# Common Fixed Point Theorems in Quasi-Gauge Space for Six Self Maps

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***Abstract*— The aim of this paper is to prove, mainly, a common fixed point theorem for six self mappings and its consequences under the condition of weakly compatible mappings in a Quasi-Gauge space.**

***Keywords*— common fixed point, weakly compatible mappings, Quasi-Gauge space.**

## I. INTRODUCTION

Rao (the first author of this paper) and Murthy [3] proved results on common fixed point of four self maps on Quasi-Gauge space, using continuity of mappings. Recently, Sharma and Tilwankar [5] pointed out that the continuity of mappings is not required to prove the results. Unfortunately their observation is not valid.

Now in this paper, we extended the results of Rao and Murthy [3] under weaker conditions.

**Definition 1.1** ([4]): A Quasi pseudo metric on a non-empty set X is a non-negative real valued function p on $X \times X$ such that

(i) $p(x,x) = 0$ for all $x \in X$,

(ii) $p(x,z) \le p(x,y) + p(y,z)$ for all $x, y, z \in X$.

**Definition 1.2** ([4]): A Quasi-Guage structure for a topological space $(X, \Im)$ is a family $\mathrm{P}$ of quasi pseudo metrics on X such that the family $\{B(x, p, \in) : x \in X, p \in \mathrm{P}, \in > 0\}$ is a subbase for $\Im$. $(B(x, p, \in)$ is the set $\{y \in X : p(x, y) < \in\})$. If a topological space $(X, \Im)$ has a Quasi- Gauge structure $\mathrm{P}$, then it is called a Quasi-Gauge space and is denoted by $(X, \mathrm{P})$.

(In the topological space $(X, \Im)$, we have the usual convergence of a sequence $\{x_n\}$ in $X$ ).

**Definition 1.3**([4]): Let $(X, \mathrm{P})$ be a Quasi-Gauge space. A sequence $\{x_n\}$ in X is left $\mathrm{P}$-Cauchy iff for each $p \in \mathrm{P}$ and $\in > 0$, there is a point $x \in X$ and a positive integer $k$ such that $p(x, x_m) < \in$ for all $m \ge k$ ( $x$ and $k$ depend on $\in$ and $p$ ).

$(X, \mathrm{P})$ is left sequentially complete if every left $\mathrm{P}$-Cauchy sequence in X is convergent.

Let $(X, \mathrm{P})$ be a Quasi-Gauge space. A sequence $\{x_n\}$ in X is right $\mathrm{P}$-Cauchy iff for each $p \in \mathrm{P}$ and $\in > 0$, there is a point $x \in X$ and a positive integer $k$ such that $p(x_m, x) < \in$ for all $m \geq k$ ( $x$ and $k$ depend on $\in$ and $p$ ).

$(X, \mathrm{P})$ is right sequentially complete if every right $\mathrm{P}$-Cauchy sequence in X is convergent.

A sequence $\{x_n\}$ in X is $\mathrm{P}$-Cauchy iff for each $p \in \mathrm{P}$ and $\in > 0$, there is a positive integer $k$ such that $p(x_m, x_n) < \in$ for all $m, n \geq k$.

**Result 1.4**([4]): Let $(X, \mathrm{P})$ be a Quasi-Gauge space. Then $X$ is a $T_0$-space iff $p(x, y) = p(y, x) = 0$ for all $p \in \mathrm{P}$ implies x = y.

**Definition 1.5**([1]): Let $(X, \mathrm{P})$ be a Quasi-Gauge space. The self maps f and g on X are said to be $(f, g)$ weak compatible if $\lim_{n\to\infty} gfx_n = fz$ whenever $\{x_n\}$ is sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ and $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ffx_n = fz$.

$f$ and $g$ are said to be weak compatible to each other if $(f, g)$ and $(g, f)$ are weak compatible.

Now, we give the following:

**Definition 1.6**: Let (X, $\mathrm{P}$) be a Quasi-Gauge space. The pair of self maps {f,g} is said to be weakly compatible iff fgx =gfx whenever fx = gx for some $x \in X$.

This is weaker than the previous one, in view of the following example.

**Example 1.7**: Let X = [0,1).(with the usual metric) Define self maps f and g on X by

$$fx = \frac{x}{2} \quad \text{and} \quad gx = \begin{cases} 0 \ if \ 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} \ if \ \frac{1}{2} \leq x < 1. \end{cases}$$

Then the pair {f, g} is weakly compatible but not (f,g) weak compatible.

For, $fx = gx \Leftrightarrow \frac{x}{2} = 0 \text{ or } \frac{1}{2} \Leftrightarrow x = 0$ since x < 1 in X.

Now fg0 = 0 = gf0. Therefore, {f, g} is weakly compatible.

Now we prove that self maps f and g are not (f, g) weak compatible.

Take $x_n = 1 - \frac{1}{n}$, for $n \in \mathrm{N}$.

$$fx_n = f\left(1 - \frac{1}{n}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right) \to \frac{1}{2} \text{ as } n \to \infty;$$

$$\text{for } n \geq 2,\ gx_n = g\left(1 - \frac{1}{n}\right) = \frac{1}{2} \text{ and so } gx_n \to \frac{1}{2} \text{ as } n \to \infty.$$

Now $$\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} f\left(\frac{1}{2}\right) = \frac{1}{4} = f\left(\frac{1}{2}\right)$$

and $$\lim_{n\to\infty} f\ fx_n = \lim_{n\to\infty} f\left(\frac{1}{2}\left(1-\frac{1}{n}\right)\right) = \lim_{n\to\infty} \frac{1}{4}\left(1-\frac{1}{n}\right) = \frac{1}{4} = f\left(\frac{1}{2}\right).$$

Consider, $gfx_n = g\left(\frac{1}{2}\left(1-\frac{1}{n}\right)\right) = 0;$ implies that

$$\lim_{n\to\infty} g\ fx_n = 0 \neq f\left(\frac{1}{2}\right).$$

Thus the claims follow.

The following is useful in establishing our results.

**Lemma 1.8**: ([2]) Let $\psi : [0,\infty) \to [0,\infty)$ be non decreasing and upper semi continuous from the right . If $\psi(t) < t$ for every t >0, then $\lim_{n\to\infty} \psi^n(t) = 0.$

Rao and Murty [3] proved the following two theorems.

**Theorem 1.9**: Let A, B, S and T be self maps on a left (right) sequentially complete Quasi-Gauge $T_0$-space (X, P) such that

(i) the ordered pairs (A,S) and (B,T) are weak compatible maps ;

(ii) T(X) $\subseteq$ A(X) and S(X) $\subseteq$ B(X);

(iii) A and B are continuous ;

(iv)
$$\begin{aligned}\max\{p^2(Sx,Ty), p^2(Ty,Sx)\} \le \phi\{&p(Ax,Sx)p(By,Ty), p(Ax,Ty)p(By,Sx),\\ &p(Ax,Sx)p(Ax,Ty), p(By,Sx)p(By,Ty),\\ &p(By,Sx)p(Ax,Sx), p(By,Ty)p(Ax,Ty)\}\end{aligned}$$

for all $x, y \in X$ and for all $p \in$ P ; where $\phi : [0,\infty)^6 \to [0,\infty)$ satisfies the following:

(v) $\phi$ is non-decreasing and upper semi continuous in each coordinate variable and for each t > 0,

$$\psi(t) = \max\{\phi\{t,0,2t,0,0,2t\}, \phi\{t,0,0,2t,2t,0\}, \phi\{0,t,0,0,0,0\}\} < t.$$

Then A, B, S and T have a unique common fixed point.

**Theorem 1.10**: Let A, B, S and T be self maps on a left (right) sequentially complete Quasi-Gauge $T_0$-space (X, P) with conditions (iv) and (v) of Theorem(1.9). Further,

(i) the ordered pairs (A,S), (S, A), (B, T) and (T, B) are weak compatible maps;

(ii) T(X) $\subseteq$ A(X) and S(X) $\subseteq$ B(X);

(iii) One of A, B, S, T is continuous ;

Then the same conclusion of Theorem (1.9) holds.

Sushal Sharma and Tilwakar [5] claimed the following, stating that the continuity of the mappings is not necessary in Theorems (1.9) and (1.10).

**Theorem 1.11**([5]). Let A, B, S and T be self maps on a left (right) sequentially complete Quasi-Gauge $T_0$- space (X, $\mathcal{P}$) such that

(i) the pairs {A, S} and {B, T} are weak compatible maps;

(ii) T(X) $\subseteq$ A(X) and S(X) $\subseteq$ B(X);

(iii)
$$\max\{p^2(Sx,Ty),\, p^2(Ty,Sx)\} \le \phi\{p(Ax,Sx)\,p(By,Ty),\, p(Ax,Ty)\,p(By,Sx),\, p(Ax,Sx)\,p(Ax,Ty),\, p(By,Sx)\,p(By,Ty),\, p(By,Sx)\,p(Ax,Sx),\, p(By,Ty)\,p(Ax,Ty)\}$$

for all $x, y \in X$ and for all $p \in \mathcal{P}$; where $\phi:[0,\infty)^6 \to [0,\infty)$ satisfies the following:

(iv) $\phi$ is non-decreasing and upper semi continuous in each coordinate variable and for each t > 0,

$$\psi(t) = \max\{\phi\{t,0,2t,0,0,2t\}, \phi\{t,0,0,2t,2t,0\}, \phi\{0,t,0,0,0,0\}, \phi\{0,0,0,0,0,t\}, \phi\{0,0,0,0,t,0\}\} < t.$$

Then A, B, S and T have a unique common fixed point.

Unfortunately in the proof they assumed that S(X) is complete which is not in the hypothesis.

Now we establish a common fixed point theorem of six self maps on a sequentially complete Quasi-Gauge $T_0$-space which generalizes Theorem (1.11) with the extra hypothesis of the completeness of a subspace of X.

## 2. MAIN RESULTS

**Theorem 2.1**: Let A, B, S, T, L and M be self maps on a left (right) sequentially complete Quasi-Gauge $T_0$-space (X, $\mathcal{P}$) such that

(i) the pairs {AL, S} and {BM, T} are weakly compatible ;

(ii) T(X) $\subseteq$ AL(X) and S(X) $\subseteq$ BM(X);

(iii) One of AL(X), BM(X), S(X) and T(X) is a complete subspace of X;

(iv) AL = LA and BM = MB;

(v) "either TB = BT or TM = MT" and "either SA = AS or SL = LS" ;

(vi)
$$\max\{p^2(Sx,Ty),\, p^2(Ty,Sx)\} \le \phi\{p(ALx,Sx)\,p(BMy,Ty),\, p(ALx,Ty)\,p(BMy,Sx),\, p(ALx,Sx)\,p(ALx,Ty),\, p(BMy,Sx)\,p(BMy,Ty),\, p(BMy,Sx)\,p(ALx,Sx),\, p(BMy,Ty)\,p(ALx,Ty)\}$$

for all $x, y \in X$ and for all $p \in \mathcal{P}$; where $\phi:[0,\infty)^6 \to [0,\infty)$ satisfies the following:

(vii) $\phi$ is non-decreasing and upper semi continuous in each coordinate variable and for each t > 0,

$$\psi(t) = \max\{\phi\{t,0,2t,0,0,2t\}, \phi\{t,0,0,2t,2t,0\}, \phi\{0,t,0,0,0,0\}, \phi\{0,0,0,0,0,t\}, \phi\{0,0,0,0,t,0\}\} < t.$$

Then A, B, S, T, L and M have a unique common fixed point.

Proof :

Let $x_0 \in X$. By (ii) we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Sx_{2n} = BMx_{2n+1} = y_{2n} \text{ (say)}$$

and

$Tx_{2n+1} = ALx_{2n+2} = y_{2n+1}$ (say), for $n = 0,1,2,\ldots$

Let $d_n = p(y_n, y_{n+1})$ and $e_n = p(y_{n+1}, y_n)$.

Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (vi), we get that

$$\max\{p^2(y_{2n}, y_{2n+1}), p^2(y_{2n+1}, y_{2n})\} \le \phi\{p(y_{2n-1}, y_{2n})p(y_{2n}, y_{2n+1}), p(y_{2n-1}, y_{2n+1})p(y_{2n}, y_{2n}), p(y_{2n-1}, y_{2n})p(y_{2n-1}, y_{2n+1}), p(y_{2n}, y_{2n})p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n})p(y_{2n-1}, y_{2n}), p(y_{2n}, y_{2n+1})p(y_{2n-1}, y_{2n+1})\}$$

i.e., $\max\{d_{2n}^2, e_{2n}^2\} \le \phi\{d_{2n-1}d_{2n}, 0, d_{2n-1}^2 + d_{2n-1}d_{2n}, 0, 0, d_{2n}d_{2n-1} + d_{2n}^2\}$.

If $d_{2n-1} < d_{2n}$ then $\max\{d_{2n}^2, e_{2n}^2\} \le \phi\{d^2{}_{2n}, 0, 2d_{2n}^2, 0, 0, 2d_{2n}^2\} < d_{2n}^2$, which is a contradiction; hence $d_{2n-1} \ge d_{2n}$.

By taking $x=x_{2n+2}$ and $y= x_{2n+1}$ in (vi), we get that

$$\max\{p^2(y_{2n+2}, y_{2n+1}), p^2(y_{2n+1}, y_{2n+2})\} \le \phi\{p(y_{2n+1}, y_{2n+2})p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+1})p(y_{2n}, y_{2n+2}), p(y_{2n+1}, y_{2n+2})p(y_{2n+1}, y_{2n+1}), p(y_{2n}, y_{2n+2})p(y_{2n}, y_{2n+1}), p(y_{2n}, y_{2n+2})p(y_{2n+1}, y_{2n+2}), p(y_{2n}, y_{2n+1})p(y_{2n+1}, y_{2n+1})\}$$

i.e., $\max\{e_{2n+1}^2, d_{2n+1}^2\} \le \phi\{d_{2n+1}d_{2n}, 0, 0, d_{2n}^2 + d_{2n}d_{2n+1}, d_{2n}d_{2n+1} + d_{2n+1}^2, 0\}$.

If $d_{2n} < d_{2n+1}$ then $\max\{e_{2n+1}^2, d_{2n+1}^2\} \le \phi\{d^2{}_{2n+1}, 0, 0, 2d_{2n+1}^2, 2d^2{}_{2n+1}, 0\} < d_{2n+1}^2$, which is a contradiction by (vii);

hence $d_{2n} \ge d_{2n+1}$.

Now, $\max\{d_{2n}^2, e_{2n}^2\} \le \phi\{d^2{}_{2n-1}, 0, 2d_{2n-1}^2, 0, 0, 2d_{2n-1}^2\} \le \psi(d_{2n-1}^2) = \psi(p^2(y_{2n-1}, y_{2n})$

and $\max\{e_{2n+1}^2, d_{2n+1}^2\} \le \phi\{d^2{}_{2n}, 0, 0, 2d_{2n}^2, 2d^2{}_{2n}, 0\} \le \psi(d_{2n}^2) = \psi(p^2(y_{2n}, y_{2n+1}))$.

So, $d_n^2 = p^2(y_n, y_{n+1}) \le \psi(p^2(y_{n-1}, y_n)) \le \cdots \le \psi^n(p^2(y_0, y_1))$ (2.1.1)

and $e_n^2 = p^2(y_{n+1}, y_n) \le \psi(p^2(y_n, y_{n-1})) \le \cdots \le \psi^n(p^2(y_1, y_0))$. (2.1.2)

By Lemma (1.8) and from (2.1.1) & (2.1.2), we get

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} e_n = 0 \qquad (2.1.3)$$

Now we prove $\{y_n\}$ is a P-Cauchy sequence. For this, it is enough to show that $\{y_{2n}\}$ is P -Cauchy. Suppose not ; then there is an $\in > 0$ such that for each positive integer 2k there exist positive integers 2m(k) and 2n(k) such that for some $p \in P$

$p(y_{2n(k)}, y_{2m(k)}) > \in$ for 2m(k) > 2n(k) > 2k (2.1.4)

and $p(y_{2m(k)}, y_{2n(k)}) > \in$ for 2m(k) > 2n(k) > 2k. (2.1.5)

For each positive even integer 2k, let 2m(k) be the least positive even integer exceeding 2n(k) and satisfying (2.1.3); hence $p(y_{2n(k)}, y_{2m(k)-2}) \le \in$.

Then for each even integer 2k,

$\in < p(y_{2n(k)}, y_{2m(k)})$

$\leq \; p(y_{2n(k)}, y_{2m(k)-2}) + p(y_{2m(k)-2}, y_{2m(k)-1}) + p(y_{2m(k)-1}, y_{2m(k)})$

$\leq p(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}$ (2.1.6)

From (2.1.3) and (2.1.5), $\lim_{k\to\infty} p(y_{2n(k)}, y_{2m(k)}) = \in$ (2.1.7)

By triangle inequality,

$p(y_{2n(k)}, y_{2m(k)}) \leq p(y_{2n(k)}, y_{2m(k)-1}) + p(y_{2m(k)-1}, y_{2m(k)})$

$= p(y_{2n(k)}, y_{2m(k)-1}) + d_{2m(k)-1}$

and

$p(y_{2n(k)}, y_{2m(k)-1}) \leq p(y_{2n(k)}, y_{2m(k)}) + p(y_{2m(k)}, y_{2m(k)-1})$

$= p(y_{2n(k)}, y_{2m(k)}) + e_{2m(k)-1}$.

These imply that $\left|p(y_{2n(k)}, y_{2m(k)}) - p(y_{2n(k)}, y_{2m(k)-1})\right| \leq \max\{d_{2m(k)-1}, e_{2m(k)-1}\}$ (2.1.8)

Similarly, we get that

$\left|p\left(y_{2n(k)+1}, y_{2m(k)-1}\right) - p\left(y_{2n(k)}, y_{2m(k)}\right)\right| \leq \max\{d_{2n(k)} + d_{2m(k)-1}, e_{2n(k)} + e_{2m(k)-1}\}$ (2.1.9)

From equations (2.1.8) and (2.1.9), by virtue of (2.1.3), we get that $\lim_{k\to\infty} p(y_{2n(k)}, y_{2m(k)-1}) = \in$

and $\lim_{k\to\infty} p(y_{2n(k)+1}, y_{2m(k)-1}) = \in$.

If $p(y_{2m(k)}, y_{2n(k)}) > \in$, proceeding as above, we get that

$$\lim_{k\to\infty} p(y_{2m(k)}, y_{2n(k)}) = \lim_{k\to\infty} p(y_{2m(k)-1}, y_{2n(k)+1}) = \lim_{k\to\infty} p(y_{2m(k)-1}, y_{2n(k)}) = \in.$$

By taking $x = x_{2m(k)}$ and $y = x_{2n(k)+1}$ in (vi), we get that

$\in < p(y_{2n(k)}, y_{2m(k)})$

$\leq p(y_{2n(k)}, y_{2n(k)+1}) + p(y_{2n(k)+1}, y_{2m(k)})$

$\leq d_{2n(k)} + \max\{p(y_{2n(k)+1}, y_{2m(k)}), p(y_{2m(k)}, y_{2n(k)+1})\}$

$= d_{2n(k)} + \max\{p(Tx_{2n(k)+1}, Sx_{2m(k)}), p(Sx_{2m(k)}, Tx_{2n(k)+1})\}$

$\leq d_{2n(k)} + [\phi\{p(y_{2m(k)-1}, y_{2m(k)}) p(y_{2n(k)}, y_{2n(k)+1}),$

$p(y_{2m(k)-1}, y_{2n(k)+1})\, p(y_{2n(k)}, y_{2m(k)}), p(y_{2m(k)-1}, y_{2m(k)}) p(y_{2m(k)-1}, y_{2n(k)+1}),$

$p(y_{2n(k)}, y_{2m(k)}) p(y_{2n(k)}, y_{2n(k)+1}), p(y_{2n(k)-1}, y_{2m(k)}) p(y_{2m(k)-1}, y_{2m(k)}),$

$p(y_{2n(k)}, y_{2n(k)+1}) p(y_{2m(k)-1}, y_{2n(k)+1})]^{1/2}$.

Since $\phi$ is upper semi continuous, as $k \to \infty$

we get $\in \leq [\varphi\{0, \in^2, 0, 0, 0, 0\}]^{\frac{1}{2}} < \in$ which is a contradiction.

Therefore, $\{y_n\}$ is a P-Cauchy sequence in X.

Since X is sequentially complete, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} y_n = z.$$

Hence, $\lim_{n\to\infty} ALx_{2n} = \lim_{n\to\infty} Tx_{2n-1} = z$

and $\lim_{n\to\infty} BMx_{2n+2} = \lim_{n\to\infty} Sx_{2n+1} = z.$

**Case I**: Suppose S(X) or BM(X) is a complete subspace of X.

Since $\{y_n\} \subseteq S(X) (\subseteq BM(X))$, there exists a point $u \in X$ such that z = BMu.

We now show that Tu = z. If not

taking $x = x_{2n}$ and y=u in (vi), we get that

$$\max\{p^2(Sx_{2n},Tu),\, p^2(Tu,Sx_{2n})\} \le \phi\{p(ALx_{2n},Sx_{2n})p(z,Tu),\, p(ALx_{2n},Tu)p(z,Sx_{2n}),\\ p(ALx_{2n},Sx_{2n})p(ALx_{2n},Tu),\, p(z,Sx_{2n})p(z,Tu),\\ p(z,Sx_{2n})p(ALx_{2n},Sx_{2n}),\, p(z,Tu)p(ALx_{2n},Tu)\}$$

$$i.e.,\ \max\{p^2(y_{2n},Tu),\, p^2(Tu,y_{2n})\} \le \phi\{p(y_{2n-1},y_{2n})p(z,Tu),\, p(y_{2n-1},Tu)p(z,y_{2n}),\\ p(y_{2n-1},y_{2n})p(y_{2n-1},Tu),\, p(z,y_{2n})p(z,Tu),\\ p(z,y_{2n})p(y_{2n-1},y_{2n}),\, p(z,Tu)p(y_{2n-1},Tu)\}.$$

Letting $n \to \infty$, we get that

$$\max\{p^2(z,Tu),p^2(Tu,z)\} \le \phi\{0,0,0,0,0,p(z,Tu)p(z,Tu)\}\\ < p(z,Tu)p(z,Tu)$$

which is a contradiction.

So, Tu = z. Thus, Tu = z = BMu.

Since {BM, T} is weakly compatible, TBMu= BMTu. i.e., Tz = BMz.

We now show that Tz = z. If not

taking $x = x_{2n}$ and y = z in (vi), we get that

$$\max\{p^2(y_{2n},Tz),\, p^2(Tz,y_{2n})\} \le \phi\{p(y_{2n-1},y_{2n})p(BMy,Tz),\, p(y_{2n-1},Tz)p(BMy,y_{2n}),\\ p(y_{2n-1},y_{2n})p(y_{2n-1},Tz),\, p(BMz,y_{2n})p(BMz,Tz),\\ p(BMz,y_{2n})p(y_{2n-1},y_{2n}),\, p(BMz,Tz)p(y_{2n-1},Tz)\}$$

$$i.e.,\ \max\{p^2(y_{2n},Tz),\, p^2(Tz,y_{2n})\} \le \phi\{p(y_{2n-1},y_{2n})p(z,Tz),\, p(y_{2n-1},Tz)p(z,y_{2n})\\ p(y_{2n-1},y_{2n})p(y_{2n-1},Tz),\, p(z,y_{2n})p(z,Tz),\\ p(z,y_{2n})p(y_{2n-1},y_{2n}),\, p(z,Tz)p(y_{2n-1},Tz)\}$$

Letting $n \to \infty$, we get that

$$\max\{p^2(z,Tz),p^2(Tz,z)\} \le \phi\{0,0,0,0,0,p(z,Tz)p(z,Tz)\}\\ < p(z,Tz)p(z,Tz)$$

which is a contradiction. So, Tz = z.

Hence, Tz = BMz = z.

Suppose TM = MT. Since BM = MB, BMMz = MBMz = Mz and TMz = MTz = Mz.

We now show that Mz = z, If not

taking $x = x_{2n}$ and y = Mz in (vi), we get that

$$\max\{p^2(Sx_{2n}, Mz), p^2(Mz, Sx_{2n})\} \le \phi\{p(ALx_{2n}, Sx_{2n})p(Mz, Mz), p(ALx_{2n}, Mz)p(Mz, Sx_{2n}), p(ALx_{2n}, Sx_{2n})p(ALx_{2n}, Mz), p(Mz, Sx_{2n})p(Mz, Mz), p(Mz, Sx_{2n})p(ALx_{2n}, Sx_{2n}), p(Mz, Mz)p(ALx_{2n}, Mz)\}.$$

Letting $n \to \infty$, we get that

$$\max\{p^2(z, Mz), p^2(Mz, z)\} \le \phi\{p(z,z)p(Mz, Mz), p(z, Mz)p(Mz, z), p(z,z)p(z, Mz), p(Mz, z)p(Mz, Mz), p(Mz, z)p(z,z), p(Mz, Mz)p(z, Mz)\}.$$

This implies that

$$\max\left\{p^2(z, Mz), p^2(Mz, z)\right\} \le \phi\{0, p(z, Mz)p(Mz, z), 0, 0, 0, 0\} < p(z, Mz)p(Mz, z).$$

We first show that p(z, Mz) = p(Mz, z) and then = 0. Suppose not;

either p(z,Mz) > p(Mz, z) $\geq$ 0 or p(Mz, z) > p(z, Mz) $\geq$ 0.

Without loss of generality, we assume that p(z,Mz) > p(Mz, z) $\geq$ 0.

Now, max{$p^2$(z, Mz), $p^2$(Mz, z)} $\leq$ $p^2$(z, Mz), which is a contradiction.

Therefore, p(z, Mz) = p(Mz, z). If the common value is not 0 then the above inequality is not valid.

Hence the common value is 0.

Since X is $T_0$ – space, follows that Mz = z.

Since z = BMz, we have Bz = z. Hence Bz = Mz = Tz = z.

Suppose TB = BT.

Since BM = MB, BMBz = BBMz = Bz and TBz = BTz = Bz.

We now show that Bz = z. If not

taking $x = x_{2n}$ and y = Bz in (vi), we get that

$$\max\{p^2(Sx_{2n}, Bz), p^2(Bz, Sx_{2n})\} \le \phi\{p(ALx_{2n}, Sx_{2n})p(Bz, Bz), p(ALx_{2n}, Bz)p(Bz, Sx_{2n}), p(ALx_{2n}, Sx_{2n})p(ALx_{2n}, Bz), p(Bz, Sx_{2n})p(Bz, Bz), p(Bz, Sx_{2n})p(ALx_{2n}, Sx_{2n}), p(Bz, Bz)p(ALx_{2n}, Bz)\}.$$

Letting $n \to \infty$, we get that

$$\max\{p^2(z, Bz), p^2(Bz, z)\} \le \phi\{p(z,z)p(Bz, Bz), p(z, Bz)p(Bz, z), p(z,z)p(z, Bz), p(Bz, z)p(Bz, Bz), p(Bz, z)p(z,z), p(Bz, Bz)p(z, Bz)\}$$

i.e., $$\max\left\{p^2(z, Bz), p^2(Bz, z)\right\} \le \phi\{0, p(z, Bz)p(Bz, z), 0, 0, 0, 0\} < p(z, Bz)p(Bz, z)$$

As above, we can prove that p(z, Bz) = p(Bz, z) = 0.

Since X is $T_0$ – space follows that Bz = z.

Since z = BMz, z = BMz = MBz = Mz. Hence z = Bz = Mz = Tz.

Since T(X) $\subseteq$ AL (X), there is a $w \in X$ such that z = ALw.

We now show that Sw = z. If not;

taking x = $w$ and y = $x_{2n+1}$ in (vi), we get that

$$\begin{aligned}\max\{p^2(Sw,Tx_{2n+1}), p^2(Tx_{2n+1},Sw)\} &\le \phi\{p(ALw,Sw)\,p(BMx_{2n+1},Tx_{2n+1}),\ p(ALw,Tx_{2n+1})\,p(BMx_{2n+1},Sw),\\ &\quad p(ALw,Sw)\,p(ALw,Tx_{2n+1}),\ p(BMx_{2n+1},Sw)\,p(BMx_{2n+1},Tx_{2n+1}),\\ &\quad p(BMx_{2n+1},Sw)\,p(ALw,Sw),\ p(BMx_{2n+1},Tx_{2n+1})\,p(ALw,Tx_{2n+1})\}.\end{aligned}$$

Letting $n \to \infty$, we get that

$$\begin{aligned}\max\{p^2(Sw,z), p^2(z,Sw)\} \le \phi\{&p(z,Sw)\,p(z,z),\ p(z,z)\,p(z,Sw),\\ &p(z,Sw)\,p(z,z),\ p(z,Sw)\,p(z,z),\\ &p(z,Sw)\,p(z,Sw),\ p(z,z)\,p(z,z)\}\end{aligned}$$

$$\begin{aligned}\text{i.e., } \max\left\{p^2(Sw,z), p^2(z,Sw)\right\} &\le \phi\{0,0,0,0,p(z,Sw)\,p(z,Sw),0\}\\ &< p(z,Sw)\,p(z,Sw)\end{aligned}$$

which is contradiction. Hence z = Sw.

Thus Sw = z = ALw. Since {AL, S} is weakly compatible, ALSw = SALw.

i.e., ALz=Sz.

We now show that Sz = z. If not ;

taking x = z and y = $x_{2n+1}$ in (vi), we get that

$$\begin{aligned}\max\{p^2(Sz,Tx_{2n+1}), p^2(Tx_{2n+1},Sz)\} &\le \phi\{p(ALz,Sz)\,p(BMx_{2n+1},Tx_{2n+1}),\ p(ALz,Tx_{2n+1})\,p(BMx_{2n+1},Sz),\\ &\quad p(ALz,Sz)\,p(ALz,Tx_{2n+1}),\ p(BMx_{2n+1},Sz)\,p(BMx_{2n+1},Tx_{2n+1}),\\ &\quad p(BMx_{2n+1},Sz)\,p(ALz,Sz),\ p(BMx_{2n+1},Tx_{2n+1})\,p(ALz,Tx_{2n+1})\}.\end{aligned}$$

Letting $n \to \infty$, we get that

$$\begin{aligned}\max\left\{p^2(Sz,z), p^2(z,Sz)\right\} &\le \phi\{0,p(Sz,z)\,p(z,Sz),0,0,0,0\}\\ &< p(Sz,z)\,p(z,Sz)\end{aligned}$$

which is a contradiction. So, Sz = z. Thus z = Sz = ALz.

Suppose SL = LS. Since AL = LA, ALLz = LALz = Lz and SLz = LSz = Lz.

We now show that Lz = z. If not,

taking x = Lz and y = $x_{2n+1}$ in (vi) , we get that

$$\max\{p^2(Lz,Tx_{2n+1}),\, p^2(Tx_{2n+1},Lz)\} \le \phi\{p(Lz,Lz)\,p(BMx_{2n+1},Tx_{2n+1}),\, p(Lz,Tx_{2n+1})\,p(BMx_{2n+1},Lz),\, p(Lz,Lz)\,p(Lz,Tx_{2n+1}),\, p(BMx_{2n+1},Lz)\,p(BMx_{2n+1},Tx_{2n+1}),\, p(BMx_{2n+1},Lz)\,p(Lz,Lz),\, p(BMx_{2n+1},Tx_{2n+1})\,p(Lz,Tx_{2n+1})\}.$$

Letting $n \to \infty$, we get that

$$\max\{p^2(Lz,z), p^2(z,Lz)\} \le \phi\{0, p(Lz,z)\,p(z,Lz), 0,0,0,0\} < p(Lz,z)\,p(z,Lz)$$

which is a contradiction. So, Lz = z. Since z = ALz , we have z = ALz = Az. Thus Az = Lz = Sz = z.

Suppose AS = SA. Since AL = LA, ALAz = AALz = Az and SAz = ASz = Az.

We now show that Az = z. If not

taking x = Az and y = $x_{2n+1}$ in (vi), we get that

$$\max\{p^2(Az,Tx_{2n+1}),\, p^2(Tx_{2n+1},Az)\} \le \phi\{p(Az,Az)\,p(BMx_{2n+1},Tx_{2n+1}),\, p(Az,Tx_{2n+1})\,p(BMx_{2n+1},Az),\, p(Az,Az)\,p(Az,Tx_{2n+1}),\, p(BMx_{2n+1},Az)\,p(BMx_{2n+1},Tx_{2n+1}),\, p(BMx_{2n+1},Az)\,p(Az,Az),\, p(BMx_{2n+1},Tx_{2n+1})\,p(Az,Tx_{2n+1})\}.$$

Letting $n \to \infty$, we get that

$$\max\{p^2(Az,z), p^2(z,Az)\} \le \phi\{0, p(Az,z)\,p(z,Az), 0,0,0,0\} < p(Az,z)\,p(z,Az)$$

which is a contradiction. Hence Az = z.

Since z = ALz, we have z = ALz = LAz = Lz. Thus z = Az = Lz = Sz.

Hence z = Az = Bz = Lz = Mz = Sz = Tz.

**Case 2**: Suppose T(X) or AL(X) is a complete subspace of X.

In this case, we first get that z = Az = Lz = Sz and then z = Bz = Mz = Tz.

Thus z is a common fixed point of A, B, S, T, L and M in X.

**Uniqueness**: suppose $z^1$ is also a common fixed point of A, B, S, T, L and M in X.

We now show that $z^1$ = z . If not

taking x = z and y = $z^1$ in (vi), we get a contradiction, so $z^1$ = z.

Hence, z is the unique common fixed point of A, B, S, T, L and M in X.

We, now, give the following example in support of our Theorem (2.1).

**Example 2.2**: Let $X = [0, \infty)$ with the usual metric and let A, B, S, T, L and M be self maps on X. Define A, B, S, T, L and M by $Ax = x$, $Bx=x^2$, $Lx = Mx = x$,

$$Sx = \begin{cases} 0 \ if \ \ x \le 3, \\ 1 \ if \ \ x > 3 \end{cases}$$

and $Tx = 0$.

Define $\phi$ by $\phi\left(t_1,t_2,t_3,t_4,t_5,t_6\right)=\dfrac{t_1+t_2+t_3+t_4+t_5+t_6}{6}$.

Define $p: X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$p(x,y) = \begin{cases} x-y \ \ if \ \ x \ge y, \\ \dfrac{y-x}{2}, \ if \ \ x \le y. \end{cases}$$

Clearly '0' is the unique common fixed point of A, B, S, T, L and M.

For, when $x \le 3$ and $y \in X$

L.H.S= max $\{p^2(Sx,Ty), p^2(Ty,Sx)\}$ = max $\{p^2(0,0), p^2(0,0)\} = 0$.

When $x > 3$ and $y \in X$.

L.H.S = max$\{p^2(1, 0), p^2(0,1)\}$= max$\{1, 1/2\}$=1.

$$\text{R.H.S.} = \begin{array}{l} \phi \ \left\{p(x,1)\,p\left(y^2,0\right), p(x,0), p\left(y^2,1\right), p(x,1)\,p(x,0), \right. \\ \left. p\left(y^2,1\right)p\left(y^2,0\right), p\left(y^2,1\right)p(x,1), p\left(y^2,0\right)p(x,0)\right\} \end{array}$$

$$= \phi \ \{(x-1)y^2, x(y^2-1), (x-1)x, (y^2-1)y^2, (y^2-1)(x-1), y^2x\}$$

$$= \frac{\{(x-1)\,y^2 + x\left(y^2-1\right)+\left(x-1\right)x+\left(y^2-1\right)y^2+\left(y^2-1\right)\left(x-1\right)y^2x\}}{6}$$

$$= \frac{\{y^2\left(2x+y^2-1\right)+\left(x-1\right)y^2+\left(x-1\right)x+y^2x\}}{6}$$

$$\ge \frac{x\left(x-1\right)}{6}$$

$$\ge \frac{3(2)}{6} = 1$$

Thus L.H.S. $\le$ R.H.S.

All the other conditions are trivially satisfied.

**Note 2.3:** By taking L = M = I (The identity map on X) in Theorem (2.1), we get the following:

**Corollary 2.4:** Let A, B, S and T be self maps on a left(right) sequentially complete Quasi gauge $T_0$- space (X, P ) such that

(i) the pairs {A,S} and {B, T} are weakly compatible ;

(ii) T(X) $\subseteq$ A(X) and S(X) $\subseteq$ B(X);

(iii) One of A(X), B(X), S(X) and T(X) is a complete subspace of X;

(iv)
$$\max\{p^2(Sx,Ty),\, p^2(Ty,Sx)\} \le \phi\{p(Ax,Sx)p(By,Ty),\, p(Ax,Ty)p(By,Sx),\, p(Ax,Sx)p(Ax,Ty),\, p(By,Sx)p(By,Ty),\, p(By,Sx)p(Ax,Sx),\, p(By,Ty)p(Ax,Ty)\}$$

for all x,y $\in$ X and for all p $\in$ P, where $\phi:[0,\infty)^6 \to (0,\infty)$ satisfies the following.

(v) $\phi$ is non-decreasing and upper semi continuous in each coordinate variable and for each t > 0,

$$\psi(t) = \max\left\{\varphi\{t,0,2t,0,0,2t\},\varphi\{t,0,0,2t,2t,0\},\varphi\{0,t,0,0,0,0\},\varphi\{0,0,0,0,0,t\},\varphi\{0,0,0,0,t,0\}\right\} < t.$$

Then A, B, S, and T have a unique common fixed point.

**Remark 2.5**: Corollary (2.3) is a revised version of Theorem (1.11).

## REFERENCES


[1] J.Antony: studies in fixed points and Quasi-Gauges., Ph.D. Thesis IIT, madras (1991).

[2] J. Matkowski: Fixed point theorems for mappings with a contractive iterate at a point, Proc. Ameer. Math, Soc., 62(1977), no.2, 344-348.

[3] I.H.N.Rao and A.Sree Rama Murty: Common fixed point of weakly compatible mappings in Quasi-Gauge space, J.Indian Acad. Math., 21 (1999), no.1, 73-87.

[4] I.L.Reilly: Quasi-Gauge spaces, J.London Math, Soc., (2), 6(1973), 481-487.

[5] Sushi Sharma and Prashant Tilwankar: Common fixed point of weakly compatible mappings in Quasi-Gauge space, Bull, Malayas, Math.Sci. Soc.,(2)35(1),(2012),155- 161.