On $b^{\#}$-Open Sets

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Abstract : Andrijivic introduced and studied the concept of b-open sets. Following this Bharathi et.al. introduced the concept of b**-open sets. Recently Indira et.al. studied the notions of *b-open sets and **b-open sets. In this paper, it has been shown that b**-open sets are precisely semi-pre-open sets or β-open sets and **b-open sets are nothing but α-open sets. Further the notion of a $b^{\#}$-open set is introduced and its basic properties are discussed.

Keywords: b-open sets, $b^{\#}$-open sets, b-continuity, $b^{\#}$-continuity, $b^{\#}$-irresoluteness.

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1. Introduction

In the year 1996, Andrijivic introduced[3] and studied b-open sets. Following this Bharathi et.al.[6] introduced the concept of b**-open sets. Recently Indira et.al.[12] studied the notions of *b-open sets and **b-open sets. However investigations have shown that b**-open sets are precisely semi-pre-open

sets[4](β-open sets in sense of [1]) and **b-open sets are nothing but α-open sets[14]. In this paper notion of a $b^{\#}$-open set is introduced and its basic properties are discussed.

2. Preliminaries

Throughout this paper X denotes a topological space on which no separation axiom is assumed. For any subset A of X, *cl*(A) denotes the closure of A and *int*(A) denotes the interior of A in the topological space X. Further X \ A denotes the complement of A in X. The following definitions and results are very useful in the subsequent sections.

Definition 2.1. A subset A of a space X is called

(i) α-open[14] if $A \subseteq int(cl(int(A)))$ and α-closed if $cl(int(cl(A))) \subseteq A$,

(ii) semi-open[13] if $A \subseteq cl(int(A))$ and semi-closed if $int(cl(A)) \subseteq A$,

(iii) pre-open[14] if $A \subseteq int(cl(A))$ and pre-closed if $cl(int((A))) \subseteq A$,

(iv) semi-pre-open[4] or β-open [1] if $A \subseteq cl(int(cl(A)))$ and semi-pre-closed or

β-closed if $int(cl(int(A))) \subseteq A$,

(v) regular open[9] if A= $int(cl(A))$ and regular closed if A = $cl(int(A))$.

For a subset A of a space X, the semi-closure (resp. pre-closure, α-closure, semi-pre-closure) of A, denoted by *scl* A (resp. *pcl* A, resp.α *cl* A, resp.*spcl* A) is the intersection of all semi-closed (resp. pre-closed, resp.α-closed, resp.semi-pre-closed) subsets of X containing A. Dually, the semi-interior (resp. pre-interior, resp.α-interior, resp. semi-pre-interior) of A, denoted by *sint* A,(resp.*pint* A, resp.α *int* A, resp. *spint* A), is the union of all semi-open (resp. pre-open, resp.α-open, resp. semi-pre-open) subsets of X contained in A. We recollect some of the relations that, together with their duals, we shall use in the sequel.

Lemma 2.2[4]. Let A be a subset of a space X. Then

(i) $\alpha cl(A) = A \cup cl(int(cl(A)))$,

(ii) $\alpha int(A) = A \cap int(cl(int(A)))$,

(iii) $scl(A) = A \cup int(cl(A))$,

(iv) $sint(A) = A \cap cl(int(A))$,

(v) $pcl(A) = A \cup cl(int(A))$,

(vi) $pint(A) = A \cap int(cl(A))$,

(vii) $spcl(A) = A \cup int(cl(int(A)))$,

(viii) $spint(A) = A \cap cl(int(cl(A)))$,

(ix) $scl(sint(A)) = sint(A) \cup int(cl(int(A)))$,

(x) $pcl(pint(A)) = pint(A) \cup cl(int(A))$,

(xi) $spcl(spint(A)) = spint(spcl(A))$,

(xii) $int(scl(A)) = pint(cl(A)) = pint(scl(A)) = scl(pint(A)) = int(cl(A))$,

(xiii) $int(pcl(A)) = scl(int(A)) = spcl(int(A)) = int(spcl(A)) = int(cl(int(A)))$.

Definition 2.3. A subset A of a space X is called

(i) b-open[3] if $A \subseteq cl(int(A)) \cup int(cl(A))$,

(ii) *b-open [12] if $A \subseteq cl(Int(A)) \cap int(cl(A))$,

(iii) b**-open [6] if $A \subseteq int(cl(int(A))) \cup cl(int(cl(A)))$,

(iv)**b-open [12] if $A \subseteq int(cl(int(A))) \cap cl(int(cl(A)))$,

(v) b-semi-open[2] if $A \subseteq cl(bint(A))$,

(vi) b-pre-open[2] if $A \subseteq int(bcl(A))$,

(vii) a p-set[19] if $cl(int(A)) \subseteq int(cl(A))$,

(viii) a q-set[20] if $int(cl(A)) \subseteq cl(int(A))$,

(ix) a t-set[21] if $int(A)= int(cl(A))$,

(x) a t*-set[12] if $cl(A)= cl(int(A))$,

(xi) a D(c, b) set [2] if $int(A) = bint(A)$.

(xii) a D(c, α) set [18] if $int(A) = \alpha int(A)$.

(xiii) a D(c, p) set [18] if $int(A) = pint(A)$.

(xiv) a D(c, β) set [8] if $int(A) = \beta int(A)$.

The complements of b-open, *b-open, b**-open , **b-open, b-semi-open and b-pre-open sets are respectively called the corresponding closed sets. However the complement of a t-set is a t*-set, the complement of a p-set is again a p-set and that of a q-set is a q-set.

Definition 2.4. A function $f : X \rightarrow Y$ is called

(i) semi-continuous [13] if $f^{-1}(V)$ is semi-open in X for each open set V of Y .

(ii) pre-continuous [14] if $f^{-1}(V)$ is pre-open in X for each open set V of Y .

(iii) semi-pre- continuous [4]if $f^{-1}(V)$ is semi-pre-open in X for each open set V of Y .

(v) p-continuous [19] if $f^{-1}(V)$ is a p-set in X for each open set V of Y .

(vi) q-continuous [20] if $f^{-1}(V)$ is a q-set in X for each open set V of Y .

(vii) b-continuous [17] if $f^{-1}(V)$ is b-open in X for each open set V of Y .

(ix) D(c, b)-continuous [2] if $f^{-1}(V)$is D(c,b)-set in X for each open set V of Y .

(x) D(c, p)-continuous [18] if $f^{-1}(V)$is D(c,p)-set in X for each open set V of Y .

(xii)D(c, β)-continuous [8] if $f^{-1}(V)$is D(c, β)-set in X for each open set V of Y .

Lemma 2.5[3]: Let A be a sub set of a space X. Then

(i) $int(bcl(A))=bcl(int(A))= int(cl(int(A)))$.

(ii) $cl(bint(A))= bint(cl(A))= cl(int(cl(A)))$.

(iii) $bint(A) = sint(A) \cup pint(A)$.

Lemma 2.6[5]: $\alpha cl(\alpha int(A))= cl(int(A))$ and $\alpha int(\alpha cl(A))= int(cl(A))$ for any sub set A of X.

Definition 2.7[11]: A space X is called extremally disconnected if $cl(A)$ is open for every open sub set A of X.

Definition 2.8: A sub set A is co-dense if the complement X \ A is dense or equivalently *int*(A)=ϕ. It is nwd(=nowhere dense) if *int*(*cl*(A))= ϕ that is if *cl*(A) is co-dense.

3. Some useful properties

In this section it is established that b** -open sets are precisely semi-pre-open sets, **b-open sets are precisely α –open sets, t* -open sets are precisely semi-open sets, t -open sets are precisely semi-closed sets, b-semi-open sets are precisely β -open sets and b-pre-open sets are precisely α -open sets.

Proposition 3.1. Let A be a sub set of a topological space X. Then the following are equivalent.

(i) A is b** -open.

(ii) A is semi-pre-open.

(iii) A is β -open.

(iv) A is b-semi-open.

Proof. For any subset A of X, *int*(A) $\subseteq$ A $\subseteq$ *cl*(A). This implies *cl*(*int*(A)) $\subseteq$ *cl*(A) that implies *int*(*cl*(*int*(A))) $\subseteq$ *int*(*cl*(A)). This proves that for any subset A of X, the relation *int*(*cl*(*int*(A))) $\subseteq$ *cl*(*int*(*cl*(A))) always holds that implies

int(*cl*(*int*(A))) $\cup$ *cl*(*int*(*cl*(A)))= *cl*(*int*(*cl*(A))). Then using Definition 2.1(iv) and Definition 2.3(iii) we get (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Now A is b-semi-open if and only if A $\subseteq$ *cl*(*bint*(A)). By using Lemma 2.5(ii), *cl*(*bint*(A))= *cl*(*int*(*cl*(A))) that implies A is b-semi-open if and only if A $\subseteq$ *cl*(*int*(*cl*(A)) if and only if A is semi-pre-open. This proves the proposition.

Proposition 3.2. Let A be a sub set of a topological space X. Then the following are equivalent.

(i) A is **b -open.

(ii) A is α-open.

(iii) A is b-pre-open.

Proof. For any subset A of X, from the proof of Proposition 3.1, it follows that *int*(*cl*(*int*(A))) $\subseteq$ *cl*(*int*(*cl*(A))) always holds that implies

int(*cl*(*int*(A))) $\cap$ *cl*(*int*(*cl*(A)))= *int*(*cl*(*int*(A))).

Then using Definition 2.1(i) and Definition 2.3(iv) we get (i) $\Leftrightarrow$ (ii) .

Now A is b-pre-open if and only if $A \subseteq int(bcl(A))$. By using Lemma 2.5(i), $int(bcl(A)) = int(cl(int(A)))$ that implies A is b-pre-open if and only if $A \subseteq int(cl(int(A))$ if and only if A is α-open that impies(ii) $\Leftrightarrow$ (iii). This proves the proposition.

Proposition 3.3: Let A be a sub set of a topological space X. Then A is t*-open if and only if it is semi-open.

Proof: A sub set A is semi-open if and only if $A \subseteq cl(int(A))$ that is if and only if $cl(A) \subseteq cl(int(A)) \subseteq cl(A)$that is if and only if $cl(A) = cl(int(A))$ that is if and only if A is a t*-set.

Corollary 3.4: Let A be a sub set of a topological space X. Then A is t-open if and only if it is semi-closed.

Proposition 3.5: Suppose A is a q-set. Then **(i)** $pcl(pint(A)) = cl(int(A))$ and (ii) $pint(pcl(A)) = int(cl(A))$.

Proof: By using Lemma 2.2(x), $pcl(pint(A)) = pint(A) \cup cl(int(A))$

$$= (A \cap int(cl(A))) \cup cl(int(A)).$$

Since A is a q-set , $int(cl(A))) \subseteq cl(int(A))$ that implies $= A \cap int(cl(A)) \subseteq cl(int(A))$ so that $pcl(pint(A)) = cl(int(A))$. This proves (i). If A is a q-set then X\A is also a q-set that implies , by using (i), $pcl(pint(X \backslash A)) = cl(int(X \backslash A))$. This proves that $pint(pcl(A)) = int(cl(A))$.

4. $b^{\#}$-open sets.

In this section $b^{\#}$-open sets are introduced and their properties are investigated.

Definition 4.1. A subset A of a space X is called $b^{\#}$-open if $A = cl(int(A)) \cup int(cl(A))$.

It is note worthy to see that every $b^{\#}$-open set is b-open set. However the converse is not true as shown in the following example.

Example 4.2. Let X={a, b, c, d} and $\tau = \{\phi$, {a}, {b}, {c}, {a,b}, {a,c}, {b, c}, {c, d}, {a, c, d}, {b, c, d}, {a, b, c}, X}. {c} is b-open but not $b^{\#}$-open.

Proposition 4.3. For a subset A of a space X , the following are equivalent:

(i) A is $b^{\#}$-open.

(ii) A is *b-closed and b-open.

(iii) A is pre-closed, semi-closed and $A = pint(A) \cup sint(A)$.

(iv) A is semi-closed and

A = $pcl(pint(A))$.

Proof: A is b$^{\#}$-open $\Leftrightarrow$ $cl(int(A)) \cup int(cl(A)) = A \Leftrightarrow cl(int(A)) \cup int(cl(A)) \subseteq A$ and $A \subseteq cl(int(A)) \cup int(cl(A))$ $\Leftrightarrow$ A is *b-closed and b-open. This proves (i) $\Leftrightarrow$ (ii).

To prove (ii) $\Rightarrow$ (iii), let A be *b-closed and b-open. Since A is *b-closed, $cl(int(A)) \cup int(cl(A)) \subseteq A$. It follows that $cl(int(A)) \subseteq A$ and $int(cl(A)) \subseteq A$. Thus A is pre-closed and semi-closed. Now since A is b-open, $A \subseteq cl(int(A)) \cup int(cl(A))$. Then by using Lemma 2.2, $pint(A) \cup sint(A) = (A \cap int(cl(A)) \cup (A \cap cl(int(A)) = A \cap (int(cl(A)) \cup (cl(int(A))) = A$.

This proves (ii) $\Rightarrow$ (iii). Now to prove (iii) $\Rightarrow$ (iv). Suppose (iii) holds.

$A = pint(A) \cup sint(A) = pint(A) \cup (A \cap cl(int(A)) = (pint(A) \cup A) \cap (pint(A) \cup cl(int(A)) = A \cap (pint(A) \cup cl(int(A)) = pint(A) \cup (A \cap cl(int(A))$.

Since A is pre-closed it follows that $A = pint(A) \cup cl(int(A)) = pcl(pint(A))$.

This proves (iii) $\Rightarrow$ (iv).

To prove (iv) $\Rightarrow$ (i). Suppose (iii) holds. Since A is semi-closed that is $int(cl(A)) \subseteq A$. Using Lemma 2.2, $A = pcl(pint(A))$

$= pint(A) \cup cl(int(A))$ $\quad = (A \cap int(cl(A))) \cup cl(int(A))$

$= int(cl(A)) \cup cl(int(A))$. This completes the proof.

Proposition 4.4. (i) If A is b$^{\#}$-open and co-dense then A is regular open.

(ii) If A is b$^{\#}$-open and nowhere dense then A is regular closed.

Proof. Suppose A is b$^{\#}$-open and co-dense. Then **A=** $cl(int(A)) \cup int(cl(A)) = \phi \cup int(cl(A)) = int(cl(A))$ that implies A is regular open. This proves (i). Now suppose A is b$^{\#}$-open and nowhere dense. Then **A =** $cl(int(A)) \cup int(cl(A)) = cl(int(A)) \cup \phi = cl(int(A))$ that implies A is regular closed. This proves (ii).

Proposition 4.5. Let (X, τ) be a topological space and τ^{α} be a collection of α-open sets in (X, τ). Then for a subset A of X A is b$^{\#}$-open in (X, τ) if and only if A is b$^{\#}$-open in (X, τ^{α}).

Proof: Follows from Definition 4.1 and Lemma 2.6.

Proposition 4.6: Suppose A is b$^{\#}$-open in (X, τ). Then

(i) A is open $\Rightarrow$ A is closed,

(ii) A is α-open $\Rightarrow$ A is regular closed and regular open.

(iii) A is α-closed $\Rightarrow$ A is closed,

(iv) A is semi-open $\Rightarrow$A is regular closed and a q- set.

(v) A is semi-closed $\Rightarrow$ A is closed,

(vi) A is pre-open $\Rightarrow$ A is regular open and a p- set.

(vii) A is *b-open $\Rightarrow$ A is both regular open and regular closed.

(viii) A is β-open $\Rightarrow$ cl(A) is regular closed.

(ix) A is a p-set $\Rightarrow$ A is regular open and β-closed.

(x) A is a q-set $\Rightarrow$ A is regular closed and β-open.

(xi) A is a D(c, b) set $\Rightarrow$ A is both open and closed.

(xii) A is a D(c, α) set $\Rightarrow$ int(A) is regular open.

(xiii) A is a D(c, p) set $\Rightarrow$ A is β-open.

(xiv) A is a D(c, β) set $\Rightarrow$ A is both open and β-open.

Proof: Let A be $b^{\#}$-open. Then

A= $cl(int(A)) \cup int(cl(A))$.

Suppose A is open. Then A= $cl(int(A)) \cup int(cl(A))= cl(A) \cup int(cl(A))=cl(A)$ that implies A is closed. This proves (i). Now to prove (ii). Suppose A is α-open. Then $cl(int(A)) \cup int(cl(A))=A \subseteq int(cl(int(A))) \subseteq cl(int(A)) \subseteq cl(int(A)) \cup int(cl(A))$ =A. This shows that A = $cl(int(A)) \cup int(cl(A)) = cl(int(A)) = int(cl(int(A)))$ that implies

$int(cl(A)) \subseteq cl(int(A))$. Further $cl(int(A)) = int(cl(int(A))) \subseteq int(cl(A))$.

This shows that A = $cl(int(A)) = int(cl(A))$. Therefore A is both regular closed and regular open. This proves (ii). To prove (iii). Suppose A is α-closed. Then $cl(int(cl(A))) \subseteq$ A= $cl(int(A)) \cup int(cl(A)) \subseteq cl(int(cl(A)))$ that implies A= $cl(int(cl(A)))$. This proves that A is closed. Thus (iii) is proved. Now suppose A is semi-open. Then A$\subseteq cl(int(A))$ that implies $cl(A) = cl(int(A))$. Therefore

Then $cl(int(A)) \cup int(cl(A))$ = A$\subseteq cl(int(A))$ that implies

$int(cl(A)) \subseteq cl(int(A))$ which further implies A is a q-set.

Since $cl(int(A)) \subseteq cl(int(A)) \cup int(cl(A))$ = A $\subseteq cl(int(A))$ it follows that A = $cl(int(A))$. This shows that A is regular closed. This proves (iv). Now suppose A is semi-closed. Then $int(cl(A)) \subseteq$ A that implies $int(A) = int(cl(A))$. Since A is $b^{\#}$-open , A=$cl(int(A)) \cup int(cl(A)) = cl(int(cl(A))) \cup int(cl(A)) = cl(int(cl(A)))$ that implies A is closed. This proves (v).

Suppose A is pre-open. Then $cl(int(A)) \cup int(cl(A)) = A \subseteq int(cl(A))$ that implies $cl(int(A)) \subseteq int(cl(A))$ which further implies A is a p-set.

Since $int(cl(A)) \subseteq cl(int(A)) \cup int(cl(A)) = A \subseteq int(cl(A))$ it follows that $A = int(cl(A))$. This shows that A is regular open. This proves (vi). Now suppose A is *b-open. Then $A \subseteq cl(int(A)) \cap int(cl(A))$. It follows that $A \subseteq cl(int(A))$ and $A \subseteq int(cl(A))$. Since A is $b^{\#}$-open we have $A= int(cl(A)) \cup cl(int(A))$ that implies

$A \subseteq cl(int(A)) \subseteq cl(int(A)) \cup int(cl(A)) = A$. This implies $A = cl(int(A))$ and hence A is regular closed. Now $A \subseteq int(cl(A)) \subseteq cl(int(A)) \cup int(cl(A)) = A$ that proves A is regular open. This proves (vii). Suppose A is β-open. Then $A \subseteq cl(int(cl(A)))$

Since A is $b^{\#}$-open we have $A= int(cl(A)) \cup cl(int(A))$ that implies $int(cl(A)) \subseteq A$. $A= int(cl(A)) \cup cl(int(A)) \subseteq cl(int(cl(A))) \subseteq cl(A)$ that implies $cl(int(cl(A))) = cl(A)$. Therefore $cl(A)$ is regular closed. This proves (viii). Suppose A is a p-set. Then $cl(int(A)) \subseteq int(cl(A))$. Since A is $b^{\#}$-open we have $A= int(cl(A)) \cup cl(int(A))$ that implies $int(cl(int(A))) \subseteq int(cl(A)) =A$. Therefore A is regular open and β-closed. This proves (ix). Now suppose A is a q-set. Then $int(cl(A))) \subseteq cl(int(A)))$. Since A is $b^{\#}$-open we have $A= int(cl(A)) \cup cl(int(A))$ that implies $A=cl(int(A))) \subseteq cl(int(cl(A)))$ and hence A is regular closed and β-open. This proves (x). Let A be a D(c, b) set. Then $int(A) =bint(A)$. Since A is $b^{\#}$-open we have $A= int(cl(A)) \cup cl(int(A))$. Since $int(A)=bint(A)$, using the Lemma 2.8,we have $int (A) = sint (A) \cup pint (A) = (A \cap cl(int(A))) \cup (A \cap int(cl(A)))$

$$= A \cap [cl(int(A)) \cup int(cl(A))]=A.$$

This shows that A is open. Also by using (i) A is closed. This proves (xi).

Now let A be a D(c, α) set then $int(A)=\alpha\, int(A)$. By Lemma 2.2(ii), $\alpha\, int(A)=A \cap int(cl(int(A)))$. Since A is $b^{\#}$-open we have $A= int(cl(A)) \cup cl(int(A))$.

$$int(A) = \alpha\, int(A)= [int(cl(A)) \cup cl(int(A))] \cap int(cl(int(A)))$$

$$=[int(cl(A)) \cap int(cl(int(A)))] \cup [cl(int(A)) \cap int(cl(int(A)))].$$

$= int(cl(int(A)))$. This proves that $int(A)$ is regular open. This proves (xii).

To prove (xiii). Let A be a D(c, p) set. Then $int(A) =pint(A)$. Since A is $b^{\#}$-open we have $A= int(cl(A)) \cup cl(int(A))$. Since $int(A)=pint(A)$,using the Lemma 2.2,we have

A= $int(cl(\mathrm{A}))\cup cl(int(\mathrm{A}))= int(cl(\mathrm{A}))\cup cl(pint(\mathrm{A}))$

$= int\,(cl(\mathrm{A}))\cup cl(A\cap int(cl(\mathrm{A}))) \subseteq int(cl(\mathrm{A}))\cup[cl(\mathrm{A})\cap cl(int(cl(\mathrm{A})))]$

$= [int(cl(\mathrm{A}))\cup cl(\mathrm{A}))] \cap [int(cl(\mathrm{A}))\cup cl(int(cl(\mathrm{A})))]= cl(\mathrm{A})\cap cl(int(cl(\mathrm{A})))= cl(int(cl(\mathrm{A})))$.

This shows that A is β-open. This proves (xiii). Now let A be a D(c, β) set then $int(\mathrm{A})= \beta\, int(\mathrm{A})$. By Lemma 2.2(ii), $\beta\, int(\mathrm{A})=\mathrm{A}\cap cl(int(cl(\mathrm{A})))$. Since A is b$^{\#}$-open we have A= $int(cl(\mathrm{A}))\cup cl(int(\mathrm{A}))$.

$int(\mathrm{A}) = \beta\, int(\mathrm{A})= [int(cl(\mathrm{A}))\cup cl(int(\mathrm{A}))] \cap cl(int(cl(\mathrm{A})))$

$=[int(cl(\mathrm{A}))\cap cl(int(cl(\mathrm{A})))]\cup[cl(int(\mathrm{A}))\cap cl(int(cl(\mathrm{A})))]$

$= int(cl(\mathrm{A}))\cup cl(int(\mathrm{A})) = \mathrm{A}$. This proves that A is open.

Again A= $int(cl(\mathrm{A}))\cup cl(int(\mathrm{A})) = int(cl(\mathrm{A}))\cup cl(\beta\, int(\mathrm{A}))= int(cl(\mathrm{A}))\cup cl(\mathrm{A}\cap cl(int(cl(\mathrm{A}))))$ $\subseteq int(cl(\mathrm{A}))\cup[cl(\mathrm{A}) \cap cl(int(cl(\mathrm{A})))]=[int(cl(\mathrm{A}))\cup cl(\mathrm{A})] \cap [int(cl(\mathrm{A})) \cup cl(int(cl(\mathrm{A})))$. $=cl(\mathrm{A}) \cap cl(int(cl(\mathrm{A})))= cl(int(cl(\mathrm{A})))$. Hence A is β–open. This proves (xiv).

This completes the proof.

Proposition 4.7: Let X be an extremally disconnected space and A be b$^{\#}$-open in (X, τ). Then A is semi-pre-open $\Rightarrow$ A is a p-set.

Proof: Let A be semi-pre-open. Then A $\subseteq cl(int(cl(\mathrm{A})))$. Since $int(cl(int(cl(\mathrm{A})))) = int(cl(\mathrm{A}))$ and A is b$^{\#}$-open we have A=$cl(int(\mathrm{A}))\cup int(cl(\mathrm{A}))\subseteq int(cl(\mathrm{A}))$ that implies $cl(int(\mathrm{A}))\cup int(cl(\mathrm{A}))=int(cl(\mathrm{A}))$. Therefore $cl(int(\mathrm{A})) \subseteq int(cl(\mathrm{A}))$. This shows that A is a p-set.

Proposition 4.8: Suppose A is a q-set. Then A is b$^{\#}$-open$\Leftrightarrow$A= $pint(pcl(\mathrm{A}))\cup pcl(pint(\mathrm{A}))$.

Proof: By using Proposition 3.5, we get $pint(pcl(\mathrm{A}))=int(cl(\mathrm{A}))$ and $pcl(pint(\mathrm{A})) = cl(int(\mathrm{A}))$. This implies that A is b$^{\#}$-open$\Leftrightarrow$ A= $int(cl(\mathrm{A}))\cup cl(int(\mathrm{A}))$ $\Leftrightarrow$ A= $pint(pcl(\mathrm{A}))\cup pcl(pint(\mathrm{A}))$.

Proposition 4.9: Suppose A is b$^{\#}$-open. Then $pcl(pint(\mathrm{A})) = pcl(\mathrm{A})$.

Proof: $pcl(pint(\mathrm{A})) = pint(\mathrm{A}) \cup cl(int(pint(\mathrm{A}))) = pint(\mathrm{A})\cup cl(int(\mathrm{A}))$

$= (\mathrm{A} \cap int(cl(\mathrm{A})))\,)\cup cl(int(\mathrm{A}))$

$=(\mathrm{A}\cup cl(int(\mathrm{A})))\cap (int(cl(\mathrm{A})) \cup cl(int(\mathrm{A})))$

$= (\mathrm{A}\cup cl(int(\mathrm{A})))\cap \mathrm{A} = \mathrm{A}\cup cl(int(\mathrm{A}))= pcl(\mathrm{A})$.

Arbitrary union and arbitrary intersection of $b^{\#}$-open sets need not be $b^{\#}$-open as shown in the following example.

Example 4.10. In the real line R, each closed interval [a, b], is $b^{\#}$-open where a<b and all other intervals are not $b^{\#}$-open. Let a < b. For each n =1,2,3., let $A_n = \left[a+\frac{1}{n}, b-\frac{1}{n}\right]$.

Then $\bigcup_{n=1}^{\infty} A_n$ = (a, b). But (a, b) is not $b^{\#}$-open. For,

$cl(int(a, b)) \cup int(cl(a, b))$= [a, b] $\cup$ (a, b)= [a, b] $\neq$ (a, b). Therefore arbitrary union of $b^{\#}$-open sets need not be $b^{\#}$-open.

Now, let $B_n = \left[a-\frac{1}{n}, a+\frac{1}{n}\right]$. Then $\bigcap_{n=1}^{\infty} B_n$ = {a}. But {a} is not $b^{\#}$-open. For,

$cl(int(\{a\})) \cup int(cl(\{a\})) = \phi \cup \phi = \phi$. Therefore arbitrary intersection of $b^{\#}$-open sets need not be $b^{\#}$-open.

Definition 4.11. A family **{** $\mathbf{A}_j$: $j \in \Delta$} of subsets of a topological space is called join related with respect to the operator γ on the power set of X if $\bigcup_{j\in\Delta} \gamma(A_j) = \gamma(\bigcup_{j\in\Delta} A_j)$ and is called meet related with respect to the operator γ if $\bigcap_{j\in\Delta} \gamma(A_j) = \gamma(\bigcap_{j\in\Delta} A_j)$

Theorem 4.12 Let X be a topological space and let $\{A_\alpha\}_{\alpha\in\Delta}$ be a join related family of $b^{\#}$-open sets with respect to the operators cl (int(.)) and int(cl(.)) on the power set of X. Then $\bigcup_{j\in\Delta} A_j$ is $b^{\#}$-open.

Proof: Let Δ be a collection of $b^{\#}$-open sets. Then by Definition 4.1, for each $j \in \Delta$, $A_j = int(cl(A_j)) \cup cl(int(A_j))$. Now, $\bigcup_{j\in\Delta} A_j = \bigcup_{j\in\Delta} [int(cl(A_j)) \cup cl(int(A_j))]$

$=[\bigcup_{j\in\Delta} int(cl(A_j))] \cup [\bigcup_{j\in\Delta} cl(int(A_j))]$. By Definition 4.11,

$\bigcup_{j\in\Delta} A_j = int(cl(\bigcup_{j\in\Delta} A_j)) \cup cl(int(\bigcup_{j\in\Delta} A_j))$. Hence arbitrary union of $b^{\#}$-open sets is $b^{\#}$-open.

Remark 4.13. The complement of a p-set is a p-set and that of a q-set is again a q-set. But the complement of a $b^{\#}$-open set is not $b^{\#}$-open as shown in the following example.

Example 4.14. In the real line topology R, take A = (- ∞, a]. Then B= (a, ∞]. It is easy to see that A is $b^{\#}$-open but X\A = B is not $b^{\#}$-open.

5. $b^{\#}$-closed sets

As seen in the previous section, the complement of a $b^{\#}$-open set need not be $b^{\#}$-open. This leads to the definition of $b^{\#}$-closed sets. In this section we define the concept of a $b^{\#}$-closed set and discuss its basic properties.

Definition 5.1. A sub set A of a space X is called $b^{\#}$-closed if X\A is $b^{\#}$-open. That is A is $b^{\#}$-closed if and only if $int(cl(A)) \cap cl(int(A))= A$.

We see that every $b^{\#}$-closed set is b-closed. But the converse is not true as shown in the following example.

Example 5.2: Let X={a, b, c, d} and $\tau =\{\phi$, {a}, {b}, {a, b}, {b, c}, {a, b, c}, X$\}$.

Let A={a, c, d}. A is b-closed but not $b^{\#}$-closed.

Proposition 5.3. For a subset A of a space X the following are equivalent.

(i) A is $b^{\#}$-closed.

(ii) A is *b-open and b-closed.

(iii) A is pre-open, semi-open and A = $pcl(A) \cap scl(A)$.

(iv) A is semi-open and A= $pint(pcl((A)))$.

Proof: Follows from Proposition 4.3 and Definition 5.1.

Proposition 5.4. Let (X, τ) be a topological space. Then for a subset A of X, A is $b^{\#}$-closed in (X, τ) if and only if A is $b^{\#}$-closed in (X, τ^{α}).

Proof: Follows from Definition 5.1 and Lemma 2.6.

Proposition 5.5: Suppose A is $b^{\#}$-closed in (X, τ). Then

(i) A is closed A is open

(ii) A is α-closed ⇒ A is regular closed and regular open.

(iii) A is α-open ⇒ A is open

(iv) A is semi-closed ⇒A is regular open and a q- set.

(v) A is semi-open ⇒ A is open

(vi) A is pre-closed ⇒ A is regular closed and a p- set.

(vii) A is *b-closed ⇒ A is both regular open and regular closed.

(viii) A is β-closed ⇒ *int*A) is regular open.

(ix) A is a p-set ⇒ A is regular closed and β-open.

(x) A is a q-set ⇒ A is regular open and β-closed.

(xi) X\A is a D(c, b) set ⇒ A is both open and closed.

(xii) X\A is a D(c, α) set ⇒ *cl*(A) is regular closed

(xiii) X\A is a D(c, p) set ⇒ A is β-closed.

(xiv) X\ A is a D(c, β) set ⇒ A is closed.

Proof: Follows from Proposition 4.6 and Definition 5.1.

Proposition 5.6: Let X be an extremally disconnected space and A be $b^{\#}$-closed in (X, τ). Then A is semi-pre-closed ⇒ A is a p-set.

Proof: Follows from Proposition 4.7.

Definition 5.7[15, 10]: A space X is called a partition space (or locally indiscrete)if every open sub set of X is closed.

Proposition 5.8: Let X be a partition space. Then every $b^{\#}$-open set is $b^{\#}$-closed.

Proof: Let A ⊆X be $b^{\#}$-open. A= *int*(*cl*(A)) ∪ *cl*(*int*(A)) = *cl*(A)∪ *int*(A) = *cl*(A) that implies A is closed. Since X is a partition space A is also open. Now *int*(*cl*(A))∩ *cl*(*int*(A))= A that implies A is $b^{\#}$-closed. This proves the proposition.

Proposition 5.9: Suppose A is a q-set. Then A is $b^{\#}$-closed⇔A= *pint*(p*cl*(A))∩*pcl*(p*int*(A)).

Proof: Follows from Proposition 4.8 and Definition 5.1.

Proposition 5.10: Suppose A is $b^{\#}$-closed. Then *pint*(*pcl*(A)) = *pint*(A).

Proof: Follows from Proposition 4.9 and Definition 5.1.

Arbitrary union and arbitrary intersection of $b^{\#}$-closed sets need not be $b^{\#}$-closed as shown in the following examples.

Example 5.11. In the real line R, each open interval (a, b) is $b^{\#}$-closed where a < b. All other non empty intervals are not $b^{\#}$-closed. Let a < b, For each n =1,2,3., let $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$.

Then $\bigcap_{n=1}^{\infty} A_n$ = {0}. But {0} is not b$^{\#}$-closed. . Therefore arbitrary intersection of b$^{\#}$-closed sets is not b$^{\#}$-closed.

Example 5.12. Let X= {a, b, c, d} and $\tau = \{\phi$, {a}, {b}, {a, b}, {b, c}, {a, b, c}, X}.

{a} and {b, c} are b$^{\#}$-closed. But {a, b, c} is not b$^{\#}$-closed. Therefore the union of b$^{\#}$-closed sets need not be b$^{\#}$-closed.

Theorem 5.13: Let X be a topological space and let $\{A_\alpha\}_{\alpha\in\Delta}$ be a meet related family of b$^{\#}$-closed sets with respect to the operators *cl*(*int*(.)) and *int*(*cl*(.)) on the power set of X. Then $\bigcap_{j\in\Delta} A_j$ is b$^{\#}$-closed.

Proof: Let Δ be a collection of b$^{\#}$-closed sets. Then by Definition 6.1, for each j $\in \Delta$,

$A_j = int(cl(A_j)) \cap cl(int(A_j))$. Now, $\bigcap_{j\in\Delta} A_j = \bigcap_{j\in\Delta} [int(cl(A_j)) \cap cl(int(A_j))]$

$=[\bigcap_{j\in\Delta} int(cl(A_j))] \cap [\bigcap_{j\in\Delta} cl(int(A_j))]$. By using Definition 4.11

$\bigcap_{j\in\Delta} A_j = int(cl(\bigcap_{j\in\Delta} A_j)) \cap cl(int(\bigcap_{j\in\Delta} A_j))$ that implies $\bigcap_{j\in\Delta} A_j$ is b$^{\#}$-closed. This proves the proposition.

6. b$^{\#}$-operators

In this section we introduce the concepts of b$^{\#}$-interior and b$^{\#}$-closure operators and some of their properties are discussed.

Definition 6.1: The b$^{\#}$-interior of A, denoted by b$^{\#}$-*int*(A), is defined to be the union of all b$^{\#}$-open sets contained in A. That is b$^{\#}$-*int*(A)= $\bigcup$ {B: B $\subseteq$ A and B is b$^{\#}$-open}. The b$^{\#}$-closure of A, denoted by b$^{\#}$-*cl*(A), is defined to be the intersection of all b$^{\#}$-closed sets containing A. That is b$^{\#}$-*cl*(A)= $\bigcap$ {B: A $\subseteq$ B and B is b$^{\#}$-closed}.

Remarks 6.2:

i. b$^{\#}$-*int*(ϕ)= ϕ,

ii. b$^{\#}$-*int*(X)= X.

iii. b$^{\#}$-*int*(A) $\subseteq$ A.

iv. $b^{\#}$-interior of a set A is not always $b^{\#}$-open.

v. If A is $b^{\#}$-open then $b^{\#}$-*int*(A)=A.

vi If $b^{\#}$-*int*(A)=A then it is not true that A is $b^{\#}$-open as seen in the following example.

Example 6.3: By Example 4.3, we can easily show that

$b^{\#}$-*int*[(a,b)]= $b^{\#}$-*int*($\bigcup_{n=1}^{\infty}\left[a+\frac{1}{n}, b-\frac{1}{n}\right]$)=(a, b) is not $b^{\#}$-open.

.

Proposition 6.4: Let X be a space. Then for any two sub sets A and B of X we have

(i) If A $\subseteq$ B then $b^{\#}$-*int*(A) $\subseteq$ $b^{\#}$-*int*(B).

(ii) $b^{\#}$-*int*($b^{\#}$-*int*(A)) $\subseteq$ $b^{\#}$-*int*(A).

(iii) X\ $b^{\#}$-*int*(A)= $b^{\#}$-*cl*(X\ A).

(iv) X \ $b^{\#}$-*cl*(A)= $b^{\#}$-*int*(X\ A).

(v) $b^{\#}$-*int*(A $\bigcap$ B)$\subseteq$ $b^{\#}$-*int*(A) $\bigcap$ $b^{\#}$-*int*(B).

(vi) $b^{\#}$-*int*(A$\bigcup$B)$\supseteq$ $b^{\#}$-*int*(A) $\bigcup$ $b^{\#}$-*int*(B).

Proof: Straight forward proof is omitted.

Remarks 6.5:

(i) $b^{\#}$-*cl*(ϕ)=ϕ,

(ii) $b^{\#}$-*cl*(X)=X.

(iii)A $\subseteq$ $b^{\#}$-*cl*(A).

(iv)$b^{\#}$-closure of a set A is not always $b^{\#}$-closed.

(v) If A is $b^{\#}$-closed then $b^{\#}$-*cl*(A)=A.

If $b^{\#}$-*cl*(A)=A then it is not true that A is $b^{\#}$-closed as seen in the following example.

Example 6.6: -By Example 5.11, $b^{\#}$-*cl*({0})= $b^{\#}$-*cl*$\left(\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right)\right)$={0} but {0} is not $b^{\#}$-closed.

Proposition 6.7: Let X be a space. Then for any sub sets A and B of X we have

(i) If A $\subseteq$ B then $b^{\#}$-*cl*(A) $\subseteq$ $b^{\#}$-*cl*(B).

(ii) $b^{\#}$-*cl*($b^{\#}$-*cl*(A))$\supseteq$ $b^{\#}$-*cl*(A).

(iii) $b^{\#}$-*cl*(A $\bigcup$ B) $\supseteq$ $b^{\#}$-*cl*(A) $\bigcup$ $b^{\#}$-*cl*(B).

(iv) $b^{\#}$-$cl(A \cap B) \subseteq b^{\#}$-$cl(A) \cap b^{\#}$-$cl(B)$.

Proof: Straight forward proof is omitted.

7. $b^{\#}$-continuity and $b^{\#}$-irresoluteness

In this section we introduce $b^{\#}$-continuous and $b^{\#}$-irresolute functions and their basic properties are studied.

Definition 7.1: Let X and Y are topological spaces. A function f: X $\rightarrow$ Y is called $b^{\#}$-continuous if $f^{-1}(V)$ is $b^{\#}$-open in X for each open set V of Y.

$b^{\#}$-continuity implies b- continuity but the reverse is not true that is shown in the following example.

Example 7.2: Let X= {a, b, c, d} and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c, d\}, X\}$. Then (X, τ) is a topology. Let Y= {1, 2, 3} and $\sigma = \{\phi, \{1\}, Y\}$. Then (Y, σ)is a topology.

Now we define a function f:X $\rightarrow$ Y by f(a)=2, f(b)=1, f(c)=1, f(d)=3. Then $f^{-1}(1)$={b, c}is b-open but not $b^{\#}$-open. Hence f is b-continuous but not $b^{\#}$-continuous.

Proposition 7.3: Let (X, τ) and (Y, σ) be two topological spaces and f be a map from X to Y. Then the following are equivalent.

(i) f is a $b^{\#}$-continuous map.

(ii) The inverse image of a closed set in Y is a $b^{\#}$-closed set in X

(iii) $b^{\#}$-$cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for every subset B of Y.

(iv) $f(b^{\#}$-$cl(A)) \subseteq cl(f(A)$ for every sub set A of X.

(v) $f^{-1}(int(B)) \subseteq b^{\#}$-$int(f^{-1}(B))$ for every sub set B of Y.

Proof: (i) $\Rightarrow$ (ii) Let A be a closed set in Y, then Y\A is open Y. Then $f^{-1}(Y \backslash A)$is $b^{\#}$-open in X. It follows that $f^{-1}(A)$ is $b^{\#}$-closed in X. To prove (ii) $\Rightarrow$ (iii). Let B be any sub set of Y. Since $cl(B)$ is closed in Y, then $f^{-1}(cl(B))$ is $b^{\#}$-closed in X. Therefore $b^{\#}$-$cl(f^{-1}(B)) \subseteq b^{\#}$-$cl(f^{-1}(cl(B)) = f^{-1}(cl(B))$. To prove (iii) $\Rightarrow$ (iv). Let A be any sub set of X. By (iii), we have $f^{-1}(cl(f(A))) \supseteq b^{\#}$-$cl(f^{-1}(f(A))) \supseteq b^{\#}$-$cl(A)$. Therefore $f(b^{\#}$-$cl(A)) \subseteq cl(f(A)$. To prove (iv) $\Rightarrow$ (v). Let B be any sub set of Y. By (iv), $f(b^{\#}$-$cl(X \backslash f^{-1}(B))) \subseteq cl(f(X \backslash f^{-1}(B))$ that implies $f(X \backslash b^{\#}$-$int(f^{-1}(B))) \subseteq cl(Y \backslash B) = Y \backslash int(B)$. Therefore we have $X \backslash b^{\#}$-$int(f^{-1}(B)) \subseteq f^{-1}(Y \backslash int(B))$ that implies $f^{-1}(int(B)) \subseteq b^{\#}$-$int(f^{-1}(B))$.

To prove (v) $\Rightarrow$ (i). Let B be any open sub set of Y. Then $f^{-1}(int(B)) \subseteq b^{\#}\text{-}int(f^{-1}(B))$. Then $f^{-1}(B) \subseteq b^{\#}\text{-}int(f^{-1}(B))$. But $b^{\#}\text{-}int(f^{-1}(B)) \subseteq f^{-1}(B)$ that implies $f^{-1}(B) = b^{\#}\text{-}int(f^{-1}(B))$. This completes the proof.

Proposition 7.4: Let (X, τ), (Y, σ) and (Z, υ)be three topological spaces. If f:X$\rightarrow$ Y is $b^{\#}$-continuous and g:Y$\rightarrow$Z is continuous , then $g_o f$:X$\rightarrow$Z is $b^{\#}$-continuous.

Proof: Straight forward.

Proposition 7.5: Let X be an extremally disconnected space. If f:X $\rightarrow$Y is $b^{\#}$-continuous and semi-pre-continuous then f is p-continuous.

Proof:Follows from Proposition 4.7.

Proposition 7.6: Let f:X $\rightarrow$Y be $b^{\#}$-continuous. Then

(i) f is continuous $\Rightarrow$ f is contra continuous.

(ii) f is semi-continuous$\Rightarrow$f is q-continuous

(iii) f is pre-continuous$\Rightarrow$ p-continuous.

(iv) f is q-continuous $\Rightarrow$ f is β-continuous.

(v) f is D(c, b) continuous $\Rightarrow$f is both continuous and contra continuous .

(vi) f is D(c, p) continuous $\Rightarrow$f is β- continuous.

(vii) f is D(c, β) continuous $\Rightarrow$f is both continuous and β- continuous.

Proof: Follows from Proposition 4.6.

Definition 7.7: Let X and Y be topological spaces. A function f: X$\rightarrow$Y is called $b^{\#}$-irresolute if $f^{-1}(V)$ is $b^{\#}$-open in X for each $b^{\#}$-open sub set V of Y.

Proposition 7.8: Let (X, τ) and (Y, σ) be two topological spaces and f be a map from X to Y. Then f is $b^{\#}$- irresolute if and only if the inverse image of a $b^{\#}$-closed set in Y is $b^{\#}$-closed in X.

Proof: Straight forward.

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