

Distance r -Coloring and Distance r -Dominator Coloring number of a graph

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Abstract— Given a positive integer r , two vertices $u, v \in V(G)$ are r -independent if $d(u, v) > r$. A partition of $V(G)$ into r -independent set is called distance r -coloring . A study of distance r -coloring and distance r -dominator coloring concepts are studied in this paper.

Keywords— distance r -chromatic number, distance r -independent color partition , distance r -independent and distance r -dominator coloring.

1. INTRODUCTION

Consider a network and a coloring scheme for the nodes. Two nodes are compatible if they receive the same color. The usual coloring scheme stipulates that adjacent nodes should not receive the same color. Such a scheme is helpful in storage problem of chemicals where two non-compatible chemicals (two chemicals which when placed nearby will cause danger) cannot be stored in the same room. The chromatic number of such a scheme will give the minimum number of storage spaces required for keeping all the chemicals without any problem. In a Institution, groups may exists. People in different groups may not be well disposed to each other. The affinity of people within a group may be determined by the closeness between the people and the number of people in the group. In the graphical model of this situation, all the nodes of the same group may

be given the same color and different groups may be given different colors. Two important aspects of graphs are partitions of vertex set into sets with a prescribed property and partition of edge set with a prescribed properties. The first gives rise to different types of colourings and the second leads to decomposition in graphs. We introduce a new coloring based on the distance. Given a positive integer r , two vertices $u, v \in V(G)$ are r -adjacent if u, v are adjacent in G^r and are r -independent if they are independent in G^r . A partition of $V(G)$ into r -independent sets is called a distance r -coloring. These are same as proper coloring in G^r . The chromatic number of G^r will coincide with the distance r -chromatic number of G .

1.1 DISTANCE r -COLORING IN GRAPHS

Definition 1.1 For any integer $r \geq 1$, a graph $G = (V, E)$ is said to be r -complete if every vertex in $V(G)$ is r -adjacent to every other vertex in $V(G)$. The maximum cardinality of a subset S of $V(G)$ such that $\langle S \rangle$ is r -complete is called r -clique number of G and is denoted by $\omega_r(G)$.

Definition 1.2 Let $r \geq 1$. Let $u, v \in V(G)$. A vertex u distance r -dominators a vertex v if

$$d(u, v) \leq r.$$

Definition 1.3 A subset I of $V(G)$ is distance r -independent set if for any $u, v \in I$, $d(u, v) > r$. The maximum cardinality of a distance r -independence set of G is called the distance r -independence number of G and is denoted by $\beta_r(G)$.

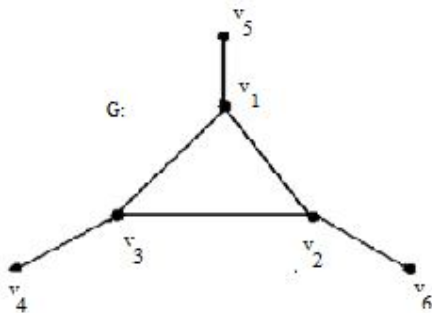
Definition 1.4 [6] A partition of $V(G)$ is called distance r -independent color partition of G if each

element of the partition is distance r -independent. The minimum cardinality of a distance r -independent color of G is called distance r -chromatic number and is denoted by $\chi_r(G)$.

Remark 1.5 Let G be a simple graph. Let $V(G) = \{v_1, v_2, \dots, v_n\}$. Let $\pi = \{\{v_1\}\}, \{\{v_2\}\}, \dots, \{\{v_n\}\}$ be a distance r -color partition of a graph G . Therefore existence of distance r -color partition of any graph is guaranteed.

Remark 1.6 Any distance r -color partition of G is a proper color partition of G but not the other way. For example,

Example 1.7



$\{v_1, v_6\}, \{v_3, v_5\}, \{v_2, v_4\}$ is a proper color partition of G , but it is not a distance 2-color partition of G .

Observation 1.8 For any graph G , $\chi(G) \leq \chi_r(G)$ 1.2 $\chi_r(G)$ for standard graphs

1. $\chi_r(\overline{K_n}) = 1$, for all r .
2. $\chi_r(K_n) = n$, for all r .
3. $\chi_r(K_{1,n}) = \begin{cases} 2, & \text{if } r = 1 \\ n + 1, & \text{if } r \geq 2 \end{cases}$
4. $\chi_r(W_n) = \begin{cases} 3, & \text{if } r = 1 \text{ and } n \text{ is odd} \\ 2, & \text{if } r = 1 \text{ and } n \text{ is even} \\ n, & \text{if } r \geq 2 \end{cases}$

$$5. \chi_r(K_{m,n}) = \begin{cases} 2, & \text{if } r = 1 \\ m + n, & \text{if } r \geq 2 \end{cases}$$

$$6. \chi_r(P_n) = \begin{cases} r + 1, & \text{if } 1 \leq r < n - 1 \\ n, & \text{if } r \geq n - 1 \end{cases}$$

$$7. \chi_r(C_n) = \begin{cases} \text{If } r < \left\lfloor \frac{n}{2} \right\rfloor \\ r + 1, & \text{if } n \equiv 0 \pmod{r + 1} \\ r + 2, & \text{otherwise} \\ n, & \text{if } r \geq \left\lfloor \frac{n}{2} \right\rfloor \end{cases}$$

8. Let $D_{m,n}$ be a double star with $m < n$.

$$\chi_r(D_{m,n}) = \begin{cases} 2, & \text{if } r = 2 \\ m + n + 2, & \text{if } r \geq 3 \\ n + 2, & \text{if } r \geq 2 \end{cases}$$

Observation 1.9 If G has diameter ≤ 2 , then

$$\chi_r(G) = \begin{cases} n, & \text{if } r > 2 \\ \chi(G), & \text{if } r = 1 \end{cases}$$

Observation 1.10 For any graph G , $1 \leq \chi_r(G) \leq n$.

Theorem 1.11 $\chi_r(G) = 1$ if and only if $G = \overline{K_n}$.

Theorem 1.12 For any graph G , $\chi_r(G) = n$ if and only if $r \geq \text{diam}(G)$.

Theorem 1.13 Let G be a graph of order n . Then

$$\frac{n}{\beta_r(G)} \leq \chi_r(G) \leq n - \beta_r(G) + 1$$

Proof: Let $\chi_r(G) = s$. Let $\pi = \{V_1, V_2, \dots, V_s\}$ be a χ_r -partition of $V(G)$. Therefore $\beta_r(G) \geq |V_i|$, for all i , ($1 \leq i \leq s$). Now $n =$

$|V_1| + |V_2| + \dots + |V_s| \leq s\beta_r(G)$. Hence $\beta_r(G)\chi_r(G) \geq n$. Let D be a β_r -set of G . Let $D = \{u_1, u_2, \dots, u_{\beta_r(G)}\}$. Let $\pi = \{D, \{u_{\beta_r+1}\}, \dots, \{u_n\}\}$.

Then π is a distance r -color partition of G .

Therefore $\pi \geq \chi_r(G)$.

That is $n - \beta_r(G) + 1 \geq \chi_r(G)$.

Corollary 1.14 Let G be a graph of order n . Then $2\sqrt{n} \leq \beta_r(G) + \chi_r(G) \leq n + 1$.

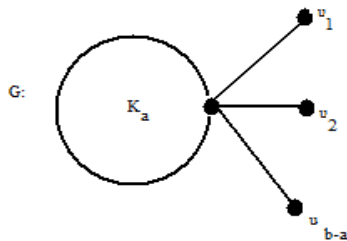
Remark 1.15 For any graph G , $\omega_r(G) \leq \chi_r(G) \leq 1 + \Delta_r(G)$.

Remark 1.16 For any graph G , if $r = \text{diam}(G)$, then $\chi_r(G) = 1 + \Delta_r(G)$.

Proof: If $r = \text{diam}(G)$, then every vertex of G is r -adjacent to every other vertex of G . Hence $\chi_r(G) = n = 1 + \Delta_r(G)$.

Proposition 1.17 Given positive integers a, b , and r with $a \leq b$, there exists a connected graph G such that $\chi(G) = a, \chi_r(G) = b$.

Proof: Case 1: $a = b$, For any $r \geq 1$. Then $\chi(K_a) = \chi_r(K_a) = a = b$.



Case 2: $a < b$. Then $r \geq 2$. Attach $(b-a)$ pendent vertices at a vertex of K_a . Let G be the resulting graph. Then $\chi(G) = a$ and $\chi_r(G) = b$.

Proposition 1.18 Let G be a connected graph of order ≥ 3 which is non complete. Let G have a full degree vertex. Then $\chi(G) \neq \chi_r(G)$, for all $r \geq 2$.

Proof: Let G be a connected graph of order ≥ 3 which is non complete. Let G have a full degree vertex. Then $\text{diam}(G) \leq 2$ and hence

$\chi_r(G) = |V(G)|$ and $\chi(G) < n$. Therefore

$\chi(G) < \chi_r(G)$.

That is $\chi(G) \neq \chi_r(G)$, for all $r \geq 2$.

Theorem 1.19 For any spanning subgraph H of G , $\chi_r(H) \leq \chi_r(G)$.

Theorem 1.20 Suppose G is disconnected.

Let $r \geq 2$. Then $\chi_r(G) = \chi(G)$ if and only if there exists a component of G which is complete and whose distance r -chromatic number is $\chi_r(G)$.

Proof: Let G be disconnected. Let G_1, G_2, \dots, G_k be the component of G . Let $r \geq 2$ and

$\chi(G) = \chi_r(G), \chi(G) = \max_{1 \leq i \leq k} \{\chi(G_i)\} = \chi(G)$ say

$\chi_r(G) = \max_{1 \leq i \leq k} \{\chi_r(G_i)\} = \chi_r(G_1)$. Without loss of

generality, let $\max_{1 \leq i \leq k} \{\chi_r(G_i)\} = \chi_r(G_s)$. Then

$\chi_r(G_s) = \chi(G_1)$.

But $\chi_r(G_1) \geq \chi(G_1) = \chi_r(G_s) \geq \chi_r(G_1)$. Therefore

$\chi_r(G_1) = \chi_r(G_s)$ and hence $\chi_r(G_1) = \chi(G_s)$.

Therefore G_1 complete and $\chi(G_1) = \chi(G) = \chi_r(G)$. Conversely, let G be disconnected and $r \geq 2$.

Let G contain a component, say G_1 which is complete and $\chi_r(G_1) = \chi_r(G)$. Since G_1 is complete, $\chi(G_1) = \chi_r(G_1) = \chi_r(G)$. Suppose

$\chi(G_1) < \chi(G_s) = \chi(G)$.

Then $\chi_r(G) < \chi(G_s) = \chi(G)$, a contradiction.

Thus $\chi(G_1) \geq \chi(G_s)$, for all s and so

$\chi(G_1) = \chi(G)$.

That is, $\chi_r(G) = \chi(G)$.

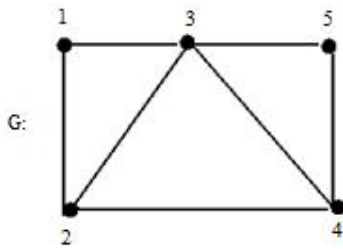
2.DISTANCE r -DOMINATOR COLORING IN GRAPHS

Definition 2.1 A distance r -color partition is called distance- r -dominator color partition if each vertex

of G distance- r -dominates every vertex of some distance- r -color class of the partition. The minimum cardinality of a distance- r -dominator color partition

is called distance- r -dominator coloring number of G and is denoted by $\chi_r^d(G)$

Example 2.2



Let $r = 1$. Then $\{ \{1,4\} , \{2,5\} , \{3\} \}$ is a dominator color partition. Therefore $\chi_r^d(G) = 3$.

Let $r = 2$. Then $\chi_r^d(G) = 5$.

Observation 2.3 If $r \geq \text{diam}(G)$, then $\chi_{dr}(G) = n$, where $n = |V(G)|$.

Observation 2.4 1. $\chi_r^d(K_n) = n$, for all r .

$$2. \chi_r^d(K_{1,n}) = \begin{cases} 2, & \text{if } r = 1 \\ n+1, & \text{if } r \geq 2 \end{cases}$$

$$3. \chi_r^d(K_{m,n}) = \begin{cases} 2, & \text{if } r = 1 \\ m+n, & \text{if } r \geq 2 \end{cases}$$

$$4. \chi_r^d(W_n) = \begin{cases} 3, & \text{if } r=1 \text{ and } n \text{ is odd} \\ 4, & \text{if } r = 1 \text{ and } n \text{ is even} \\ n, & \text{if } r \geq 2 \end{cases}$$

$$5. \chi_r^d(P_n) = \begin{cases} \left\lceil \frac{n}{2r+1} \right\rceil + r, & \text{if } r \leq n-2 \\ n, & \text{if } r \geq n-1 \end{cases}$$

$$6. \chi_r^d(C_n) = \begin{cases} \left\lceil \frac{n}{2r+1} \right\rceil + r + 1, & \text{if } r < \left\lfloor \frac{n}{2} \right\rfloor \\ n, & \text{if } r \geq \left\lfloor \frac{n}{2} \right\rfloor \end{cases}$$

Observation 2.5 For any graph G , $2 \leq \chi_r^d(G) \leq n$. The upper bound is sharp if $r = \text{diam}(G)$.

Observation 2.6 For any graph G , $\chi_r(G) \leq \chi_r^d(G)$.

Theorem 2.7 For any graph G , $\max\{ \chi_r(G), \gamma_r(G) \leq \chi_r^d(G) \leq \chi_r(G) + \gamma_r(G) \}$

Proof: Let $\{V_1, V_2, \dots, V_T\}$ be a $\chi_r^d(G)$ - partition of $V(G)$. Let $D = \{x_1, x_2, \dots, x_t\}$ be a subset of V . Let $v \in V - D$. Then v distance- r -dominates V_j , for some j . Therefore $d(v, x_j) \leq r$. Hence x_j distance- r -dominates v . Therefore D is a distance- r -dominating set of G . Therefore $\gamma_r(G) \leq |D| = t = \chi_r^d(G)$.

Let π be a χ_r -partition of G . Assign colors $\chi_r(G) + 1, \dots, \chi_r(G) + \gamma_r(G)$ to the vertices of a minimum distance- r -dominating set $D = \{v_1, v_2, \dots, v_{\gamma_r(G)}\}$ of G , leaving the rest of the vertices colored as before.

Let $\pi_1 = \{V_1, V_2, \dots, V_{\chi_r(G)}, V_{\chi_r(G)+1}, \dots, V_{\chi_r(G)+\gamma_r(G)}\}$, where $V_{\chi_r(G)+1}, \dots, V_{\chi_r(G)+\gamma_r(G)}$ are singletons. Let $v \in (V(G) - D)$. Then there exists $v_j \in D$ such that $d(v, v_j) \leq r$. Therefore v distance- r -dominates the color class $V_{\chi_r(G)+j}$. Therefore π_1 is a distance- r -dominator color partition of G . Hence $\chi_r^d(G) \leq |\pi_1| = \chi_r(G) + \gamma_r(G)$.

Remark 2.8 The lower bound is obtained when $G = K_n$.

Observation 2.9 If $r = \text{diam}(G)$, then $\chi_r^d(G) = \chi_r(G) + \gamma_r(G)$.

4. CONCLUSION

We have made study of distance r -coloring and distance r -dominator coloring in a graph. It is further continued in our subsequent investigation in this direction. Storage problem of chemical and other application are also attempted.

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