Common Fixed Points in Fuzzy Metric Spaces

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Abstract: This paper presents some modified common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces.

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1.Introduction

Fuzzy set was defined by Zadeh [7]. Michalek [5] introduced fuzzy metric space, Veermani [2] P.Balasubramaniam[1], Zun-Quan Xia And Fang-Fang Guo [8] ,Common fixed points of four mappings in a fuzzy metric spaces and modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many researchers have obtained common fixed point theorems for mappings satisfying different types introduced the new concept continuous mappings and established some common fixed point theorems open problem on the existence of contractive definition which generates a fixed point but does not force the mappings to be continuous at the fixed point this paper presents some common fixed point theorems.

2 Preliminary Notes

Definition 2.1 [7] A fuzzy set A in X is a function with domain X and values in [0,1].

Definition 2.2 [6] A binary operation $*$: $[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous t-norms if $*$is satisfying conditions:

(1) $*$is an commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0,1]$;

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$, and $a,b,c,d \in [0,1]$.

(5) a $*$ b=min{a,b}

Definition 2.3 [2] A 3-tuple (X,M,$*$) is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions, for all x,y,z ϵ X, s,t>0,

(f1) M(x,y,t) > 0;
(f2) M(x,y,t) = 1 if and only if x = y;
(f_3) M(x,y,t) = M(y,x,t);
(f_4) M(x,y,t)$*$ M(y,z,s) ≤ M(x,z,t+s) ;

(f_5) M(x,y,.): (0,∞)→(0,1] is continuous.

Then M is called a fuzzy metric on X. Then M(x,y,t) denotes the degree of nearness between x and y with respect to t.

Definition 2.4[2]Let (X,d) be a metric space. Denote a $*$ b = ab for all a, b ϵ [0,1] and M_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}.$$

Then (X, M_d, $*$) is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.

Definition 2.5[3] Two self mappings f and g of a fuzzy metric space (X,M,$*$) are called compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ whenever {x_n} is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X.

Definition 2.6[1] Twoself mappings f and g of a fuzzy metric space (X,M,$*$) are called reciprocally continuous on X if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$ whenever {x_n} is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X.

Lemma 2.8[4] Let X be a set, f,g owcself maps of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

3 Main Results

Theorem 3.1 Let (X, M, $*$) be a complete fuzzy metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) such that

$$M(Px,Ry,qt) \geq \min\{ M(Sx,Ty,t), M(Sx,Px,t), M(Ry,Ty,t), M(Px,Ty,t), M(Ry,Sx,t), M(Px,Ry,t), M(Px,Px,t)\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

For all x,y ϵ X and for all t>o, then there exists a unique point w ϵ X such that Pw = Sw = w and a unique point zϵX such that Rz = Tz = z. Moreover z = w so that there is a unique common fixed point of P,R,S and T.

Proof : Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px = Sx and Ry = Ty. We claim that Px = Ry. If not, by inequality (1)

$$M(Px,Ry,qt) \geq \min\{ M(Sx,Ty,t), M(Sx,Px,t), M(Ry,Ty,t), M(Px,Ty,t), M(Ry,Sx,t), M(Px,Ry,t), M(Px,Px,t)\}$$

$$\begin{aligned} M(Px,Ry,qt) &\geq \min\{ M(Px,Ry,t), M(Px,Px,t), M(Ty,Ty,t), M(Px,Ry,t), M(Ry,Px,t), M(Px,Ry,t),M(Px,Px,t)\} \\ &\geq \min\{ M(Px,Ry,t), M(Px,Px,t), M(Ty,Ty,t), M(Px,Ry,t), M(Px,Ry,t),M(Px,Ry,t),1\} \\ &= M(Px,Ry,t). \end{aligned}$$

Therefore Px = Ry, i.e. Px = Sx = Ry = Ty. Suppose that there is a another point z such that Pz = Sz then by (1) we have Pz = Sz = Ry = Ty, so Px=Pz and w = Px = Sx is the unique point of coincidence of P and S.By Lemma 2.8 w is the only common fixed point of P and S. Similarly there is a unique point z ϵ X such that z = Rz = Tz.

Assume that w $\neq$ z. we have

$$\begin{aligned} M(w,z,qt) &= M(Pw,Rz,qt) \\ &\geq \min\{ M(Sw,Tz,t), M(Sw,Pw,t), M(Rz,Tz,t), M(Pw,Tz,t), M(Rz,Sw,t), M(Pw,Rz,t), M(Pw,Pw,t)\} \\ &\geq \min\{ M(w,z,t), M(w,w,t), M(z,z,t), M(w,z,t), M(z,w,t), M(w,z,t), M(w,w,t)\} \\ &= M(w,z,t). \end{aligned}$$

Therefore we have z = w and z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds.

Theorem 3.2 Let (X, M, $*$) be a complete fuzzy metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc. If there exists qϵ(0,1) such that

$$M(Px,Ry,qt) \geq \emptyset(\min\{ M(Sx,Ty,t), M(Sx,Px,t), M(Ry,Ty,t), M(Px,Ty,t), M(Ry,Sx,t), M(Px,Ry,t), M(Px,Px,t)\}) \quad \ldots\ldots\ldots\ldots\ldots(2)$$

For all x,y ϵ X and Ø: [0,1]→[0,1] such that Ø(t) > t for all 0<t<1, then there exists a unique common fixed point of P,R,S and T.

Proof : Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px = Sx and Ry = Ty. We claim that Px = Ry. If not, by inequality (2)

$$\begin{aligned} M(Px,Ry,qt) &\geq \emptyset(\min\{ M(Sx,Ty,t), M(Sx,Px,t), M(Ry,Ty,t), M(Px,Ty,t), M(Ry,Sx,t), M(Px,Ry,t), M(Px,Px,t)\}) \\ &> \emptyset(M(Px,Ry,t)). \qquad \text{From Theorem 3.1} \\ &= M(Px,Ry,t). \end{aligned}$$

Assume that w $\neq$ z. we have

$$\begin{aligned} M(w,z,qt) &= M(Pw,Rz,qt) \\ &\geq \min\{ M(Sw,Tz,t), M(Sw,Pw,t), M(Rz,Tz,t), M(Pw,Tz,t), M(Rz,Sw,t), M(Pw,Rz,t),M(Pw,Pw,t)\} \end{aligned}$$

$=M(w,z,t)$. From Theorem 3.1

Therefore we have $z = w$ and z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds.

Theorem 3.3 Let $(X, M, *)$ be a complete fuzzy metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists $q \in (0,1)$ such that

$$M(Px,Ry,qt) \geq \emptyset(M(Sx,Ty,t), M(Sx,Px,t), M(Ry,Ty,t), M(Px,Ty,t), M(Ry,Sx,t), M(Px,Ry,t), M(Px,Px,t)) \quad \ldots\ldots\ldots\ldots\ldots(3)$$

For all $x,y \in X$ and $\emptyset: [0,1]^7 \rightarrow [0,1]$ such that $\emptyset(t,1,1,t,t,1,t) > t$ for all $0<t<1$, then there exists a unique common fixed point of P,R,S and T.

Proof: Let the pairs {P,S} and {R,T} be owc, so there are points $x,y \in X$ such that $Px = Sx$ and $Ry = Ty$. We claim that $Px = Ry$. If not, by inequality (3)

$$M(Px,Ry,qt) \geq \emptyset(M(Sx,Ty,t), M(Sx,Px,t), M(Ry,Ty,t), M(Px,Ty,t), M(Ry,Sx,t), M(Px,Ry,t),M(Px,Px,t))$$

$$\begin{aligned} M(Px,Ry,qt) &\geq \emptyset(M(Px,Ry,t), M(Px,Px,t), M(Ty,Ty,t), M(Px,Ry,t), M(Ry,Px,t), M(Px,Ry,t),M(Px,Px,t)) \\ &= \emptyset(M(Px,Ry,t), M(Px,Px,t), M(Ty,Ty,t), M(Px,Ry,t), M(Px,Ry,t),M(Px,Ry,t), 1) \\ &= \emptyset(M(Px,Ry,t), 1, 1, M(Px,Ry,t), M(Px,Ry,t),M(Px,Ry,t), M(Px,Ry,t)) \\ &> M(Px,Ry,t). \end{aligned}$$

A contradiction, therefore $Px = Ry$, i.e. $Px = Sx = Ry = Ty$. Suppose that there is a another point z such that $Pz = Sz$ then by (3) we have $Pz = Sz = Ry = Ty$, so $Px=Pz$ and $w = Px = Sx$ is the unique point of coincidence of P and S.By Lemma 2.8 w is the only common fixed point of P and S. Similarly there is a unique point $z \in X$ such that $z = Rz = Tz$. Thus z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds from (3).

Theorem 3.4 Let $(X, M, *)$ be a complete fuzzy metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc. If there exists $q \in (0,1)$ for all $x, y \in X$ and $t > 0$

$$M(Px,Ry,qt) \geq M(Sx,Ty,t) * M(Sx,Px,t) * M(Ry,Ty,t) * M(Px,Ty,t) * M(Ry,Sx,t) * M(Px,Ry,t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Then there exists a unique common fixed point of P,R,S and T.

Proof: Let the pairs {P,S} and {R,T} be owc, so there are points $x,y \in X$ such that $Px = Sx$ and $Ry = Ty$. We claim that $Px = Ry$. If not, by inequality (4)

We have

$$M(Px,Ry,qt) \geq M(Sx,Ty,t) * M(Sx,Px,t) * M(Ry,Ty,t) * M(Px,Ty,t) * M(Ry,Sx,t) * M(Px,Ry,t)$$

$$= M(Px,Ry,t)* \; M(Px,Px,t)* \; M(Ty,Ty,t)* \; M(Px,Ry,t)* \; M(Ry,Px,t)* \; M(Px,Ry,t)$$

$$= M(Px,Ry,t)* \; 1* \; 1* \; M(Px,Ry,t)* \; M(Ry,Px,t)* \; M(Px,Ry,t)$$

$$> M(Px,Ry,t).$$

Thus we have $Px = Ry$, i.e. $Px = Sx = Ry = Ty$. Suppose that there is a another point z such that $Pz = Sz$ then by (4) we have $Pz = Sz = Ry = Ty$, so $Px=Pz$ and $w = Px = Sx$ is the unique point of coincidence of P and S.Similarly there is a unique point $z \in X$ such that $z = Rz = Tz$.Thus w is a common fixed point of P,R,S and T.

Corollary 3.5 Let $(X, M, *)$ be a complete fuzzy metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc. If there exists $q\epsilon(0,1)$ for all $x,y \in X$ and $t > 0$

$$M(Px,Ry,qt) \geq M(Sx,Ty,t)* \; M(Sx,Px,t)* \; M(Ry,Ty,t)* \; M(Px,Ty,t)* \; M(Ry,Sx,2t)* \; M(Px,Ry,t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Then there existsa unique common fixed point of P,R,S and T.

Proof: We have

$$M(Px,Ry,qt) \geq M(Sx,Ty,t)* \; M(Sx,Px,t)* \; M(Ry,Ty,t)* \; M(Px,Ty,t)* \; M(Ry,Sx,2t)* \; M(Px,Ry,t)$$

$$\geq M(Sx,Ty,t)* \; M(Sx,Px,t)* \; M(Ry,Ty,t)* \; M(Px,Ty,t)* \; M(Sx,Ty,t)* \; M(Ty,Ry,t)* \; M(Px,Ry,t)$$

$$\geq M(Sx,Ty,t)* \; M(Sx,Px,t)* \; M(Ry,Ty,t)* \; M(Px,Ty,t) * \; M(Px,Ry,t)$$

$$= M(Px,Ry,t)* \; M(Px,Px,t)* \; M(Ty,Ty,t)* \; M(Px,Ry,t)* \; M(Ry,Px,t)* \; M(Px,Ry,t)$$

$$= M(Px,Ry,t)* \; 1* \; 1* \; M(Px,Ry,t)* \; M(Ry,Px,t)* \; M(Px,Ry,t)$$

$$> M(Px,Ry,t).$$

And therefore from theorem 3.4, P, R,S and T have a common fixed point.

Theorem 3.7 Let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self-mappings S and T of X have a common fixed point in X if and only if there exites a self mapping P of X such that the following conditions are satisfied

(i) $PX \subset TX \cap SX$

(ii) The pairs {P,S} and {P,T} are weakly compatible,
(iii) There exists a point $q\epsilon(0,1)$ such that for all $x,y \epsilon X$ and $t > 0$

$$M(Px,Py,qt) \geq M(Sx,Ty,t) * M(Sx,Px,t) * M(Py,Ty,t) * M(Px,Ty,t) * M(Py,Sx,t) \qquad \ldots\ldots\ldots\ldots(6)$$

Then P,S and T have a unique common fixed point.

Proof: Since compatible implies ows, the result follows from Theorem 3.4

Theorem 3.8 Let (X, M, *) be a complete fuzzy metric space and let P and R be self-mappings of X. Let the P and R are owc. If there exists $q\epsilon(0,1)$ for all $x,y \epsilon X$ and $t > 0$

$$M(Sx,Sy,qt) \geq \alpha M(Px,Py,t)+\beta \min\{M(Px,Py,t), M(Sx,Px,t), M(Sy,Py,t)\} \qquad \ldots\ldots\ldots\ldots(7)$$

For all $x,y \epsilon X$ where $\alpha,\beta > 0$, $\alpha+\beta > 1$. Then P and S have a unique common fixed point.

Proof: Let the pairs {P,S} be owc, so there are points $x \epsilon X$ such that $Px = Sx$. Suppose that exist another point $y \epsilon X$ for which $Py = Sy$. We claim that $Sx = Sy$. By inequality (8)
We have

$$M(Sx,Sy,qt) \geq \alpha M(Px,Py,t)+ \beta \min\{M(Px,Py,t), M(Sx,Px,t), M(Sy,Py,t),\}$$

$$=\alpha M(Sx,Sy,t)+ \beta \min\{M(Sx,Sy,t), M(Sx,Sx,t), M(Sy,Sy,t)\}$$

$$=(\alpha+\beta)M(Sx,Sy,t)$$

A contradiction, since $(\alpha+\beta) > 1$.Therefore $Sx = Sy$. Therefore $Px = Py$ and Px is unique.
From lemma 2.8 , P and S have a unique fixed point.

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