\right.\right.$$

$$\times\left(\frac{\cos P(1-A)}{(1-\cos P\ell_i)}+\frac{\cos\overline{P}(1-A)}{(1-\cos\overline{P}\ell_i)}\right)+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{\cos\left(\frac{P+\overline{P}}{2}\right)(1-A)}{\left(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell_i\right)}$$

$$\left.\left.+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{\cos\left(\frac{P-\overline{P}}{2}\right)(1-A)}{\left(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell_i\right)}\right)+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{1}{\ell_1}\right\}\times\frac{10.3_{\ell_2}^{(1)}}{\ell_2},$$

$$\text{(iii) } \Delta_{\ell_2}^{-1}\sin^4 p\ell_2\cos^4 q\ell_2=-\frac{(-1)^{\frac{4}{2}}}{2^9}\left\{\left(\sum_{s_2=0}^{\frac{4-2}{2}}\frac{4^{(s_2)}}{s_2!}\left(\sum_{t=0}^{4}\sum_{A\in t(L_2)}\sum_{s_1=0}^{\frac{4-2}{2}}\frac{4^{(s_1)}}{s_1!}(-1)^{(4-t)+s_1}\right.\right.\right.$$

$$\times\left(\frac{\cos P(1.9-A)}{\prod_{i=1}^{2}(1-\cos P\ell_i)}+\frac{\cos\overline{P}(1.9-A)}{\prod_{i=1}^{2}(1-\cos\overline{P}\ell_i)}\right)+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{\cos\left(\frac{P+\overline{P}}{2}\right)(2.1-A)}{\prod_{i=1}^{2}\left(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell_i\right)}$$

$$\left.\left.+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{\cos\left(\frac{P-\overline{P}}{2}\right)(1.9-A)}{\prod_{i=1}^{2}\left(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell_i\right)}\right)+2\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{1.9_{\ell_2}^{(2)}}{2\ell_1\ell_2}\right\},$$

(the terms for $2(J_4)=\{1,2\}$ )

$$\text{(iv) } \Delta_{\ell_1}^{-1}\sin^4 p\ell_1\cos^4 q\ell_1\times\frac{1.9^{(1)}_{\ell_2}}{\ell_2}=\frac{(-1)^{\frac{4}{2}}}{2^8}\Bigg\{\Bigg(\sum_{s_2=0}^{\frac{4-2}{2}}\frac{4^{(s_2)}}{s_2!}\Bigg(\sum_{t=0}^{1}\sum_{A\in t(L_2)}\sum_{s_1=0}^{\frac{4-2}{2}}\frac{4^{(s_1)}}{s_1!}$$

$$\times(-1)^{(1-t)+s_1}\left(\frac{\cos P(1-A)}{(1-\cos P\ell_i)}+\frac{\cos\overline{P}(1-A)}{(1-\cos\overline{P}\ell_i)}\right)+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{\cos\left(\frac{P+\overline{P}}{2}\right)(1-A)}{\left(1-\cos\left(\frac{P+\overline{P}}{2}\right)\ell_i\right)}\Bigg)$$

$$+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{\cos\left(\frac{P-\overline{P}}{2}\right)(1-A)}{\left(1-\cos\left(\frac{P-\overline{P}}{2}\right)\ell_i\right)})+\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{4^{\left(\frac{4}{2}\right)}}{\left(\frac{4}{2}\right)!}\frac{1}{\ell_1}\Bigg\}\times\frac{1.9^{(1)}_{\ell_2}}{\ell_2}.$$

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