Common Fixed Points for Generalized Weakly Contractive Mappings in G-Metric Spaces

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***Abstract*: The existence of points of coincidence and common fixed points for a pair of self mappings satisfying generalized weakly contractive conditions in G-metric spaces is proved. Our results are generalization and extension of several well-known recent results related to fixed point theory.**

***Keywords*: G-metric space, altering distance function, point of coincidence, common fixed point.**

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I. Introduction

Metric fixed point theory plays an important role in mathematics and applied sciences. Some generalizations of the usual notion of a metric space have been proposed by several authors. One such generalization is a G-metric space initiated by Mustafa and Sims [11]. Moreover, they presented several interesting and useful facts about G-metric spaces, illustrated with appropriate examples. Thereafter, a series of articles about G-metric spaces have been dedicated to the improvement of fixed point theory. In 1997, Alber and Guerre-Delabriere [2] introduced the concept of weakly contractive mappings in Hilbert spaces and proved some fixed point theorems in this setting. Rhoades [16] showed that most of the results claimed by Alber and Guerre-Delabriere [2] are also valid for any metric spaces. In a very recent paper [1], the author established some fixed point theorems for mappings satisfying generalized weakly contractive conditions in G-metric spaces. In this work, we obtain sufficient conditions for existence of points of coincidence and common fixed points for a pair of self mappings in G-metric spaces under weakly contractive conditions related to altering distance functions. Our results generalize and extend some results of [1], [4], [12], [13], [16], [20]. Finally, some examples are presented to illustrate our results.

II. Preliminaries

We present some basic definitions and useful results for G-metric spaces that will be needed in the sequel.

Definition 2.1.[11] Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G_1) $G(x, y, z) = 0$ *if* $x = y = z$,

(G_2) $0 < G(x, x, y)$, *for all* $x, y \in X$, *with* $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$, *with* $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (*symmetry in all three variables*),

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Proposition 2.2.[11] Let (X,G) be a G-metric space. Then for any $x, y, z,$ and $a \in X$, it follows that

(1) $if\ G(x,y,z)=0\ \ then\ \ x=y=z,$

(2) $G(x,y,z) \le G(x,x,y)+G(x,x,z),$

(3) $G(x,y,y) \le 2G(y,x,x),$

(4) $G(x,y,z) \le G(x,a,z)+G(a,y,z),$

(5) $G(x,y,z) \le \frac{2}{3}\left(G(x,y,a)+G(x,a,z)+G(a,y,z)\right),$

(6) $G(x,y,z) \le G(x,a,a)+G(y,a,a)+G(z,a,a).$

Definition 2.3.[11] Let (X,G) be a G-metric space, let (x_n) be a sequence of points of X, we say that (x_n) is G-convergent to x if $\lim_{n,m\to\infty} G(x,x_n,x_m)=0$; that is, for any $\varepsilon>0$, there exists $n_0 \in N$ such that $G(x,x_n,x_m)<\varepsilon$ for all $n,m \ge n_0$. We refer to x as the limit of the sequence (x_n) and $x_n \to x$.

Proposition 2.4.[11] Let (X,G) be a G-metric space. Then, the following are equivalent:

(1) $(x_n)\ is\ G-convergent\ \ to\ x.$

(2) $G(x_n,x_n,x) \to 0,\ \ as\ n \to \infty.$

(3) $G(x_n,x,x) \to 0,\ \ as\ n \to \infty.$

(4) $G(x_m,x_n,x) \to 0,\ \ as\ m,n \to \infty.$

Definition 2.5.[11] Let (X,G) be a G-metric space, a sequence (x_n) is called G-Cauchy if given $\varepsilon>0$, there is $n_0 \in N$ such that $G(x_n,x_m,x_l)<\varepsilon$ for all $n,m,l \ge n_0$ that is if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Proposition 2.6.[11] In a G-metric space (X,G), the following are equivalent.

(1) $The\ \ sequence\ (x_n)\ is\ G-Cauchy.$

(2) $For\ every\ \varepsilon>0,\ there\ \ exists\ n_0 \in N\ such\ \ that\ \ G(x_n,x_m,x_m)<\varepsilon\ \ for\ all\ n,m \ge n_0.$

Definition 2.7.[11] Let (X,G) and (X',G') be G-metric spaces and let $f:(X,G)\to(X',G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\varepsilon>0$, there exists $\delta>0$ such that $x,y \in X$; $G(a,x,y)<\delta$ implies $G'(fa, fx, fy)<\varepsilon$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 2.8.[11] Let (X,G) and (X',G') be G-metric spaces, then a function $f: X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, (fx_n) is G-convergent to fx.

Proposition 2.9.[11] Let (X,G) be a G-metric space. Then, the function $G(x,y,z)$ is jointly continuous in all three of its variables.

Definition 2.10.[11] A G-metric space (X,G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

Definition 2.11.[10] A mapping $f:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) f is continuous and nondecreasing,

(ii) $f(t)=0 \Leftrightarrow t=0.$

Definition 2.12.[3] Let T and S be self mappings of a set X. If $w=Tx=Sx$ for some x in X, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S.

Definition 2.13.[9] The mappings $T,S:X\to X$ are said to be weakly compatible, if for every $x\in X$, the following holds:

$$T(Sx)=S(Tx) \textit{ whenever } Sx=Tx.$$

Remark 2.14.[15] The concept of weak compatibility is known to be the most general among all commutative concepts in fixed point theory. For example every pair of weakly commuting self maps and each pair of compatible self maps are weakly compatible, but the reverse is not always true. In fact, the notion of weakly compatible maps is more general than compatibility of type (A), compatibility of type (B), compatibility of type (C) and compatibility of type (P). For a review of those notions of commutability, see [8].

Proposition 2.15.[3] Let S and T be weakly compatible self maps of a nonempty set X. If S and T have a unique point of coincidence $y=Sx=Tx$, then y is the unique common fixed point of S and T.

Definition 2.16.[13] Let (X,G) be a G-metric space and T be a self mapping on X. Then T is called expansive if there exists a constant $a>1$ such that for all $x,y,z\in X$, we have

$G(Tx,Ty,Tz)\geq aG(x,y,z).$

Theorem 2.17.[16] Let (X,d) be a complete metric space, and let $T:X\to X$ satisfies

$$d(Tx,Ty)\leq d(x,y)-\varphi(d(x,y))$$

for all $x,y\in X$. If $\varphi:[0,\infty)\to[0,\infty)$ is a continuous and nondecreasing function with $\varphi(t)>0$ for all $t\in(0,\infty)$ and $\varphi(0)=0$, then T has a unique fixed point.

III. Main Results

Our first main result is the following.

Theorem 3.1. Let (X,G) be a G-metric space and let ψ, φ be altering distance functions. Let the mappings T, $f:X\to X$ satisfy

$$\psi(G(Tx,Ty,Ty))\leq\psi(G(fx,fy,fy))-\varphi(G(fx,fy,fy)) \tag{3.1}$$

for all $x,y\in X$. If $T(X)\subseteq f(X)$ and $f(X)$ is a G-complete sunspace of X, then T and f have a unique point of coincidence. Moreover, if T and f are weakly compatible, then T and f have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and we construct a sequence (fx_n) by $fx_n = Tx_{n-1}, n \in N$. This is possible since $T(X) \subseteq f(X)$. We assume that $fx_{n+1} \neq fx_n$ for all $n \in N$. If $fx_{n+1} = fx_n$, for some n, then $Tx_n = fx_n = v$, say. This shows that v is a point of coincidence of T and f.

For any $n \in N$, we have by using (3.1) that

$$\begin{aligned}\psi(G(fx_n, fx_{n+1}, fx_{n+1})) &= \psi(G(Tx_{n-1}, Tx_n, Tx_n)) \\ &\le \psi(G(fx_{n-1}, fx_n, fx_n)) - \varphi(G(fx_{n-1}, fx_n, fx_n)) \quad (3.2) \\ &\le \psi(G(fx_{n-1}, fx_n, fx_n)). \quad (3.3)\end{aligned}$$

We claim that

$$G(fx_n, fx_{n+1}, fx_{n+1}) = G(fx_{n-1}, fx_n, fx_n).$$

For, otherwise $G(fx_{n-1}, fx_n, fx_n) < G(fx_n, fx_{n+1}, fx_{n+1})$ implies that

$$\psi(G(fx_{n-1}, fx_n, fx_n)) \le \psi(G(fx_n, fx_{n+1}, fx_{n+1})), \ \psi \text{ being nondecreasing.}$$

This together with (3.3) imply that

$$\psi(G(fx_n, fx_{n+1}, fx_{n+1})) = \psi(G(fx_{n-1}, fx_n, fx_n)).$$

Thus, we obtain from (3.2) that

$$\varphi(G(fx_{n-1}, fx_n, fx_n)) = 0.$$

By definition of φ, we have $G(fx_{n-1}, fx_n, fx_n) = 0$ and so $fx_n = fx_{n-1}$ which is a contraction. This shows that

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le G(fx_{n-1}, fx_n, fx_n). \quad (3.4)$$

Put $t_n = G(fx_n, fx_{n+1}, fx_{n+1})$, then from (3.4), we get $0 \le t_n \le t_{n-1}$. Thus, the sequence (t_n) is non-increasing and bounded from below. Hence it converges to some $r \ge 0$.

Taking limit as $n \to \infty$ in (3.2) and using continuity of ψ and φ, we obtain

$$\psi(r) \le \psi(r) - \varphi(r),$$

which implies that, $\psi(r) = 0$. Hence by a property of φ, we get $r = 0$. Then, we have

$$\lim_{n\to\infty} G(fx_n, fx_{n+1}, fx_{n+1}) = 0. \quad (3.5)$$

For $n, m \in N$, $n < m$, we have by repeated use of the rectangle inequality and condition (3.5) that

$$\begin{aligned}G(fx_n, fx_m, fx_m) &\le G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_{n+1}, fx_{n+2}, fx_{n+2}) + \cdots + G(fx_{m-1}, fx_m, fx_m) \\ &\to 0 \ as \ n \to \infty.\end{aligned}$$

So, it follows that

$$\lim G(fx_n, fx_m, fx_m) = 0 \ as \ m, n \to \infty.$$

For $n, m, l \in N$, (G_5) implies that

$$G(fx_n, fx_m, fx_l) \le G(fx_n, fx_m, fx_m) + G(fx_l, fx_m, fx_m).$$

Taking limit as $n, m, l \to \infty$, we get $G(fx_n, fx_m, fx_l) \to 0$. This shows that (fx_n) is a G-Cauchy sequence in $f(X)$. Since $f(X)$ is G-complete, there exist $u, v \in X$ such that $fx_n \to v = fu$.

Again, by using (3.1)

$$\begin{aligned}\psi(G(fx_{n+1}, Tu, Tu)) &= \psi(G(Tx_n, Tu, Tu)) \\ &\le \psi(G(fx_n, fu, fu)) - \varphi(G(fx_n, fu, fu)). \quad (3.6)\end{aligned}$$

Taking limit as $n \to \infty$ and using continuity of ψ, φ, we obtain from condition (3.6) that

$$\psi(G(fu, Tu, Tu)) = 0$$

which implies that, $G(fu, Tu, Tu) = 0$ and so $fu = Tu = v$. Hence v becomes a point of coincidence of T and f.

For uniqueness, suppose there exists another point $w \in X$ such that $fx = Tx = w$ for some $x \in X$.

Then,

$$\begin{aligned}\psi(G(v,w,w)) &= \psi(G(Tu,Tx,Tx)) \\ &\le \psi(G(fu, fx, fx)) - \varphi(G(fu, fx, fx)) \\ &= \psi(G(v,w,w)) - \varphi(G(v,w,w)).\end{aligned}$$

So, it must be the case that, $\varphi(G(v,w,w)) = 0$ and $G(v,w,w) = 0$, yielding that $v = w$.

If T and f are weakly compatible, then by Proposition 2.15, T and f have a unique common fixed point in X.

Corollary 3.2 is a generalization of the main part of the result [[1], Theorem 2.1].

Corollary 3.2. Let (X,G) be a complete G-metric space and let ψ, φ be altering distance functions. Suppose the mapping $T : X \to X$ satisfies

$$\psi(G(Tx,Ty,Ty)) \le \psi(G(x,y,y)) - \varphi(G(x,y,y)) \qquad (3.7)$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. Taking $f = I$, the identity mapping in Theorem 3.1, we obtain the desired conclusion.

The following Corollary is a generalization of the result [[4], Theorem 2].

Corollary 3.3. Let (X,G) be a complete G-metric space and let φ be an altering distance function. Suppose the mapping $T : X \to X$ satisfies

$$G(Tx,Ty,Ty) \le G(x,y,y) - \varphi(G(x,y,y))$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. The proof can be obtained from the proof of Theorem 3.1 by considering $f = I$ and $\psi(t) = t$ for all $t \in [0,\infty)$.

Remark 3.4. Theorem 3.1 is an extension and generalization of Theorem 2.17 in metric spaces to G-metric spaces.

Corollary 3.5.[12] Let (X,G) be a complete G-metric space and let the mapping $T : X \to X$ satisfies

$$G(Tx,Ty,Ty) \le kG(x,y,y)$$

for all $x, y \in X$, where $0 \le k < 1$. Then T has a unique fixed point in X.

Proof. The proof follows from the Theorem 3.1 by taking $f = I$ and $\psi(t) = t$, $\varphi(t) = (1-k)t$ where $0 \le k < 1$ is a constant.

As an application of Corollary 3.2, we have the following results.

Theorem 3.6. Let (X,G) be a complete G-metric space and let the mapping $T : X \to X$ be such that

$$G(Tx,Ty,Ty) \le \frac{G^2(x,y,y)}{1+G(x,y,y)} \qquad (3.8)$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Proof. We define $\psi,\varphi:[0,\infty)\to[0,\infty)$ by $\psi(t)=t$ and $\varphi(t)=\dfrac{t}{1+t}$. Then ψ and φ are altering distance functions. We can rewrite (3.8) as follows:

$$\psi(G(Tx,Ty,Ty))\le\psi(G(x,y,y))-\varphi(G(x,y,y))$$

for all $x,y\in X$, which is condition (3.7) of Corollary 3.2. Now Corollary 3.2 applies to obtain the desired conclusion.

Theorem 3.7. Let (X,G) be a complete G-metric space and let $T:X\to X$ be an onto mapping satisfying

$$\psi(G(Tx,Ty,Ty))-\varphi(G(Tx,Ty,Ty))\ge\psi(G(x,y,y)) \tag{3.9}$$

for all $x,y\in X$, where ψ, φ are altering distance functions. Then T has a unique fixed point in X.

Proof. If $x\ne y$ and $Tx=Ty$, then from (3.9) we have

$$0\ge\psi(G(x,y,y))$$

which implies that $\psi(G(x,y,y))=0$ and so $G(x,y,y)=0$, yielding that $x=y$, a contradiction. Therefore, T is one to one. Also, it is onto. Hence T^{-1} exists and let $S=T^{-1}$.

Now applying condition (3.9), we obtain

$$\begin{aligned}\psi(G(x,y,y))-\varphi(G(x,y,y))&=\psi(G(T(Sx),T(Sy),T(Sy)))-\varphi(G(T(Sx),T(Sy),T(Sy)))\\&\ge\psi(G(Sx,Sy,Sy))\end{aligned}$$

for all $x,y\in X$.

So, it must be the case that

$$\psi(G(Sx,Sy,Sy))\le\psi(G(x,y,y))-\varphi(G(x,y,y))$$

for all $x,y\in X$.

This shows that S satisfies condition (3.7) of Corollary 3.2. Hence by Corollary 3.2, S has a unique fixed point, say $u\in X$. Then,

$$Su=u=T(Su)=Tu$$

which gives that u is also a fixed point of T.

For uniqueness, let v be another fixed point of T. Then,

$$Tv=v=S(Tv)$$

implies that Tv is also a fixed point of S. But u is the unique fixed point of S. So, $u=Tv=v$ and therefore u is the unique fixed point of T.

The following Corollary is the result [[13], Theorem 2.2].

Corollary 3.8. Let (X,G) be a complete G-metric space and $T:X\to X$ be an onto mapping satisfying

$$G(Tx,Ty,Ty)\ge kG(x,y,y) \tag{3.10}$$

for all $x,y\in X$, where $k>1$ is a constant . Then T has a unique fixed point in X.

Proof. Taking $\psi(t)=t$, $\varphi(t)=\left(1-\dfrac{1}{k}\right)t$ for all $t\in[0,\infty)$, we can rewrite condition (3.10) as follows:

$$\psi(G(Tx,Ty,Ty))-\varphi(G(Tx,Ty,Ty))\ge\psi(G(x,y,y))$$

for all $x,y\in X$, which is condition (3.9) of Theorem 3.7. Thus, the result follows from Theorem 3.7.

Remark 3.9. It is worth mentioning that the mappings satisfying condition (3.10) form a bigger category than the one of expansive mappings.

Remark 3.10. We see that Theorem 3.7 is a generalization of the result [[13], Theorem 2.2]. Furthermore, it is an extension of the result [[20], Theorem 1] in metric spaces to G-metric spaces.

Corollary 3.11. Let (X,G) be a complete G-metric space and $T:X\to X$ be an onto mapping satisfying

$$\frac{G^2(Tx,Ty,Ty)}{1+G(Tx,Ty,Ty)}\geq G(x,y,y) \tag{3.11}$$

for all $x,y\in X$. Then T has a unique fixed point in X.

Proof. Taking $\psi(t)=t$, and $\varphi(t)=\dfrac{t}{1+t}$ for all $t\in[0,\infty)$, we can rewrite condition (3.11) as follows:

$$\psi(G(Tx,Ty,Ty))-\varphi(G(Tx,Ty,Ty))\geq\psi(G(x,y,y))$$

for all $x,y\in X$, which is condition (3.9) of Theorem 3.7. Thus, the conclusion of the Corollary follows from Theorem 3.7.

We give an example to illustrate Theorem 3.1.

Example 3.12. Let $X=[0,1]$ and define $G:X\times X\times X\to R^+$ by

$$G(x,y,z)=|x-y|+|y-z|+|z-x| \ \textit{for all } x,y,z\in X.$$

Then (X,G) is a G-metric space. Let $T,f:X\to X$ be defined by

$$Tx=\frac{x}{4}-\frac{x^2}{8},\ fx=\frac{x}{4}\ \textit{for all } x\in X.$$

Let $\psi,\varphi:[0,\infty)\to[0,\infty)$ be such that

$$\psi(t)=\frac{t}{2},\varphi(t)=\frac{t^2}{2},\ \textit{for all } t\in[0,\infty).$$

It is a simple task to show that ψ,φ are altering distance functions.

Let $x,y\in X$. Without loss of generality, we assume that $x\geq y$. Then,

$$\begin{aligned}\psi(G(Tx,Ty,Ty))&=\frac{1}{2}G(Tx,Ty,Ty)=|Tx-Ty|\\&=\left|\left(\frac{x}{4}-\frac{x^2}{8}\right)-\left(\frac{y}{4}-\frac{y^2}{8}\right)\right|\\&=\left(\frac{x}{4}-\frac{y}{4}\right)-\frac{1}{8}\left(x^2-y^2\right)\\&\leq\psi(G(fx,fy,fy))-\frac{1}{8}(x-y)^2\\&=\psi(G(fx,fy,fy))-\varphi(G(fx,fy,fy)).\end{aligned}$$

Thus, condition (3.1) of Theorem 3.1 is satisfied. Moreover, $T(X)\subseteq f(X)$, $f(X)$ is G-complete and T,f are weakly compatible. Thus we have all the conditions of Theorem 3.1 and $0\in X$ is the unique common fixed point of T and f.

We furnish an example to illustrate the fact that T is onto in Theorem 3.7 is necessary.

Example 3.13. Let $X = R$ and define $G : X \times X \times X \to R^{+}$ by

$$G(x, y, z) = |x - y| + |y - z| + |z - x| \text{ for all } x, y, z \in X.$$

Then (X, G) is a complete G-metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 3x - 2 & \text{for } x \le 0, \\ 3x + 2 & \text{for } x > 0. \end{cases}$$

Now,

$$\begin{aligned} |Tx - Ty| &= |3x - 2 - 3y - 2| \text{ for } x \le 0 \text{ and } y > 0 \\ &= 3\left|y + \frac{4}{3} - x\right| \text{ for } x \le 0 \text{ and } y > 0 \\ &\ge 3|y - x| \text{ for } x \le 0 \text{ and } y > 0 \\ &= 3|x - y| \text{ for } x \le 0 \text{ and } y > 0. \end{aligned}$$

Also,

$$|Tx - Ty| = 3|x - y| \text{ for } x, y \le 0$$

and

$$|Tx - Ty| = 3|x - y| \text{ for } x, y > 0.$$

Thus,

$$|Tx - Ty| \ge 3|x - y| \text{ for } x, y \in X.$$

Therefore, we have

$$G(Tx, Ty, Ty) = 2|Tx - Ty| \ge 6|x - y| = 3G(x, y, y) \text{ for all } x, y \in X.$$

Define $\psi, \varphi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}, \varphi(t) = \frac{t}{3}$, for all $t \in [0, \infty)$. Then ψ, φ are altering distance functions such that

$$\psi(G(Tx, Ty, Ty)) = \varphi(G(Tx, Ty, Ty)) = \frac{1}{6} G(Tx, Ty, Ty) \ge \frac{1}{2} G(x, y, y) = \psi(G(x, y, y))$$

for all $x, y \in X$. Thus, we have all the conditions of Theorem 3.7 except the surjective condition. We observe that T has got no fixed point in X.

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