# Numerical Analysis of the Fuzzy Integro-Differential Equations using Single-Term Haar Wavelet Series

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***Abstract*— This paper presents numerical analysis of the fuzzy integro-differential equations (FIDE) using Single Term Haar Wavelet Series (STHWS) method [6-9] is considered. The obtained discrete solutions using STHWS are compared with the exact solutions of the FIDE and Trapezoidal quadrature rules (TQR) method [10 ] with suitable example. Table and graph is presented to show the efficiency of this method.**

***Keywords*— Trapezoidal quadrature rules method, Haar wavelets, Single-term Haar wavelet series, integro-differential equations, Fuzzy integro-differential equations.**

## I. Introduction

The topics of fuzzy differential equations (FDE) and fuzzy integral equations (FIE) in both theoretical and numerical points of view have been developed in recent years. Prior to discussing fuzzy integro-differential equations (FIDE) and their numerical treatments, it is necessary to present a brief introduction of the previous works about FDE and FIE. When a physical system is modelled under the differential sense; it finally gives a fuzzy differential equation, a fuzzy integral equation or a fuzzy integro-differential equation and hence, the solution of integro-differential equations have a major role in the fields of science and engineering. Nonlinear integro-differential equations are usually hard to solve analytically and exact solutions are scarce. Therefore, they have been of great interest by several authors [1-3].

The technique that we used is the single-term Haar wavelet series method (STHWS), which is based on Haar wavelet series expansion. STHWS method is different from the traditional high order Haar wavelet series method. When requires symbolic computation of necessary derivatives of the data function and is computationally expensive for higher order. Intrinsically, the STHWS method evaluates the approximate solution by the finite Haar wavelet series. But, in the STHWS method the derivative is not computed directly. Instead, the relative derivatives are calculated by an iteration procedure. It is introduced by Sekar and his team of researchers [4-9] in a study about electrical circuits. In this way, Allahviranloo et al. [2] proposed FDTM for solving first order fuzzy differential equation under strongly H-differentiability. Moreover, Arikoglu et al. [3] has been proposed differential transform method for solving integro-differential equations.

The structure of paper is organized as follows; In section 2, some basic definitions of Haar wavelets and STHWS which will be used later are brought. In section 3, we discuss the properties of the Haar wavelets. In section 4, we discuss the remarks of the STHWS. We shall propose general form of FIDE in section 5. The proposed method is implemented to a suitable example in section 6 and finally, conclusion is drawn in section 7.

## II. Haar Wavelet series and Single-Term Haar Wavelet Series

Any function $y(t)$ which is square integrable in the interval $(0,1)$ can be expanded into a Haar series of infinite terms.

$$y(t) = \sum_{0}^{\infty} c_i h_i(t), \quad t \in (0,1) \tag{1}$$

Using the orthogonality relationship of Haar wavelets,

$$\int_0^1 h_i(t)h_l(t)dt \quad = 2^{-j}\delta_{il} = \begin{Bmatrix} 2^{-j} & i = l = 2^j + k \\ 0 & i \neq l \end{Bmatrix}$$

The Haar coefficients $c_i$ can be determined by

$$c_i = 2^j \int_0^1 y(t)h_i(t)dt$$

Usually, the series expansion equation (1) contains infinite terms for a general function $y\ (t)$ If $y\ (t)$ is either piecewise constant or may be approximated by piecewise constant segments then equation (1) will be terminated at a finite number of terms; that is,

$$y\ (t) \approx \sum_{i=0}^{m-1} c_i(t)h_i(t) = \tilde{c}_m^T \tilde{h}_m(t) \underline{\underline{\Delta}} \hat{y}\ (t), \quad t \in [0,1) \tag{2}$$

Where the subscript $T$ means transposition and

$$\tilde{c}_m \underline{\underline{\Delta}} [c_0, c_1, ......, c_{m-1}]^T \tag{3}$$

$$\tilde{h}_m \underline{\underline{\Delta}} [h_0(t), h_1(t), ......, h_{m-1}(t)]^T \tag{4}$$

$m$ is chosen to be $2^j$ for the positive integer $j$. Define the square Haar matrix of dimension as $m \times m$ as

$$H_{m\times m} \underline{\underline{\Delta}} [\tilde{h}_m(1/2m), \tilde{h}_m(3/m), ......, \tilde{h}_m((2m-1)/2m)]. \tag{5}$$

Therefore, equation (2) can be represented as

$$[\tilde{y}(1/2m), \tilde{y}(3/m), ......, \tilde{y}((2m-1)/2m)] = \tilde{c}_m{}^T H_{m\times m} \quad . \tag{6}$$

It obvious that

$$\tilde{c}_m{}^T = [\tilde{y}(1/2m), \tilde{y}(3/m), ......, \tilde{y}((2m-1)/2m)]\, H^{-1}{}_{m\times m} \quad . \tag{7}$$

Equation (7) called the forward transform, transforms the time function $\hat{y}(t)$ into the coefficient vector $\tilde{c}_m^T$; Equation (6) called the inverse transform, recovers $\hat{y}(t)$ from $c_m^T$. Since $H_{m\times m}$ and $H_{m\times m}^{-1}$ contain many zeros, the Haar transform is much faster than the Fourier transform, and even faster than the Walsh transform.

For example, consider the case $m = 4$. The Haar wavelets can be expressed as

$$h_o(t) = \langle 1, \quad 1, \quad 1, \quad 1 \rangle$$
$$h_o(t) = \langle 1, \quad 1, \quad -1, \quad -1 \rangle$$
$$h_0(t) = \langle 1 \quad -1 \quad 0 \quad 0 \ \rangle,$$
$$h_0(t) = \langle 0, \quad 0, \quad 1, \quad -1 \rangle,$$

Where $\langle \alpha_{0,} \alpha_{1,}, ......, \alpha_{m-1,} \rangle$, means that the function has the value $\alpha_i$ at $t \in [i/m.(i+1)/m)$, $i = 0,1,2,.....,m-1$. Suppose that $\tilde{y}(t) = \langle 8, \quad 6, \quad 7, \quad 3 \rangle$. Then it can be represented by

$$\hat{y}(t) = 6h_0(t) + h_1(t) + h_2(t) + 2h_3(t) = \tilde{c}_4^T H_{4\times 4},$$
$$H_{4\times 4} \underline{\underline{\Delta}} [\tilde{h}_4(1/8), \tilde{h}_4(3/8), \tilde{h}_4(5/8), \tilde{h}_4(7/8)].$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The Haar coefficient $c_i$, can be obtained by applying equation (3.10) directly,

$$\tilde{c}_4^{\,T} \underline{\underline{\Delta}} [c_0, c_1, c_2, c_3] = \tilde{y}(t) H^{-1}{}_{m\times m} = [6, 1, 1, 2] \quad .$$

$$H_{4\times4}^{-1} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 2 \\ 1 & -1 & 0 & -2 \end{bmatrix}$$

In practical applications, a small number of terms increases the calculation speed and saves memory storage; a large number of terms improve resolution accuracy. Therefore, a trade-off between calculation speed, memory saving, and resolution accuracy must be considered in this analysis.

## III. Properties of haar wavelets

In the wavelet analysis for a dynamic system, all relevant functions need to be transformed into Haar series. Since differentiation of Haar wavelets results always in impulse functions, this must be avoided; instead, integration of Haar wavelets is preferred. In turn the integration of Haar wavelets should be expandable into Haar series with Haar coefficient matrix $P$

$$\int_0^t \tilde{h}_m(\tau) d\tau \approx P_{m\times m} \tilde{h}_m(t), \quad t \in [0,1)$$

The $m \times m$ square matrix $P$ is called the operational matrix of integration and $\tilde{h}_m(t)$ is defined in equation (4), with

$$P_{m\times m} = (1/2m) \begin{bmatrix} 2mP_{(m/2)\times(m/2)} & -H_{(m/2)\times(m/2)} \\ H^{-1}_{(m/2)\times(m/2)} & 0_{(m/2)\times(m/2)} \end{bmatrix}, \quad P_{1\times1} = 1/2 \tag{8}$$

And $H_{m\times m}$ defined in equation (5)

In the integration of the adjoint equations, it is necessary to integrate Haar wavelets from 1 to *t*. Figure 1 shows the backward integration functions $\int_1^t h_1(\tau) d\tau$ .

In general,

$$\int_1^t \tilde{h}_m(\tau) d\tau \approx S_{m\times m} \tilde{h}_m(t) \quad t \in [0,1)$$

Where

$$S_{m\times m} = (1/2m) \begin{bmatrix} 2mS_{(m/2)\times(m/2)} & -H_{(m/2)\times(m/2)} \\ H^{-1}_{(m/2)\times(m/2)} & 0_{(m/2)\times(m/2)} \end{bmatrix}, \quad S_{1\times1} = 1/2 \tag{9}$$

With $H_{m\times m}$ defined in equation (5). From the comparison of $P_{m\times m}$ in equation (8) with $S_{m\times m}$ in equation (9), it is seen that these two matrices are the same for any $m$, except $m=2$; indeed, $P_{1\times 1}=1/2$, while $S_{1\times 1}=-1/2$. $S_{m\times m}$ called the operational matrix of backward integration. Figure 1 also shows that

$$\int_1^t h_0(\tau)d\tau = \int_1^t h_0(\tau)d\tau - 1$$

$$\int_1^t h_i(\tau)d\tau = \int_1^t h_i(\tau)d\tau \quad i=1,2,.....,m-1$$

In the study of time-varying systems via Haar wavelets, it is need to evaluate $\tilde{h}_m(t)\tilde{h}_m^T(t)$. Let the product of $\tilde{h}_m(t)$ and $\tilde{h}_m^T(t)$ be called the Haar product matrix $M_{m\times m}(t)$ That is,

$$\tilde{h}_m(t)\tilde{h}_m^T(t) \underline{\underline{\Delta}} M_{m\times m}(t) \tag{10}$$

The basic multiplication properties of Haar wavelets are as follows:

(i) For any two Haar wavelets $h_n(t)$ and $h_l(t)$, with $n<l$,

$$h_n(t)h_l(t) = \rho h_l(t), \tag{11}$$

$$\rho = h_n(2^{-i}(q+1/2)) \qquad = \begin{cases} 1, & 2^{i-j}k \le q < 2^{i-j}(k+1/2) \\ -1, & 2^{i-j}(k+1/2) \le q < 2^{i-j}(k+1) \\ 0, & otherwise \end{cases}$$

$$\begin{aligned} n &= 2^j + k, \quad j \ge 0, \ 0 \le k < 2^j, \\ l &= 2^i + q, \quad i \ge 0, \ 0 \le q < 2i. \end{aligned} \tag{12}$$

ii) The square of any Haar wavelet is a block pulse with magnitude of 1 during both the positive and negative half waves of the Haar wavelet

## IV. Remarks of the Single-Term Haar Wavelet Series

Equation (11) means that, when $n<l$ the product $h_n(t)h_l(t)$ equals $h_l(t)$ if $h_l(t)$ occurs during the first positive half wave of $h_n(t)$; and it equals $-h_l(t)$ if $h_l(t)$ occurs during the second negative half wave of $h_n(t)$. The product $h_n(t)h_l(t)$ must be zero when these two wavelets have no overlaps.

In the case of $n$ and $l$ defined in equation (12), with $i=j$ but $q \ne k$, meaning that $h_n(t)$ and $h_l(t)$ have the same dilations but different shifts, then

$$h_n(t)h_l(t) = 0.$$

For notation simplification, let

$$\tilde{h}_a(t)\underline{\underline{\Delta}}[h_0(t), h_1(t), ......, h_{m/2-1}(t)]^T = \tilde{h}_{m/2}(t),$$

$$\tilde{h}_b(t)\underline{\underline{\Delta}}[h_{m/2}(t), h_{m/2+1}(t), ......, h_{m-1}(t)]^T.$$

The matrix $M_{m\times m}(t)$ in equation (10) can be derived easily as follows

$$M_{m\times m}(t)=\begin{bmatrix} M_{(m/2)\times(m/2)}(t) & H_{(m/2)\times(m/2)}diag[\tilde{h}_b(t)] \\ diag[\tilde{h}_b(t)]H^{T}{}_{(m/2)\times(m/2)} & diag[H^{-1}{}_{(m/2)\times(m/2)}\tilde{h}_a(t)] \end{bmatrix},\ M_{1\times 1}(t)=h_0(t). \quad (13)$$

With the above recursive formulas, one can evaluate $M_{m\times m}(t)$ for any $M = 2^j$, $j$ a positive integer. The matrix $M_{m\times m}(t)$ satisfies

$$M_{m\times m}(t)\tilde{c}_m \equiv C_{m\times m}(t)\tilde{h}_m(t) \quad (14)$$

where the coefficient vector $C_m$ is defined in equation (3). By equation (13) and equation (14), the coefficient matrix $C_{m\times m}$ has the following form

$$C_{m\times m}(t)=\begin{bmatrix} C_{(m/2)\times(m/2)} & H_{(m/2)\times(m/2)}diag[\tilde{c}_b] \\ diag[\tilde{c}_b]H^{-1}{}_{(m/2)\times(m/2)} & diag[\tilde{c}_b{}^{T}H_{(m/2)\times(m/2)}] \end{bmatrix},\ C_{1\times 1}(t)=C_0.$$

Where

$$\tilde{c}_a \underline{\underline{\Delta}}[c_0, c_1, \ldots\ldots, c_{m/2-1}]^T = \tilde{c}_{m/2},$$

$$\tilde{c}_b(t)\underline{\underline{\Delta}}[c_{m/2}, c_{m/2+1}, \ldots\ldots, c_{m-1}(t)]^T.$$

Equation (14) is an important relationship for the study of time-varying systems.

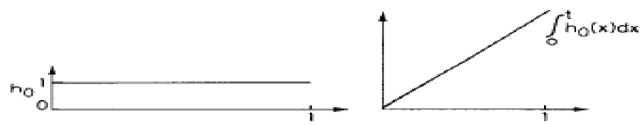


Figure 1 First Haar function and corresponding integral

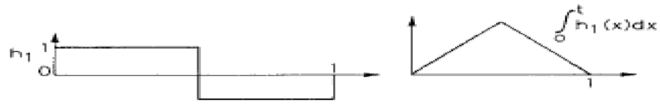


Figure 2 Second Haar function and corresponding integral

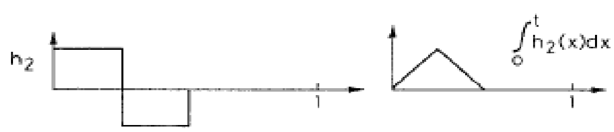


Figure 3 Third Haar function and corresponding integral

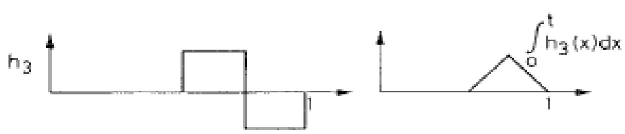


Figure 4 Fourth Haar function and corresponding integral

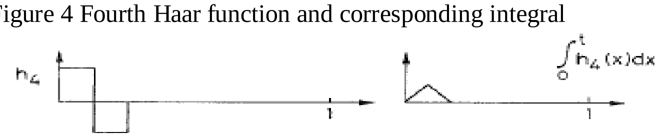


Figure 5 Fifth Haar function and corresponding integral

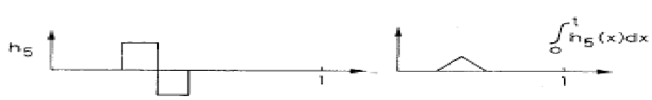

Figure 6 sixth Haar function and corresponding integral

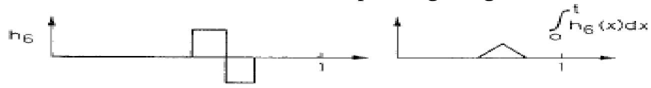


Figure 7 Seventh Haar function and corresponding integral

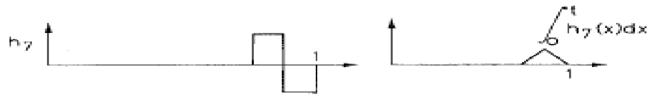


Figure 8 Eighth Haar function and corresponding integral

In STHWS, the operational matrix P in equation (8) becomes P = 1/2. The single-term Haar wavelets method is an extension of the single-term algorithm $P_{1\times 1} = 1/2$, which avoid the inverse of the big matrix induced by the Kronecker product. This approach is applicable for any transform with piecewise constant basis and one can take the advantages of its fast, local, and multiplicative properties to solve any kind of problems.

## V. Fuzzy Integro-Differential Equations

We consider the fuzzy integro-differential equations is of the form

$$\left.\begin{aligned} \frac{dX(t)}{dt} &= f\left(t, X(t), \int_{t_0}^{t} k(t,s,X(s))ds\right), \ , t \in T = [t_0, b], \\ X(t_0) &= X_0 \end{aligned}\right\}$$

Where $f : T \times L_2 \times L_2 \to L_2$, $k : T^2 \times L_2 \to L_2$ are m.s. continuous fuzzy mappings with respect to $t, s, t_0, b \in R$, $X_0 \in L_2$.

## VI. Numerical Results

In this section, the exact solution and approximated solution obtained by STHWS method and trapezoidal quadrature rules method with n=10. To show the efficiency of the STHW, we have considered the following problem taken from [10], with step size $\pi/20$ along with the exact solutions. The absolute errors between them are tabulated and are presented in Table 1. To distinguish the effect of the errors in accordance with the exact solutions, graphical representations are given for selected step size and are presented in Fig. 9 for the following problem, using three dimensional effects.

*Example 6.1*

Consider the fuzzy number A along with the r-cuts $[A]^r = [r^2 + r, 4 - r^3 - r]$ for $r \in [0,1]$. Let the functions $k : [0,1] \times [0,1] \to R$ and $f : [0,1] \to R_F$ be given by

$$k(x,t) = 0.1\sin\left(\frac{x}{2}\right)\sin(t)$$

$$f(x) = \left(\frac{1}{2}\cos\left(\frac{x}{2}\right) - 0.1\sin^2\left(\frac{x}{2}\right) + \frac{0.1}{3}\sin\left(\frac{x}{2}\right)\sin\left(\frac{3x}{2}\right)\right).A$$

Then the fuzzy integro-differential equation

$$\left.\begin{array}{l} y'(x) = f(x) + \int_0^x k(x,t).y(t)dt,, \ , x \in \left[0, \frac{\pi}{2}\right], \\ y(0) = 0 \end{array}\right\}$$

Has the exact solution $y(x) = \sin\left(\frac{x}{2}\right).A.$

TABLE I

| x | Example 6.1 | |
|---|---|---|
| | TQR Error | STHWS Error |
| $\pi/20$ | 0.3178E-06 | 0.5356E-11 |
| $2\pi/20$ | 0.2964E-05 | 0.9213E-10 |
| $3\pi/20$ | 0.9982E-05 | 0.6001E-10 |
| $4\pi/20$ | 0.2287E-04 | 0.9991E-09 |
| $5\pi/20$ | 0.4241E-04 | 0.5862E-09 |
| $6\pi/20$ | 0.6849E-04 | 0.1634E-09 |
| $7\pi/20$ | 0.1003E-03 | 0.9268E-08 |
| $8\pi/20$ | 0.1356E-03 | 0.7567E-08 |
| $9\pi/20$ | 0.1724E-03 | 0.4327E-08 |
| $\pi/2$ | 0.2077E-03 | 0.1002E-08 |

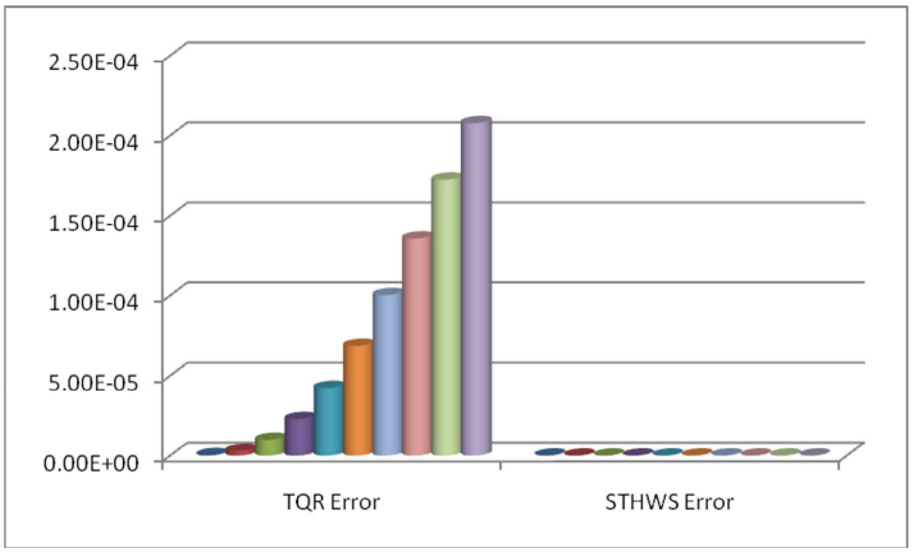


Fig. 9 Error estimation of Example 6.1

## VII. Conclusions

The obtained results (approximate solutions) of the FIDE show that the STHWS works well for finding the solution. From the table 1, we can observe that for most of the time intervals, the absolute error is less (almost no error) in the STHWS when compared to the TQR, which yields a little error, along with the exact solutions. From the figure 9, it can be predicted that the error is very less in STHWS when compared to the TQR method discussed by Zeinali et al. [10]. Hence the STHWS is more suitable for studying the FIDE.

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