Existential and Uniqueness Results for Boundary Value Problems associated with Nonlinear Singular Interface Problems on Time Scales using Fixed Point Theorems

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The authors dedicate this work to the Founder Chancellor of Sri Sathya Sai Institute of Higher Learning, Bhagwan Sri Sathya Sai Baba.

Abstract. In this paper we present existential and uniqueness results for BVPs associated with 4^{th} order nonlinear singular interface problems on Time Scales. We discuss these results using the classical fixed point theorems of Banach and Schauder.

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1. Introduction

In literature we find a class of problems wherein two different differential equations are defined on adjacent intervals with a common point of interface. We term these problems as interface problems.

If the interface problem has a well defined boundary, we call the problem to be a *regular boundary value problem(RBVP).* The interface problem with a boundary that has singularity at the end points is called a *singular boundary value problem(SBVP).* If there is a singularity at the point of interface, we term the problem to be a *singular interface problem(SIP).* Solving these type of boundary value problems with singularities remains a challenge for mathematicians.

While *regular* boundary value problems, those over finite intervals with well-behaved coefficients pose no difficulties, the problems wherein the domain of the problem is not well defined, or the continuity and/or smoothness of the functions, coefficients involved are not guaranteed in some parts of the domain, sometimes in the boundary or parts of the boundary are difficult to tackle. There are quite a number of different approaches that we come across in the literature to tackle these *singular* problems [1],[4],[6],[8],[9],[10].

In literature, we see that work has been done on initial and boundary value problems associated with a pair of linear differential operators with conditions at the interface for both regular and singular cases. Some publications include [15]-[22],[26]-[31].

The singular interface problem requires a special mention. In this case existing theory based on the conventional analysis may not come handy.

We feel that the new framework of dynamic equations on time scales(an arbitrary closed subset of real numbers)[2] with facilities of the two jump operators with various definitions of continuity and derivatives makes one's job simple to study these singular interface problems. These dynamic equations are nothing but the differential equations when $\mathbb{T} = \mathbb{R}$ and are difference equations when $\mathbb{T} = \mathbb{Z}$.

Our preliminary investigation about the feasibility of this study for linear second order interface problems has resulted in the work [16, 17, 25].

From the above we observe that substantial amount of work has been done for regular and singular boundary value problems involving linear differential operators. It is clear that there is a need for these singular interface problems to be discussed for the case where the problem involves nonlinear differential operators.

A systematic study of Initial Value Problems, Boundary Value Problems and eigen Value problems associated with these nonlinear singular interface problems involving nonlinear second order pair of dynamic equations is done in [34]-[41].

In this paper we study the existence and uniqueness of solution for a fourth order Boundary Value Problem associated with these nonlinear singular interface problems. Schauder and Banach's fixed point theorems are used for proving the exsistential and uniqueness results.

2. Mathematical Preliminaries

Definition 2.1. Let $\mathbb{T}$ be a time scale(an arbitrary closed subset of real numbers). For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is *right-scattered,* while $\rho(t) < t$ we say that t is *left-scattered.* Points that are right-scattered and left-scattered at the same time are called *isolated.* Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense,* and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense.* Points that are right-dense and left-dense at the same time are called *dense.* Finally, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Definition 2.2. $\mathbb{T}^\kappa = \left\{ \begin{array}{l} \mathbb{T} - \{m\} \text{ if sup } \mathbb{T} < \infty \\ \mathbb{T} \text{ if sup } \mathbb{T} = \infty \end{array} \right\}$ where m is the left scattered maximum of $\mathbb{T}$.

Definition 2.3. Let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\epsilon > 0$ there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon\, |\sigma(t) - s| \text{ for all } s \in \mathcal{N}.$$

Definition 2.4. For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided f^Δ is differentiable on $\mathbb{T}^{\kappa 2} = (\mathbb{T}^\kappa)^\kappa$ with derivative $f^{\Delta\Delta} = (f^\Delta)^\Delta : \mathbb{T}^{\kappa 2} \to \mathbb{R}$. Similarly we define the higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa n} \to \mathbb{R}$.

Theorem 2.5. *(Banach Contraction Mapping Theorem)*
If $T : X \to X$ is contractive on a complete metric space X then T has a unique fixed point in X.

Theorem 2.6. *(Schauder's Fixed Point Thorem)*
Let L be a convex subset of a normed linear space E. Then each compact map $T : L \to L$ has a fixed point.

Let $I = [c, d]$ with $c < \rho(d)$. We define $I_c = [c, \infty)$ in case $sup\mathbb{T} = +\infty$. By $\mathcal{C}^B_{TS}(I_a)$ we mean the linear space of all continuous functions $f : I_c \to \mathbb{R}$ such that $sup_{t \in I_c}|f(t)| < \infty$.

Now we quote the time scales version of the Arzela-Ascoli theorem [1].

Theorem 2.7. *(Arzela-Ascoli Theorem)*
Let X be a subset of $\mathcal{C}^B_{TS}(I_a)$ having the following properties.
(i) X is bounded.
(ii) On every compact subinterval J of $[c, \infty)$ we have: For any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in X$.
(iii) For every $\epsilon > 0$ there exists $b \in I_c$ such that $t_1, t_2 \in [b, \infty)$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in X$.
Then X is relatively compact.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. Let $C(\mathbb{T})$ denote the space of all continuous functions on the time scale $\mathbb{T}$.

Definition 2.8. By $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we mean that $t_1 \in \mathbb{T}_1$ and $t_2 \in \mathbb{T}_2$ with the product topology on $\mathbb{T}_1 \times \mathbb{T}_2$.

Definition 2.9. Let $Z(\mathbb{T})$ be a function space on the time scale $\mathbb{T}$. For $x \in Z(\mathbb{T})$ we define

$$\|x\| = sup_{t \in \mathbb{T}}|x(t)|.$$

Definition 2.10. By $(x_1, x_2) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$ where X and Y are function spaces, we mean that $x_1 \in X(\mathbb{T}_1)$ and $x_2 \in Y(\mathbb{T}_2)$ with the product topology on $X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$. For $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we define

$$(x_1, x_2)(t_1, t_2) = (x_1(t_1), x_2(t_2)).$$

Definition 2.11. For $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, we define

$$\|(t_1, t_2)\| = \|t_1\| + \|t_2\| = |t_1| + |t_2|.$$

Definition 2.12. For $(x_1, x_2) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$, we define

$$\begin{aligned} \|(x_1, x_2)\| &= \|x_1\| + \|x_2\| \\ &= sup_{t_1 \in \mathbb{T}_1} |x_1(t_1)| + sup_{t_2 \in \mathbb{T}_2} |x_2(t_2)|. \end{aligned}$$

Definition 2.13. Let $(y_{11}, y_{12}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$. We say that (y_{11}, y_{12}) is continuous on $\mathbb{T}_1 \times \mathbb{T}_2$ if for $\epsilon > 0$ there exists $\delta > 0$ such that for arbitrarily fixed $(t_{01}, t_{02}) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that

$$\begin{aligned} \| & (t_1, t_2) - (t_{01}, t_{02})\| < \delta \\ \Rightarrow & \|(y_{11}, y_{12})(t_1, t_2) - (y_{11}, y_{12})(t_{01}, t_{02})\| < \epsilon. \end{aligned} \tag{2.1}$$

Definition 2.14. A sequence $(y_{n1}, y_{n2}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is said to be cauchy sequence if for every $\epsilon > 0$ there exists N such that $\forall n1, n2, m1, m2 > N$ implies

$$\|(x_{n1}, x_{n2}) - (x_{m1}, x_{m2})\| < \epsilon.$$

Definition 2.15. A sequence $(y_{n1}, y_{n2}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$ is said to be equicontinuous if for every $\epsilon > 0$ there is a $\delta > 0$, depending only on ϵ, such that for all (y_{n1}, y_{n2}) and all $(t_1, t_2), (t_1^{'}, t_2^{'}) \in \mathbb{T}_1 \times \mathbb{T}_2$ satisfying

$$\begin{aligned} \| & (t_1, t_2) - (t_1^{'}, t_2^{'})\| < \delta \\ \Rightarrow & \|(y_{n1}, y_{n2})(t_1, t_2) - (y_{n1}, y_{n2})(t_1^{'}, t_2^{'})\| < \epsilon. \end{aligned} \tag{2.2}$$

Definition 2.16. The space $X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$ is said to convex if for every $(y_{11}, y_{12}), (y_{21}, y_{22}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$, we have

$$\alpha(y_{11}, y_{12}) + (1 - \alpha)(y_{21}, y_{22}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2) \text{ for } 0 < \alpha < 1.$$

3. Existence and Uniqueness of Solution for a Fourth Order Boundary Value Problem associated with Nonlinear Singular Interface Problem

3.1. Definition of the Problem

Let $\mathbb{T}_1 = [0, a]_{\mathbb{T}}, \mathbb{T}_2 = [\sigma(a), b]_{\mathbb{T}}$ where $a, \sigma(a), b < +\infty$. Also let (f_1, f_2) be non-linear function tuple in $\mathcal{C}(\mathbb{T}_1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$. Let us consider the following BVP associated with singular interface problem(BVP-SIP).

$$y_1^{\Delta\Delta\Delta\Delta}(t) = f_1(t, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}), \ t \in \mathbb{T}_1 \tag{3.1}$$

$$y_2^{\Delta\Delta\Delta\Delta}(t) = f_2(t, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}), \ t \in {\mathbb{T}_2}^{\kappa^4} \tag{3.2}$$

with the initial conditions

$$y_1(0) = 0 \quad (3.3)$$
$$y_1^{\Delta}(0) = 0 \quad (3.4)$$
$$y_1^{\Delta\Delta}(0) = 0 \quad (3.5)$$

and boundary condition

$$y_2^{\Delta\Delta\Delta}(\rho^3(b)) = 0 \quad (3.6)$$

followed by the matching interface conditions

$$\gamma_1 y_1(a) = \gamma_2 y_2(\sigma(a)), \quad (3.7)$$
$$\gamma_3 y_1^{\Delta}(a) = \gamma_4 y_2^{\Delta}(\sigma(a)), \quad (3.8)$$
$$\gamma_5 y_1^{\Delta\Delta}(a) = \gamma_6 y_2^{\Delta\Delta}(\sigma(a)), \quad (3.9)$$
$$\gamma_7 y_1^{\Delta\Delta\Delta}(a) = \gamma_8 y_2^{\Delta\Delta\Delta}(\sigma(a)), \gamma_i > 0, \ i = 1,2,3,4,5,6,7,8. \quad (3.10)$$

3.2. Existence of Solution using Schauder's Fixed Point Theorem

In this section we prove the existence of solution for the BVP-SIP using Schauder's fixed point theorem.

Theorem 3.1. *If (f_1, f_2) is continuous and bounded, then there exists atleast one solution for the 4th order BVP-SIP.(3.1)-(3.10)*

Proof. Case I Let $t \in \mathbb{T}_1$
Then,

$$\begin{aligned}
y_1^{\Delta\Delta\Delta\Delta}(t) &= f_1(t, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) \\
y_1^{\Delta\Delta\Delta}(t) &= \int_0^t f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s + c_{11} \\
y_1^{\Delta\Delta}(t) &= \int_0^t \int_0^m f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta m + \int_0^t c_{11}\Delta s + c_{12} \\
y_1^{\Delta}(t) &= \int_0^t \int_0^m \int_0^n f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta n\Delta m \\
&+ \int_0^t \int_0^m c_{11}\Delta s\Delta m + \int_0^t c_{12}\Delta s + c_{13} \\
y_1(t) &= \int_0^t \int_0^m \int_0^n \int_0^p f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&+ \int_0^t \int_0^m \int_0^n c_{11}\Delta s\Delta n\Delta m + \int_0^t \int_0^m c_{12}\Delta s\Delta m + \int_0^t c_{13}\Delta s + c_{14}
\end{aligned}$$

where c_{11} , c_{12} ,c_{13} and $c_{13}$are constants to be determined. By using the initial conditions (3.3),(3.4),(3.5) we get

$$y_1(0) = 0 \quad \Rightarrow \quad c_{14} = 0$$
$$y_1^{\Delta}(0) = 0 \quad \Rightarrow \quad c_{13} = 0$$
$$y_1^{\Delta\Delta}(0) = 0 \quad \Rightarrow \quad c_{12} = 0$$

Hence

$$\begin{aligned} y_1(t) &= \int_0^t\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\ &+ \int_0^t\int_0^m\int_0^n c_{11}\Delta s\Delta n\Delta m \end{aligned}$$

Case II Let $t\in\mathbb{T}_2$

$$\begin{aligned} y_2^{\Delta\Delta\Delta\Delta}(t) &= f_2(t,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta}) \\ y_2^{\Delta\Delta\Delta}(t) &= \int_{\sigma(a)}^t f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s + c_{21} \\ y_2^{\Delta\Delta}(t) &= \int_{\sigma(a)}^t\int_{\sigma(a)}^m f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta m + \int_{\sigma(a)}^t c_{21}\Delta s + c_{22} \\ y_2^{\Delta}(t) &= \int_{\sigma(a)}^t\int_{\sigma(a)}^m\int_{\sigma(a)}^n f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta n\Delta m \\ &+ \int_{\sigma(a)}^t\int_{\sigma(a)}^m c_{21}\Delta s\Delta m + \int_{\sigma(a)}^t c_{22}\Delta s + c_{23} \\ y_2(t) &= \int_{\sigma(a)}^t\int_{\sigma(a)}^m\int_{\sigma(a)}^n\int_{\sigma(a)}^p f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\ &+ \int_{\sigma(a)}^t\int_{\sigma(a)}^m\int_{\sigma(a)}^n c_{21}\Delta s\Delta n\Delta m \\ &+ \int_{\sigma(a)}^t\int_{\sigma(a)}^m c_{22}\Delta s\Delta m + \int_{\sigma(a)}^t c_{23}\Delta s + c_{24} \end{aligned}$$

where c_{21} , c_{22}, c_{23} and $c_{24}$are constants to be determined. Now, by (3.6), we have

$$c_{21} = -\int_{\sigma(a)}^{\rho^2(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s$$

Now, by (3.7), we get

$$\begin{aligned} \gamma_1 y_1(a) &= \gamma_2 y_2(\sigma(a)) \\ \Rightarrow c_{24} &= \frac{\gamma_1}{\gamma_2}\Big(\int_0^a\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\ &+ \int_0^a\int_0^m\int_0^n c_{11}\Delta s\Delta n\Delta m\Big). \end{aligned}$$

Also, by (3.8), we get

$$\begin{aligned} \gamma_3 y_1^{\Delta}(a) &= \gamma_4 y_2^{\Delta}(\sigma(a)) \\ \Rightarrow c_{23} &= \frac{\gamma_3}{\gamma_4}\Big(\int_0^a\int_0^m\int_0^n f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta n\Delta \\ &+ \int_0^a\int_0^m c_{11}\Delta s\Delta m\Big). \end{aligned}$$

Also,by (3.9),we get

$$\begin{aligned}\gamma_5 y_1^{\Delta\Delta}(a) &= \gamma_6 y_2^{\Delta\Delta}(\sigma(a))\\ \Rightarrow c_{22} &= \frac{\gamma_5}{\gamma_6}\Bigg(\int_0^a\int_0^m f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta m\\ &+ \int_0^a c_{11}\Delta s\Bigg).\end{aligned}$$

Also,by (3.10),we get

$$\begin{aligned}\gamma_7 y_1^{\Delta\Delta}(a) &= \gamma_8 y_2^{\Delta\Delta}(\sigma(a))\\ \Rightarrow c_{21} &= \frac{\gamma_7}{\gamma_8}\Bigg(\int_0^a f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s + c_{11}\Bigg).\\ \Rightarrow c_{11} &= \frac{\gamma_8}{\gamma_7}c_{21} - \int_0^a f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\end{aligned}$$

Hence,

$$\begin{aligned}y_1(t) &= \int_0^t\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\\ &- \frac{\gamma_8}{\gamma_7}\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\\ &- \int_o^t\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\end{aligned}$$

$$
\begin{aligned}
y_2(t) &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{\sigma(a)}^{n}\int_{\sigma(a)}^{p} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{\sigma(a)}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta m\Delta n \\
&+ \frac{\gamma_1}{\gamma_2}\int_{o}^{a}\int_{0}^{m}\int_{0}^{n}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_1}{\gamma_2}\frac{\gamma_8}{\gamma_7}\int_{0}^{a}\int_{0}^{m}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_1}{\gamma_2}\int_{0}^{a}\int_{0}^{m}\int_{0}^{n}\int_{0}^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&+ \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t}\int_{o}^{a}\int_{0}^{n}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t}\int_{0}^{a}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t}\int_{0}^{a}\int_{0}^{n}\int_{0}^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&+ \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{0}^{a}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{0}^{a}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m}\int_{0}^{a}\int_{0}^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m.
\end{aligned}
$$

We now define the integral operator $T : C(\mathbb{T}_1) \times C(\mathbb{T}_2) \to C(\mathbb{T}_1) \times C(\mathbb{T}_2)$.

$$
\begin{aligned}
T(y_1,y_2) &= \Bigg(\int_0^{t_1}\int_0^{m}\int_0^{n}\int_0^{p} f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_8}{\gamma_7}\int_o^{t_1}\int_0^{m}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \int_o^{t_1}\int_0^{m}\int_0^{n}\int_0^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m, \\
&\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{'} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m' \\
&- \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^2(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m' \\
&+ \frac{\gamma_1}{\gamma_2}\int_o^{a}\int_0^{m}\int_0^{n}\int_0^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_1}{\gamma_2}\frac{\gamma_8}{\gamma_7}\int_0^{a}\int_0^{m}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- \frac{\gamma_1}{\gamma_2}\int_0^{a}\int_0^{m}\int_0^{n}\int_0^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&+ \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_o^{a}\int_0^{n}\int_0^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
&- \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
&- \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_0^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
&+ \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^{a}\int_0^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
&- \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^{a}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
&- \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^{a}\int_0^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Bigg).
\end{aligned}
$$

where $t_1, m, r, d \in \mathbb{T}_1$ and $t_2, m', r', d' \in \mathbb{T}_2$. It is clear that (y_1, y_2) is a solution of BVP-SIP iff (y_1, y_2) solves the operator equation $(y_1, y_2) = T(y_1, y_2)$. In other words a fixed point for the operator $(y_1, y_2) = T(y_1, y_2)$ is a solution for the BVP-SIP.

We use Schauder's fixed point theorem to show the existence of a solution.

Claim 3.2. The space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is convex.

Let $(y_{11}, y_{12}), (y_{21}, y_{22}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$. For the space to be convex we need to show that

$$\alpha(y_{11}, y_{12}) + (1-\alpha)(y_{21}, y_{22}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2) \text{ for } \alpha < 1.$$

Since $(y_{11}, y_{12}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$(Definition(2.13)) for fixed $(t_{01}, t_{02}) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $\forall \epsilon > 0 \ \exists \ \delta > 0$ such that whenever

$$\begin{aligned}
\| \quad & (t_1, t_2) - (t_{01}, t_{02})\| < \delta (> 0) \\
i.e., \quad & |t_1 - t_{01}| + |t_2 - t_{02}| < \delta \\
\Rightarrow \quad & \|(y_{11}(t_1), y_{12}(t_2)) - (y_{11}(t_{01}), y_{12}(t_{02}))\| < \frac{\epsilon}{2\alpha} \\
i.e., \quad & sup_{t_1 \in \mathbb{T}_1} |y_{11}(t_1) - y_{11}(t_{01})| + sup_{t_2 \in \mathbb{T}_2} |y_{12}(t_2) - y_{12}(t_{02})| < \frac{\epsilon}{2\alpha} \quad (3.11)
\end{aligned}$$

Similarly we can show that

$$sup_{t_1 \in \mathbb{T}_1} |y_{21}(t_1) - y_{21}(t_{01})| + sup_{t_2 \in \mathbb{T}_2} |y_{22}(t_2) - y_{22}(t_{02})| < \frac{\epsilon}{2\alpha}. \quad (3.12)$$

Now, let us consider

$$\begin{aligned}
&\Big[\alpha(y_{11}, y_{12}) + (1-\alpha)(y_{21}, y_{22})\Big] \\
&\qquad = \Big[(\alpha y_{11}, \alpha y_{12}) + ((1-\alpha) y_{21}, (1-\alpha) y_{22})\Big] \\
&\qquad = \Big[(\alpha y_{11} + (1-\alpha) y_{21}, \alpha y_{12} + (1-\alpha) y_{22})\Big]
\end{aligned}$$

We see that $\|\{(\alpha y_{11} + (1-\alpha) y_{21})(t_1), (\alpha y_{12} + (1-\alpha) y_{22})(t_2)\}$
$-\{(\alpha y_{11} + (1-\alpha) y_{21})(t_{01}), (\alpha y_{12} + (1-\alpha) y_{22})(t_{02})\}\|$

$$\begin{aligned}
\leq \quad & sup_{t_1 \in \mathbb{T}_1} \Big|(\alpha y_{11} + (1-\alpha) y_{21})(t_1) - (\alpha y_{11} + (1-\alpha) y_{21})(t_{01})\Big| \\
+ \quad & sup_{t_2 \in \mathbb{T}_2} \Big|(\alpha y_{12} + (1-\alpha) y_{22})(t_2) - (\alpha y_{12} + (1-\alpha) y_{22})(t_{02})\Big| \\
\leq \quad & sup_{t_1 \in \mathbb{T}_1} \alpha |y_{11}(t_1) - y_{11}(t_{01})| + sup_{t_2 \in \mathbb{T}_2} \alpha |y_{12}(t_2) - y_{12}(t_{02})| \\
+ \quad & sup_{t_1 \in \mathbb{T}_1} (1-\alpha) |y_{21}(t_1) - y_{21}(t_{01})| + sup_{t_2 \in \mathbb{T}_2} (1-\alpha) |y_{22}(t_2) - y_{22}(t_{02})| \\
< \quad & \alpha \frac{\epsilon}{2\alpha} + (1-\alpha) \frac{\epsilon}{2(1-\alpha)} = \epsilon
\end{aligned}$$

whenever, $|t_1 - t_{01}| + |t_2 - t_{02}| < \delta$. Hence $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is convex.

Claim 3.3. T is a completely continuous map.

We first show that T is continuous. We prove it by showing that T preserves convergence.

Indeed let (y_{n1}, y_{n2}) be a sequence of functions in $\mathcal{C}(\mathbb{T}_1) \times \mathcal{C}(\mathbb{T}_2)$ such that

$$lim_{n \to \infty} \|(y_{n1}, y_{n2}) - (y_1, y_2)\| \to 0.$$

The above equation implies that

$$\begin{aligned} lim_{n\to\infty}\|(y_{n1}-y_1, y_{n2}-y_2)\| &\to 0\\ i.e., lim_{n\to\infty} sup_{t_1\in\mathbb{T}_1}|(y_{n1}-y_1)(t_1)| &\to 0\\ \text{and } lim_{n\to\infty} sup_{t_2\in\mathbb{T}_2}|(y_{n2}-y_2)(t_2)| &\to 0. \end{aligned}$$

Let us consider

$\|T(y_{n1}, y_{n2}) - T(y_1, y_2)\|$

$$\begin{aligned}
= &\ sup_{t_1\in\mathbb{T}_1}\Bigg|\Bigg(\int_0^{t_1}\int_0^m\int_0^n\int_0^p f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\\
- &\int_0^{t_1}\int_0^m\int_0^n\int_0^p f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg)\\
- &\ \frac{\gamma_8}{\gamma_7}\Bigg(\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s, y_{n2}, y_{n2}^{\Delta}, y_{n2}^{\Delta\Delta}, y_{n2}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\\
- &\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg)\\
- &\Bigg(\int_o^{t_1}\int_0^m\int_0^n\int_0^a |f_1(s, y_{n1}, y_{n1}^{\Delta}, y_{n1}^{\Delta\Delta}, y_{n1}^{\Delta\Delta\Delta}))\Delta s\Delta p\Delta n\Delta m\\
- &\int_o^{t_1}\int_0^m\int_0^n\int_0^a f_1(s, y_1, , y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\Bigg)\Bigg|\\
+ &\ sup_{t_2\in\mathbb{T}_2}\Bigg|\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s, y_{n2}, y_{n2}^{\Delta}, y_{n2}^{\Delta\Delta}, y_{n2}^{\Delta\Delta\Delta})\\
- &\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Bigg)
\end{aligned}$$

$$
\begin{aligned}
- & \Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^2(b)} f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m' \\
- & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^2(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Big) \\
- & \frac{\gamma_1}{\gamma_2}\Big(\int_0^a\int_0^m\int_0^n\int_0^a f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta}))\Delta s\Delta p\Delta n\Delta m \\
- & \int_0^a\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big) \\
- & \frac{\gamma_1}{\gamma_2}\frac{\gamma_8}{\gamma_7}\Big(\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
- & \int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big) \\
- & \frac{\gamma_1}{\gamma_2}\Big(\int_0^a\int_0^m\int_0^n\int_0^a f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
- & \int_0^a\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big) \\
+ & \frac{\gamma_3}{\gamma_4}\Big(\int_{\sigma(a)}^{t_2}\int_o^a\int_0^n\int_0^p f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
- & \int_{\sigma(a)}^{t_2}\int_o^a\int_0^n\int_0^p f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Big) \\
- & \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\Big(\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
- & \int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Big) \\
- & \frac{\gamma_3}{\gamma_4}\Big(\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_0^a f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
- & \int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Big) \\
+ & \frac{\gamma_5}{\gamma_6}\Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
- & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Big) \\
- & \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
- & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Big)
\end{aligned}
$$

$$
\begin{aligned}
- & \frac{\gamma_5}{\gamma_6}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_0^a f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Bigg)\Bigg|.
\end{aligned}
$$

Since (f_1,f_2) is continuous on $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$ we have

$$
\begin{aligned}
lim_{n\to\infty}|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})-f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})| &\to 0,\\
lim_{n\to\infty}|f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta})-f_1(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})| &\to 0.
\end{aligned}
$$

Now $\|T(y_{n1},y_{n2})-T(y_1,y_2)\|$

$$
\begin{aligned}
\leq\ & sup_{t_1\in\mathbb{T}_1}\Bigg(\int_0^{t_1}\int_0^m\int_0^n\int_0^p|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta}) \\
- & f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m \\
- & \frac{\gamma_8}{\gamma_7}\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta}) \\
- & f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m \\
- & \int_o^{t_1}\int_0^m\int_0^n\int_0^a|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta}) \\
- & f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\Bigg) \\
+\ & sup_{t_2\in\mathbb{T}_2}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{n^{'}}\int_{\sigma(a)}^{p^{'}}|f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta}) \\
- & f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p'\Delta n^{'}\Delta m^{'} \\
- & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{n^{'}}\int_{\sigma(a)}^{\rho^2(b)}|f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta}) \\
- & f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p^{'}\Delta n^{'}\Delta m^{'} \\
- & \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_0^a|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta}) \\
- & f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m \\
- & \frac{\gamma_1}{\gamma_2}\frac{\gamma_8}{\gamma_7}\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta}) \\
- & f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m \\
- & \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_0^a|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta}) \\
- & f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_{o}^{a}\int_{0}^{n}\int_{0}^{p}|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\\
&- \quad f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m^{'}\\
&- \quad \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_{0}^{a}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta})\\
&- \quad f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m^{'}\\
&- \quad \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_{0}^{a}\int_{0}^{n}\int_{0}^{a}|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\\
&- \quad f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m^{'}\\
&+ \quad \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{0}^{p}|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\\
&- \quad f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n^{'}\Delta m^{'}\\
&- \quad \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,y_{n2},y_{n2}^{\Delta},y_{n2}^{\Delta\Delta},y_{n2}^{\Delta\Delta\Delta})\\
&- \quad f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n^{'}\Delta m^{'}\\
&- \quad \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{0}^{a}|f_1(s,y_{n1},y_{n1}^{\Delta},y_{n1}^{\Delta\Delta},y_{n1}^{\Delta\Delta\Delta})\\
&- \quad f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n^{'}\Delta m^{'}\bigg).
\end{aligned}
$$

Hence, $lim_{n\to\infty}\|T(y_{n1},y_{n2})-T(y_1,y_2)\|\to 0$ proving that T is continuous.

Let

$$
\begin{aligned}
f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta}) &\leq M_1, \text{ for some } M_1>0, \forall s\in\mathbb{T}_1,\\
f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta}) &\leq M_2, \text{ for some } M_2>0, \forall s\in\mathbb{T}_2.
\end{aligned}
$$

We now show that $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is bounded and equicontinuous subset of $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$. Let us assume that $\|(y_1,y_2)\|\leq M$. Then

$$
\begin{aligned}
& T(y_1,y_2)\\
\leq\; & sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^{m}\int_0^{n}\int_0^{p}|f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\\
+\; & sup_{t_1\in\mathbb{T}_1}\frac{\gamma_8}{\gamma_7}\int_o^{t_1}\int_0^{m}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\\
+\; & sup_{t_1\in\mathbb{T}_1}\int_o^{t_1}\int_0^{m}\int_0^{n}\int_0^{a}|f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m,\\
+\; & sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{'}|f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p'\Delta n'\Delta m'\\
+\; & sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^2(b)}|f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p'\Delta n'\Delta m'\\
+\; & \frac{\gamma_1}{\gamma_2}\int_o^{a}\int_0^{m}\int_0^{n}\int_0^{p}|f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\\
+\; & \frac{\gamma_1}{\gamma_2}\frac{\gamma_8}{\gamma_7}\int_0^{a}\int_0^{m}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\\
+\; & \frac{\gamma_1}{\gamma_2}\int_0^{a}\int_0^{m}\int_0^{n}\int_0^{a}f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\\
+\; & sup_{t_2\in\mathbb{T}_2}\frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_o^{a}\int_0^{n}\int_0^{p}|f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m'\\
+\; & sup_{t_2\in\mathbb{T}_2}\frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m'\\
+\; & sup_{t_2\in\mathbb{T}_2}\frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_0^{a}|f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m'\\
+\; & sup_{t_2\in\mathbb{T}_2}\frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^{a}\int_0^{p}|f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n'\Delta m'\\
+\; & sup_{t_2\in\mathbb{T}_2}\frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^{a}\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n'\Delta m'\\
+\; & sup_{t_2\in\mathbb{T}_2}\frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^{a}\int_0^{a}|f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n'\Delta m'.
\end{aligned}
$$

Since (f_1,f_2) is bounded we can conclude that there exists a $K>0$ independent of choice of (y_1,y_2) such that $\|T(y_1,y_2)\|\leq K$. Hence, $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is bounded.

We next prove that $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is equicontinuous subset of $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$. We need to show that $\forall\epsilon>0\ \exists\ \delta>0$ such that whenever

$$\begin{aligned} \| \quad & (t_1,t_2)-(t_1{}',t_2{}')\| < \delta \\ \Rightarrow \quad & \|T(y_1(t_1),y_2(t_2))-T(y_1(t_1{}'),y_2(t_2'))\| < \epsilon. \end{aligned}$$

Let us assume that $|t_1-t_1{}'|+|t_2-t_2{}'|<\delta$. We see that

$T(y_1(t_1),y_2(t_2))-T(y_1(t_1{}'),y_2(t_2'))$

$$\begin{aligned} = \quad & \Bigg(\Big(\int_0^{t_1}\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\ & -\int_0^{t_1{}'}\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big) \\ - \quad & \frac{\gamma_8}{\gamma_7}\Big(\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\ & -\int_o^{t_1{}'}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big) \\ - \quad & \Big(\int_o^{t_1}\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\ & -\int_o^{t_1{}'}\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big), \end{aligned}$$

$$
\begin{aligned}
& \Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{'} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{'} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Big) \\
- & \Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^2(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^2(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Big) \\
+ & \frac{\gamma_1}{\gamma_2}\Big(\int_{o}^{a}\int_{0}^{m}\int_{0}^{n}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
& -\int_{o}^{a}\int_{0}^{m}\int_{0}^{n}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big) \\
- & \frac{\gamma_1}{\gamma_2}\frac{\gamma_8}{\gamma_7}\Big(\int_{0}^{a}\int_{0}^{m}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
& -\int_{0}^{a}\int_{0}^{m}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big) \\
- & \frac{\gamma_1}{\gamma_2}\Big(\int_{0}^{a}\int_{0}^{m}\int_{0}^{n}\int_{0}^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
& -\int_{0}^{a}\int_{0}^{m}\int_{0}^{n}\int_{0}^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big) \\
+ & \frac{\gamma_3}{\gamma_4}\Big(\int_{\sigma(a)}^{t_2}\int_{o}^{a}\int_{0}^{n}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{o}^{a}\int_{0}^{n}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Big) \\
- & \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\Big(\int_{\sigma(a)}^{t_2}\int_{0}^{a}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{0}^{a}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Big) \\
- & \frac{\gamma_3}{\gamma_4}\Big(\int_{\sigma(a)}^{t_2}\int_{0}^{a}\int_{0}^{n}\int_{0}^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{0}^{a}\int_{0}^{n}\int_{0}^{a} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Big) \\
+ & \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\Big(\int_{\sigma(a)}^{m'}\int_{0}^{a}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{0}^{a}\int_{0}^{p} f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Big)
\end{aligned}
$$

$$
\begin{aligned}
- & \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
& -\int_{\sigma(a)}^{t_2{}'}\int_{\sigma(a)}^{m'}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Bigg) \\
- & \frac{\gamma_5}{\gamma_6}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
& -\sigma(a)^{t_2{}'}\int_{\sigma(a)}^{m'}\int_0^a\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Bigg)\Bigg).
\end{aligned}
$$

We consider each term separately

Now

$$
\begin{aligned}
&\int_0^{t_1}\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&-\int_0^{t_1'}\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \int_0^{t_1}\int_0^m\int_0^n\int_0^p M_1\Delta s\Delta p\Delta n\Delta m-\int_0^{t_1'}\int_0^m\int_0^n\int_0^p M_1\Delta s\Delta p\Delta n\Delta m \\
=\ & \int_0^{t_1}\int_0^m\int_0^n M_1 p\Delta p\Delta n\Delta m-\int_0^{t_1'}\int_0^m\int_0^n M_1 p\Delta p\Delta n\Delta m \\
=\ & \int_0^{t_1}\int_0^m M_1\frac{n^2}{2}\Delta n\Delta m-\int_0^{t_1'}\int_0^m M_1\frac{n^2}{2}\Delta n\Delta m \\
=\ & \int_0^{t_1} M_1\frac{m^3}{6}\Delta m-\int_0^{t_1'} M_1\frac{m^3}{6}\Delta m \\
=\ & \frac{M_1}{24}\left(t_1^4-t_1'^4\right) \\
=\ & \frac{M_1}{24}\left((t_1^2-t_1'^2)(t_1^2+t_1'^2)\right) \\
.\ =\ & \frac{M_1}{24}(t_1-t_1')\left((t_1+t_1').(t_1^2+t_1'^2)\right)
\end{aligned}
$$

Hence, whenever $|t_1-t_1'|<\delta$ we have

$$
\begin{aligned}
&\Bigg|\int_0^{t_1}\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&-\int_0^{t_1'}\int_0^m\int_0^n\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg|<\frac{\epsilon}{11}.
\end{aligned}
$$

Now,

$$\frac{\gamma_8}{\gamma_7}\bigg(\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m$$
$$-\int_o^{{t_1}'}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\bigg)$$

$$\begin{aligned}
&\leq \frac{\gamma_8}{\gamma_7}\bigg(\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} M_2\Delta s\Delta p\Delta n\Delta m \\
&\quad -\int_o^{{t_1}'}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} M_2\Delta s\Delta p\Delta n\Delta m\bigg) \\
&\leq M_2\frac{\gamma_8}{\gamma_7}\bigg(\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} \Delta s\Delta p\Delta n\Delta m \\
&\quad -\int_o^{{t_1}'}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} \Delta s\Delta p\Delta n\Delta m\bigg) \\
&= M_2\frac{\gamma_8}{\gamma_7}\bigg(\rho^3(b)-\sigma(a)\bigg)\bigg(\int_o^{t_1}\int_0^m\int_0^n \Delta p\Delta n\Delta m \\
&\quad -\int_o^{{t_1}'}\int_0^m\int_0^n \Delta p\Delta n\Delta m\bigg) \\
&= \frac{M_2}{6}\frac{\gamma_8}{\gamma_7}\bigg(\rho^3(b)-\sigma(a)\bigg)\bigg((t_1-t_1^{'})\bigg)\bigg(t_1^2+t_1{t_1}'+{t_1}'^2\bigg).
\end{aligned}$$

Hence , whenever $|t_1-t_1^{'}|<\delta$ we have

$$\bigg|\frac{\gamma_8}{\gamma_7}\bigg(\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m$$
$$-\int_o^{{t_1}'}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\bigg)\bigg|<\frac{\epsilon}{11}.$$

Now,

$$\bigg|\int_o^{t_1}\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m$$
$$-\int_o^{{t_1}'}\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\bigg|$$

$$\begin{aligned}
&\leq && M_1\Big|\int_o^{t_1}\int_0^m\int_0^n\int_0^a \Delta s\Delta p\Delta n\Delta m-\int_o^{{t_1}'}\int_0^m\int_0^n\int_0^a \Delta s\Delta p\Delta n\Delta m\Big| \\
&= && M_1 a\Big|\frac{t_1^3}{6}-\frac{{t_1}'^3}{6}\Big| \\
&= && M_1\frac{a}{6}|t_1^2+{t_1}'^2+t_1{t_1}'||t_1-{t_1}'|
\end{aligned}$$

Hence , whenever $|t_1-t_1'|<\delta$ we have

$$\Big|\int_o^{t_1}\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m$$
$$-\int_o^{{t_1}'}\int_0^m\int_0^n\int_0^a f_1(s,y_1,,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Big|<\frac{\epsilon}{11}$$

Also,

$$\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'$$
$$-\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'$$

$$\begin{aligned}
&\leq && \Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} M_2\Delta s\Delta p'\Delta n'\Delta m' \\
& && -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} M_2\Delta s\Delta p'\Delta n'\Delta m'\Big) \\
&= && \Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'} M_2(p'-\sigma(a))\Delta p'\Delta n'\Delta m' \\
& && -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'} M_2(p'-\sigma(a))\Delta p'\Delta n'\Delta m'\Big) \\
&= && \Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} M_2\Big(\frac{n'^2}{2}-\sigma(a)n'+\frac{\sigma(a)^2}{2}\Big)\Delta n'\Delta m'
\end{aligned}$$

$$
\begin{aligned}
& -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'} M_2\Big(\frac{n'^2}{2}-\sigma(a)n'+\frac{\sigma(a)^2}{2}\Big)\Delta n'\Delta m'\Big) \\
= \; & \Big(\int_{\sigma(a)}^{t_2} M_2\Big(\frac{m'^3}{6}+\frac{\sigma(a)^2m'}{2}-\frac{\sigma(a)m'^2}{2}-\frac{\sigma(a)^3}{6}\Big)\Delta m' \\
& -\int_{\sigma(a)}^{t_2'} M_2\Big(\frac{m'^3}{6}+\frac{\sigma(a)^2m'}{2}-\frac{\sigma(a)m'^2}{2}-\frac{\sigma(a)^3}{6}\Big)\Delta m'\Big) \\
= \; & M_2\Big|\Big(\frac{t_2^2+t_2'^2}{24}\Big)\Big(t_2+t_2'\Big)+\frac{\sigma(a)^2}{4}\Big((t_2+t_2')\Big) \\
& -\frac{\sigma(a)}{6}\Big(t_2^2+t_2'^2+t_1t_2'\Big)\Big||t_2-t_2'|
\end{aligned}
$$

Hence, whenever $|t_2-t_2'|<\delta$ we have

$$
\begin{aligned}
& \Big|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta'\Delta n'\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Big|<\frac{\epsilon}{11}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
& \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'
\end{aligned}
$$

$$
\begin{aligned}
\leq \; & \Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)} M_2\Delta s\Delta p'\Delta n'\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)} M_2\Delta s\Delta p'\Delta n'\Delta m'\Big) \\
= \; & \Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'} M_2\Big(\rho^3(b)-\sigma(a)\Big)\Delta p'\Delta n'\Delta m' \\
& -\int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'} M_2(\rho^3(b)-\sigma(a))\Delta p'\Delta n'\Delta m'\Big) \\
= \; & M_2(\rho^3(b)-\sigma(a))(t_2-t_2')\Big(\frac{1}{6}(t_2+t_2'+t_2t_2')-\frac{\sigma(a)}{2}(t_2+t_2')+\frac{\sigma(a)^2}{2}\Big)
\end{aligned}
$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$ we have

$$\Big|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{n^{'}}\int_{\sigma(a)}^{\rho^3(b)} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta p^{'}\Delta n^{'}\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m^{'}}\int_{\sigma(a)}^{n^{'}}\int_{\sigma(a)}^{\rho^3(b)} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta})\Delta s\Delta p^{'}\Delta n^{'}\Delta m^{'}\Big| < \frac{\epsilon}{11}.$$

Now,

$$\frac{\gamma_3}{\gamma_4}\Big(\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_0^p f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_0^a\int_0^n\int_0^p f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\Big)$$

$$\begin{aligned}
&= \frac{\gamma_3}{\gamma_4}M_1\Big(\int_{\sigma(a)}^{t_2}\int_0^a \frac{n^2}{2}\Delta n\Delta m^{'} - \int_{\sigma(a)}^{t_2^{'}}\int_0^a \frac{n^2}{2}\Delta n\Delta m^{'}\Big)\\
&= \frac{\gamma_3}{\gamma_4}M_1\frac{a^3}{6}\Big(\int_{\sigma(a)}^{t_2}\Delta m^{'} - \int_{\sigma(a)}^{t_2^{'}}\Delta m^{'}\Big)\\
&= \frac{\gamma_3}{\gamma_4}M_1\frac{a^3}{6}\Big(t_2 - \sigma(a) - (t_2^{'} - \sigma(a))\Big)\\
&= \frac{\gamma_3}{\gamma_4}M_1\frac{a^3}{6}(t_2 - t_2^{'})
\end{aligned}$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$ we have

$$\frac{\gamma_3}{\gamma_4}\Big|\int_{\sigma(a)}^{t_2}\int_0^a\int_0^{n^{'}}\int_0^{p^{'}} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta p^{'}\Delta n^{'}\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_0^a\int_0^{n^{'}}\int_0^{p^{'}} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta p^{'}\Delta n^{'}\Delta m^{'}\Big| < \frac{\epsilon}{11}.$$

Now

$$\frac{\gamma_5}{\gamma_6}\Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_0^p f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_0 m^{'} f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Big)$$

$$
\begin{aligned}
&= \frac{\gamma_5}{\gamma_6} M_1 \Big(\int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{m'} \int_0^a p\Delta p \Delta n^{'} \Delta m^{'} - \int_{\sigma(a)}^{t_2^{'}} \int_{\sigma(a)}^{m'} \int_0^a p\Delta p \Delta n^{'} \Delta m^{'} \Big) \\
&= \frac{\gamma_5}{\gamma_6} M_1 \frac{a^2}{2} \Big(\frac{t_2^2}{2} - \frac{t_2'^2}{2} - \sigma(a)(t_2 - t_2{}') \Big) \\
&= \frac{\gamma_5}{\gamma_6} M_1 \frac{a^2}{2} (t_2 - t_2{}') \Big(\frac{t_2 + t_2{}'}{2} - \sigma(a) \Big)
\end{aligned}
$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$ we have

$$
\begin{aligned}
&\Big| \frac{\gamma_5}{\gamma_6} \Big(\int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{m'} \int_0^a \int_0^p f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) \Delta s \Delta p \Delta n^{'} \Delta m^{'} \\
&- \int_{\sigma(a)}^{t_2^{'}} \int_{\sigma(a)}^{m'} \int_0^a \int_0 m' f_1(s, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) \Delta s \Delta p \Delta n^{'} \Delta m^{'} \Big) \Big| < \frac{\epsilon}{11}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\frac{\gamma_3}{\gamma_4} \frac{\gamma_8}{\gamma_7} \Big(\int_{\sigma(a)}^{t_2} \int_0^a \int_0^n \int_{\sigma(a)}^{\rho^3(b)} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) \Delta s \Delta p \Delta n \Delta m^{'} \\
&- \int_{\sigma(a)}^{t_2^{'}} \int_0^a \int_0^n \int_{\sigma(a)}^{\rho^3(b)} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) \Delta s \Delta p \Delta n \Delta m^{'} \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\gamma_3}{\gamma_4} \frac{\gamma_8}{\gamma_7} M_2 \Big(\rho^3(b) - \sigma(a) \Big) \Big(\int_{\sigma(a)}^{t_2} \int_0^a n \Delta n \Delta m^{'} \\
&\quad - \int_{\sigma(a)}^{t_2'} \int_0^a n \Delta n \Delta m^{'} \Big) \\
&= \frac{\gamma_3}{\gamma_4} \frac{\gamma_8}{\gamma_7} M_2 \frac{a^2}{2} \Big(\rho^3(b) - \sigma(a) \Big) \Big(\int_{\sigma(a)}^{t_2} \Delta m^{'} - \int_{\sigma(a)}^{t_2'} \Delta m^{'} \Big) \\
&= \frac{\gamma_3}{\gamma_4} \frac{\gamma_8}{\gamma_7} M_2 \Big(\rho^3(b) - \sigma(a) \Big) \frac{a^2}{2} \Big((t_2 - \sigma(a) - (t_2^{'} - \sigma(a))) \Big) \\
&= \frac{\gamma_3}{\gamma_4} \frac{\gamma_8}{\gamma_7} M_2 \Big(\rho^3(b) - \sigma(a) \Big) \frac{a^2}{2} (t_2 - t_2^{'})
\end{aligned}
$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$ we have

$$
\begin{aligned}
&\frac{\gamma_3}{\gamma_4} \frac{\gamma_8}{\gamma_7} \Big| \int_{\sigma(a)}^{t_2} \int_0^a \int_0^{n'} \int_{\sigma(a)}^{\rho^3(b)} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) \Delta s \Delta p^{'} \Delta n^{'} \Delta m^{'} \\
&- \int_{\sigma(a)}^{t_2^{'}} \int_0^a \int_0^{n'} \int_{\sigma(a)}^{\rho^3(b)} f_2(s, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) \Delta s \Delta p' \Delta n^{'} \Delta m^{'} \Big| < \frac{\epsilon}{11}.
\end{aligned}
$$

Now

$$\frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,y_2,y_2^{\Delta},y_2^{\Delta\Delta},y_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Bigg)$$

$$= \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}M_2\Big(\rho^3(b)-\sigma(a)\Big)\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a \Delta p\Delta n^{'}\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_0^a \Delta p\Delta n^{'}\Delta m^{'}\Bigg)$$
$$= \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}M_2a\Big(\rho^3(b)-\sigma(a)\Big)\Bigg(\frac{t_2^2}{2}-\frac{t_2'^2}{2}-\sigma(a)(t_2-t_2')\Bigg)$$
$$= \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}M_2a\Big(\rho^3(b)-\sigma(a)\Big)(t_2-t_2')\Bigg(\frac{t_2+t_2'}{2}-\sigma(a)\Bigg)$$

Hence, whenever $|t_2-t_2^{'}|<\delta$ we have

$$\Bigg|\frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_0^a\int_0 m' f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Bigg)\Bigg|<\frac{\epsilon}{11}.$$

Now,

$$\frac{\gamma_3}{\gamma_4}\Bigg(\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_0^a f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_0^a\int_0^n\int_0^a f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\Bigg)$$
$$= \frac{\gamma_3}{\gamma_4}M_1\frac{a^3}{2}(t_2-t_2^{'})$$

Hence, whenever $|t_2-t_2^{'}|<\delta$ we have

$$\frac{\gamma_3}{\gamma_4}\Bigg|\int_{\sigma(a)}^{t_2}\int_0^a\int_0^{n'}\int_0^{p'} f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p^{'}\Delta n^{'}\Delta m^{'}$$
$$-\int_{\sigma(a)}^{t_2^{'}}\int_0^a\int_0^{n'}\int_0^{p'} f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n^{'}\Delta m^{'}\Bigg|<\frac{\epsilon}{11}.$$

Now

$$\frac{\gamma_5}{\gamma_6}\Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^a f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}$$

$$-\int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_0^a\int_0 m' f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Big)$$

$$\begin{aligned}
&= \frac{\gamma_5}{\gamma_6}M_1 a\Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a \Delta p\Delta n^{'}\Delta m^{'} - \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_0^a \Delta p\Delta n^{'}\Delta m^{'}\Big)\\
&= \frac{\gamma_5}{\gamma_6}M_1 a^2\Big(\frac{t_2^2}{2}-\frac{t_2'^2}{2}-\sigma(a)(t_2-t_2')\Big)\\
&= \frac{\gamma_5}{\gamma_6}M_1 a^2(t_2-t_2')\Big(\frac{t_2+t_2'}{2}-\sigma(a)\Big)
\end{aligned}$$

Hence, whenever $|t_2-t_2^{'}|<\delta$ we have

$$\Big|\frac{\gamma_5}{\gamma_6}\Big(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}$$

$$-\int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'}\int_0^a\int_0 m' f_1(s,y_1,y_1^{\Delta},y_1^{\Delta\Delta},y_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Big)\Big|<\frac{\epsilon}{11}.$$

So, we see that

$$\begin{aligned}
\|T(y_1(t_1),y_2(t_2))-T(y_1(t_1^{'}),y_2(t_2^{'}))\| &< \frac{\epsilon}{11}+\frac{\epsilon}{11}+\frac{\epsilon}{11}+\frac{\epsilon}{11}+\frac{\epsilon}{11}+\frac{\epsilon}{11}\\
&+ \frac{\epsilon}{11}+\frac{\epsilon}{11}+\frac{\epsilon}{11}+\frac{\epsilon}{11}+\frac{\epsilon}{11}=\epsilon
\end{aligned}$$

whenever

$$\begin{aligned}
\| &\quad (t_1,t_2)-(t_1^{'},t_2^{'})\|<\delta\\
i.e., &\quad |t_1-t_1^{'}|+|t_2-t_2^{'}|<\delta.
\end{aligned}$$

So, $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is equicontinuous subset of $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$.

Condition(iii) of Arzela-Acoli theorem can be seen from that fact that any $g_i(s,z_i)\in\mathcal{C}(\mathbb{T}_i\times\mathcal{C}(\mathbb{T}_i)),\ i=1,2,$ is uniformly continuous on $\mathbb{T}_i$ as $\mathbb{T}_i$ is compact(since closed and bounded).

Thus T is compact by Arzela-Ascoli theorem. So from Schauder's fixed point theorem(2.6) a fixed point exists for the operator equation $(y_1,y_2)=Ty$. Hence a solution exists for the BVP-SIP. □

3.3. Existence and Uniqueness of Solution using Banach's Fixed Point Theorem

In this section, we prove the existence of a unique solution of BVP-SIP with certain restricted conditions on the interface constants. We use the Banach contraction principle.

Theorem 3.4. *Let* $\int_{\mathbb{T}_i} f_i \Delta t < \infty$ *and* $f_i,\ i = 1, 2$ *satisfy*

$$\begin{aligned} |f_1(t, y_1, y_1^{\Delta}, y_1^{\Delta\Delta}, y_1^{\Delta\Delta\Delta}) - f_1(t, z_1, z_1^{\Delta}, z_1^{\Delta\Delta}, z_1^{\Delta\Delta\Delta})| &\leq |y_1 - z_1| \text{ for all } t \in \mathbb{T}_1 \\ y_1(t), z_1(t) \in \mathbb{R} & \qquad (3.13) \\ |f_2(t, y_2, y_2^{\Delta}, y_2^{\Delta\Delta}, y_2^{\Delta\Delta\Delta}) - f_2(t, z_2, z_2^{\Delta}, z_2^{\Delta\Delta}, z_2^{\Delta\Delta\Delta})| &\leq |y_2 - z_2| \text{ for all } t \in \mathbb{T}_2 \\ y_2(t), z_2(t) \in \mathbb{R} & \qquad (3.14) \end{aligned}$$

and

$$\begin{aligned} & \frac{a^4}{24} + \frac{a^4}{6} + \frac{\gamma_3}{\gamma_4}\frac{a^3}{6}\Big(\rho^3(b) - \sigma(a)\Big) + \frac{\gamma_5}{\gamma_6}\Big(\frac{a^2\rho^3(b)^2}{4} - \frac{a^2\sigma(a)\rho^3(b)}{2} + \frac{a^2\sigma(a)^2}{4}\Big) \\ + & \frac{\gamma_3}{\gamma_4}\frac{a^3}{2}\Big(\rho^3(b) - \sigma(a)\Big) + \frac{\gamma_5}{\gamma_6}\Big(\frac{a^2\rho^3(b)^2}{2} - a^2\sigma(a)\rho^3(b) + \frac{a^2\sigma(a)^2}{2}\Big) \\ + & \frac{\gamma_3}{\gamma_4}\frac{a^3}{6}\Big(\rho^3(b) - \sigma(a)\Big) + \frac{\gamma_1}{\gamma_2}\frac{a^4}{6} < 1 \qquad (3.15) \\ & \frac{a^3}{6}\frac{\gamma_8}{\gamma_7}\Big(\rho^3(b) - \sigma(a)\Big) \\ + & \Big(\frac{\rho^3(b)^4}{24} - \frac{\sigma(a)\rho^3(b)^3}{6} + \frac{\sigma(a)^2\rho^3(b)^2}{4} - \frac{\sigma(a)^3\rho^3(b)}{6} + \frac{\sigma(a)^4}{24}\Big) \\ + & \Big(\rho^3(b) - \sigma(a)\Big)\Big(\frac{\rho^3(b)^3}{6} - \frac{\sigma(a)\rho^3(b)^2}{2} + \frac{\sigma(a)^2\rho^3(b)}{2} - \frac{\sigma(a)^3}{6}\Big) \\ + & \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\frac{a^2}{2}\Big(\rho^3(b) - \sigma(a)\Big)\Big(\rho^3(b) - \sigma(a)\Big) \\ + & \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}a\Big(\rho^3(b) - \sigma(a)\Big)\Big(\frac{\rho^3(b)^2}{2} - \sigma(a)\rho^3(b) + \frac{\sigma(a)^2}{2}\Big) \\ + & \frac{\gamma_1}{\gamma_2}\Big(\rho^3(b) - \sigma(a)\Big)\frac{a^3}{6} < 1 \qquad (3.16) \end{aligned}$$

Then the BVP-SIP has a unique solution.

Proof. We use Banach Contraction Mapping Theorem (2.5) to prove the existence of unique fixed point for the operator equation.

Claim 3.5. The space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is complete.

Let (x_{n1}, x_{n2}) be a cauchy sequence in $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ and let $\epsilon > 0$ be given. Let $(t_{01}, t_{o2}) \in \mathbb{T}_1 \times \mathbb{T}_2$ be fixed. From the definition of continuity (2.13) for $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we have

$$\|(t_1, t_2) - (t_{01}, t_{o2})\| < \delta(> 0)$$

implies that

$$\|(x_{n1}(t_1), x_{n2}(t_2)) - (x_{n1}(t_{01}), x_{n2}(t_{02}))\| < \frac{\epsilon}{3}.$$

i.e.,

$$\begin{aligned} \| & (t_1 - t_{01}, t_2 - t_{02})\| \\ = & |t_1 - t_{01}| + |t_2 - t_{02}| < \delta \end{aligned}$$

$$\begin{aligned} \Rightarrow & \|(x_{n1}(t_1) - x_{n1}(t_{01}), x_{n2}(t_2) - x_{n2}(t_{02}))\| \\ = & sup_{t_1 \in \mathbb{T}_1} |x_{n1}(t_1) - x_{n1}(t_{01})| \\ + & sup_{t_2 \in \mathbb{T}_2} |x_{n2}(t_2) - x_{n2}(t_{02})| < \frac{\epsilon}{3}. \end{aligned} \tag{3.17}$$

Now we let $lim_{n\to\infty}(x_{n1}, x_{n2}) \to (x_1, x_2)$. Then $\exists\, \mathcal{N} > 0$ such that $\forall n1, n2 > \mathcal{N}$ we have

$$\begin{aligned} \| & (x_{n1}, x_{n2}) - (x_1, x_2)\| \\ = & \|(x_{n1} - x_1, x_{n2} - x_{n2})\| \\ = & sup_{t_1 \in \mathbb{T}_1} |(x_{n1} - x_1)(t_1)| \\ + & sup_{t_2 \in \mathbb{T}_2} |(x_{n2} - x_2)(t_2)| < \frac{\epsilon}{3}. \end{aligned} \tag{3.18}$$

For the space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ to be complete we need to show that (x_1, x_2) belongs to the space. That is

$$\begin{aligned} \| & (t_1, t_2) - (t_{01}, t_{02})\| < \delta \\ \Rightarrow & \|(x_1(t_1), x_2(t_2)) - (x_1(t_{01}), x_2(t_{02}))\| < \epsilon. \end{aligned}$$

Now let us consider $\|(x_1(t_1), x_2(t_2)) - (x_1(t_{01}), x_2(t_{02}))\|$

$$\begin{aligned} = & \|(x_1(t_1) - x_1(t_{01}), x_2(t_2) - x_2(t_{02}))\| \\ = & sup_{t_1 \in \mathbb{T}_1} |x_1(t_1) - x_1(t_{01})| + sup_{t_2 \in \mathbb{T}_2} |x_2(t_2) - x_2(t_{02})| \\ = & sup_{t_1 \in \mathbb{T}_1} |x_1(t_1) - x_{n1}(t_1) + x_{n1}(t_1) - x_{n1}(t_{01}) + x_{n1}(t_{01}) - x_1(t_{01})| \\ + & sup_{t_2 \in \mathbb{T}_2} |x_2(t_2) - x_{n2}(t_2) + x_{n2}(t_2) - x_{n2}(t_{02}) + x_{n2}(t_{02}) - x_2(t_{02})| \\ \leq & sup_{t_1 \in \mathbb{T}_1} |x_1(t_1) - x_{n1}(t_1)| + sup_{t_2 \in \mathbb{T}_2} |x_2(t_2) - x_{n2}(t_2)| \\ + & sup_{t_1 \in \mathbb{T}_1} |x_{n1}(t_1) - x_{n1}(t_{01})| + sup_{t_2 \in \mathbb{T}_2} |x_{n2}(t_2) - x_{n2}(t_{02})| \\ + & sup_{t_1 \in \mathbb{T}_1} |(x_{n1} - x_1)(t_{01})| + sup_{t_2 \in \mathbb{T}_2} |(x_{n2} - x_2)(t_{02})| \\ < & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \text{ (since (3.17), (3.18))} \\ = & \epsilon. \end{aligned}$$

Hence, $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is complete.

Claim 3.6. The map T is a contraction.

We see that

$$\|T(x_1,x_2)-T(z_1,z_2)\|$$

$$\begin{aligned}
= \quad & sup_{t_1\in\mathbb{T}_1}\Bigg|\Bigg(\int_0^{t_1}\int_0^m\int_0^n\int_0^p f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
- \quad & \int_0^{t_1}\int_0^m\int_0^n\int_0^p f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg) \\
- \quad & \frac{\gamma_8}{\gamma_7}\Bigg(\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
- \quad & \int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg) \\
- \quad & \Bigg(\int_o^{t_1}\int_0^m\int_0^n\int_0^a |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta}))\Delta s\Delta p\Delta n\Delta m \\
- \quad & \int_o^{t_1}\int_0^m\int_0^n\int_0^a f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\Bigg)\Bigg| \\
+ \quad & sup_{t_2\in\mathbb{T}_2}\Bigg|\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta}) \\
- \quad & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Bigg) \\
- \quad & \Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^2(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m' \\
- \quad & \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^2(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Bigg) \\
- \quad & \frac{\gamma_1}{\gamma_2}\Bigg(\int_0^a\int_0^m\int_0^n\int_0^a f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta}))\Delta s\Delta p\Delta n\Delta m \\
- \quad & \int_0^a\int_0^m\int_0^n\int_0^a f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg) \\
- \quad & \frac{\gamma_1}{\gamma_2}\frac{\gamma_8}{\gamma_7}\Bigg(\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
- \quad & \int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg) \\
- \quad & \frac{\gamma_1}{\gamma_2}\Bigg(\int_0^a\int_0^m\int_0^n\int_0^a f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
- \quad & \int_0^a\int_0^m\int_0^n\int_0^a f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\gamma_3}{\gamma_4}\Bigg(\int_{\sigma(a)}^{t_2}\int_{o}^{a}\int_{0}^{n}\int_{0}^{p} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\\
&- \int_{\sigma(a)}^{t_2}\int_{o}^{a}\int_{0}^{n}\int_{0}^{p} f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\Bigg)\\
&- \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\Bigg(\int_{\sigma(a)}^{t_2}\int_{0}^{a}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\\
&- \int_{\sigma(a)}^{t_2}\int_{0}^{a}\int_{0}^{n}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\Bigg)\\
&- \frac{\gamma_3}{\gamma_4}\Bigg(\int_{\sigma(a)}^{t_2}\int_{0}^{a}\int_{0}^{n}\int_{0}^{a} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\\
&- \int_{\sigma(a)}^{t_2}\int_{0}^{a}\int_{0}^{n}\int_{0}^{a} f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\Bigg)\\
&+ \frac{\gamma_5}{\gamma_6}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{0}^{p} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\\
&- \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{0}^{p} f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Bigg)\\
&- \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\\
&- \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Bigg)\\
&- \frac{\gamma_5}{\gamma_6}\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{0}^{a} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\\
&- \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_{0}^{a}\int_{0}^{a} f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Bigg)\Bigg|.
\end{aligned}
$$

Now let us consider each of the term in the above equation separately.

Now

$$
\begin{aligned}
&sup_{t_1\in\mathbb{T}_1}\Bigg|\int_{0}^{t_1}\int_{0}^{m}\int_{0}^{n}\int_{0}^{p} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\\
&-\int_{0}^{t_1}\int_{0}^{m}\int_{0}^{n}\int_{0}^{p} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg)\Bigg|
\end{aligned}
$$

$$\begin{aligned}
&\leq && sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^{m}\int_0^{n}\int_0^{p}|f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta}) \\
& && -f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m \\
&= && sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^{m}\int_0^{n}|x_1-z_1|p\Delta p\Delta n\Delta m \text{ (since (3.13))} \\
&= && sup_{t_1\in\mathbb{T}_1}sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\int_0^{t_1}\int_0^{m}\frac{n^2}{2}\Delta n\Delta m \\
&= && sup_{t_1\in\mathbb{T}_1}sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\int_0^{t_1}\frac{m^3}{6}\Delta m \\
&= && sup_{t_1\in\mathbb{T}_1}sup_{t_1\in\mathbb{T}_1}sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{t_1^4}{24}
\end{aligned}$$

So we have

$$\begin{aligned}
& && sup_{t_1\in\mathbb{T}_1}\bigg|\int_0^{t_1}\int_0^{m}\int_0^{n}\int_0^{p}f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- && \int_0^{t_1}\int_0^{m}\int_0^{n}\int_0^{p}f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\bigg)\bigg| \\
&\leq && sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\ sup_{t_1\in\mathbb{T}_1}\frac{{t_1}^4}{24}. \qquad (3.19)
\end{aligned}$$

Now,

$$\begin{aligned}
& && sup_{t_1\in\mathbb{T}_1}\frac{\gamma_8}{\gamma_7}\bigg(\int_o^{t_1}\int_0^{m}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)}f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
&- && \int_o^{t_1}\int_0^{m}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)}f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\bigg)\bigg|
\end{aligned}$$

$$\begin{aligned}
&\leq && \frac{\gamma_8}{\gamma_7}\bigg(\int_o^{t_1}\int_0^{m}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta}) \\
& && -f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\bigg) \\
&\leq && \frac{\gamma_8}{\gamma_7}\bigg(\int_o^{t_1}\int_0^{m}\int_0^{n}\int_{\sigma(a)}^{\rho^3(b)}|x_2-z_2|\Delta s\Delta p\Delta n\Delta m\bigg) \text{ (since (3.14))} \\
&= && sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|sup_{t_1\in\mathbb{T}_1}\frac{\gamma_8}{\gamma_7}\bigg(\rho^3(b)-\sigma(a)\bigg)\int_o^{t_1}\int_0^{m}\int_0^{n}\Delta p\Delta n\Delta m \\
&= && sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|sup_{t_1\in\mathbb{T}_1}\frac{t_1^3}{6}\frac{\gamma_8}{\gamma_7}\bigg(\rho^3(b)-\sigma(a)\bigg)
\end{aligned}$$

So we have

$$\begin{aligned}
& sup_{t_1\in\mathbb{T}_1}\frac{\gamma_8}{\gamma_7}\Bigg(\int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m \\
- & \int_o^{t_1}\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg)\Bigg| \\
\leq & sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|sup_{t_1\in\mathbb{T}_1}\frac{t_1^3}{6}\frac{\gamma_8}{\gamma_7}\Bigg(\rho^3(b)-\sigma(a)\Bigg). \qquad (3.20)
\end{aligned}$$

Now,

$$\begin{aligned}
& \Bigg|\Bigg(\int_o^{t_1}\int_0^m\int_0^n\int_0^a |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta}))\Delta s\Delta p\Delta n\Delta m \\
- & \int_o^{t_1}\int_0^m\int_0^n\int_0^a f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\Bigg)\Bigg|
\end{aligned}$$

$$\begin{aligned}
\leq & \Bigg|\int_o^{t_1}\int_0^m\int_0^n\int_0^a |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta}) \\
- & f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\Bigg| \\
\leq & \int_o^{t_1}\int_0^m\int_0^n\int_0^a |x_1-z_1|\Delta s\Delta p\Delta n\Delta m (\text{since } (3.13)) \\
= & sup_{t_1\in\mathbb{T}_1}sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|a\frac{t_1^3}{6}
\end{aligned}$$

$$\begin{aligned}
& sup_{t_1\in\mathbb{T}_1}\Bigg|\int_o^{t_1}\int_0^m\int_0^n\int_0^a |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta}))\Delta s\Delta p\Delta n\Delta m \\
- & \int_o^{t_1}\int_0^m\int_0^n\int_0^a f_1(s,z_1,,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\Bigg| \\
\leq & sup_{t_1\in\mathbb{T}_1}sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|a\frac{t_1^3}{6}. \qquad (3.21)
\end{aligned}$$

Also,

$$\begin{aligned}
sup_{t_2\in\mathbb{T}_2}\Bigg|\Bigg(&\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta n'\Delta m' \\
&-\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Bigg)\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leq \quad sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'}|f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\\
&- \quad f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})|\Delta s\Delta p'\Delta n'\Delta m'\\
&\leq \quad sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'}|x_2-z_2|\Delta s\Delta p'\Delta n'\Delta m' \text{ (since (3.14))}\\
&\leq \quad sup_{t_2\in\mathbb{T}_2}\left(sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\right)\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'}\Delta s\Delta p'\Delta n'\Delta m'\\
&= \quad sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}(p'-\sigma(a))\Delta p'\Delta n'\Delta m'.\\
&= \quad sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\left(\frac{n'^2}{2}-\sigma(a)n'+\frac{\sigma(a)^2}{2}\right)\Delta n'\Delta m'.\\
&= \quad sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\left(\frac{m'^3}{6}-\frac{\sigma(a)m'^2}{2}+\frac{\sigma(a)^2m'}{2}-\frac{\sigma(a)^3}{6}\right)\Delta m'.\\
&= \quad sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2^4}{24}-\frac{\sigma(a)t_2^3}{6}+\frac{\sigma(a)^2t_2^2}{4}-\frac{\sigma(a)^3t_2}{6}+\frac{\sigma(a)^4}{24}\right)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&sup_{t_2\in\mathbb{T}_2}\Bigg|\Bigg(\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'}f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta n'\Delta m'\\
&-\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{p'}f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Bigg)\Bigg|\\
&\leq sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\ sup_{t_2\in\mathbb{T}_2}\left(\frac{t_2^4}{24}-\frac{\sigma(a)t_2^3}{6}+\frac{\sigma(a)^2t_2^2}{4}-\frac{\sigma(a)^3t_2}{6}+\frac{\sigma(a)^4}{24}\right) \quad (3.22)
\end{aligned}$$

Now

$$\begin{aligned}
&sup_{t_2\in\mathbb{T}_2}\Bigg|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)}f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\\
&- \quad \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)}f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leq \quad sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\\
&- \quad f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})|\Delta s\Delta p'\Delta n'\Delta m'\\
&\leq \quad sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|sup_{t_2\in\mathbb{T}_2}\left(\rho^3(b)-\sigma(a)\right)\left(\frac{t_2^3}{6}-\frac{\sigma(a)t_2^2}{2}+\frac{\sigma(a)^2t_2}{2}-\frac{\sigma(a)^3}{6}\right)
\end{aligned}$$

which implies that

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'$$

$$- \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_{\sigma(a)}^{n'}\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p'\Delta n'\Delta m'\Bigg|$$

$$\leq sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|sup_{t_2\in\mathbb{T}_2}\Big(\rho^3(b)-\sigma(a)\Big)\Big(\frac{t_2^3}{6}-\frac{\sigma(a)t_2^2}{2}+\frac{\sigma(a)^2t_2}{2}-\frac{\sigma(a)^3}{6}\Big). \tag{3.23}$$

Now

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_0^{p} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'$$

$$- \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_0^{p} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Bigg|$$

$$\begin{aligned}
&\leq \frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_0^{p}|f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\\
&- f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m'\\
&\leq \frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_0^{p}|x_1-z_1|\Delta s\Delta p\Delta n'\Delta m'\\
&\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{6}\int_{\sigma(a)}^{t_2}\Delta m'\\
&= sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{6}\Big(t_2-\sigma(a)\Big)
\end{aligned}$$

which implies that

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_0^{p} f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'$$

$$- \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^{a}\int_0^{n}\int_0^{p} f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Bigg|$$

$$\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{6}\Big(t_2-\sigma(a)\Big). \tag{3.24}$$

Now

$$
\begin{aligned}
& sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
- & \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Bigg| \\
\leq & \frac{\gamma_5}{\gamma_6} sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta}) \\
- & f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n'\Delta m' \\
\leq & sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_5}{\gamma_6} sup_{t_2\in\mathbb{T}_2}\left(\frac{a^2t_2^2}{4}-\frac{a^2\sigma(a)t_2}{2}+\frac{a^2\sigma(a)^2}{4}\right)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m' \\
- & \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^p f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Bigg| \\
& \leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_5}{\gamma_6} sup_{t_2\in\mathbb{T}_2}\left(\frac{a^2t_2^2}{4}-\frac{a^2\sigma(a)t_2}{2}+\frac{a^2\sigma(a)^2}{4}\right). \quad (3.25)
\end{aligned}
$$

Now

$$
\begin{aligned}
& sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m' \\
- & \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Bigg| \\
\leq & \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7} sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_{\sigma(a)}^{\rho^3(b)} |f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta}) \\
- & f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m' \\
\leq & \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7} sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_{\sigma(a)}^{\rho^3(b)} |x_2-z_2|\Delta s\Delta p\Delta n'\Delta m' \\
\leq & sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7} sup_{t_2\in\mathbb{T}_2}\Big(\rho^3(b)-\sigma(a)\Big)\frac{a^2}{2}\int_{\sigma(a)}^{t_2}\Delta m' \\
= & sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7} sup_{t_2\in\mathbb{T}_2}\frac{a^2}{2}\Big(\rho^3(b)-\sigma(a)\Big)\Big(t_2-\sigma(a)\Big)
\end{aligned}
$$

which implies that

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}$$
$$- \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\Bigg|$$
$$\leq sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}sup_{t_2\in\mathbb{T}_2}\frac{a^2}{2}\Big(\rho^3(b)-\sigma(a)\Big)\Big(t_2-\sigma(a)\Big). \quad (3.26)$$

Now

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}$$
$$- \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Bigg|$$

$$\begin{aligned}
&\leq \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_{\sigma(a)}^{\rho^3(b)}|f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\\
&- f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n^{'}\Delta m^{'}\\
&\leq sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}a\Big(\rho^3(b)-\sigma(a)\Big)sup_{t_2\in\mathbb{T}_2}\Big(\frac{t_2^2}{2}-\sigma(a)t_2+\frac{\sigma(a)^2}{2}\Big)
\end{aligned}$$

which implies that

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}$$
$$- \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m^{'}}\int_0^a\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_2^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n^{'}\Delta m^{'}\Bigg|$$
$$\leq sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}a\Big(\rho^3(b)-\sigma(a)\Big)sup_{t_2\in\mathbb{T}_2}\Big(\frac{t_2^2}{2}-\sigma(a)t_2+\frac{\sigma(a)^2}{2}\Big). \quad (3.27)$$

Similarly,

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_0^a f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}$$
$$- \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_0^a f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m^{'}\Bigg|$$
$$\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{2}\Big(t_2-\sigma(a)\Big). \quad (3.28)$$

and

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^a f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'$$

$$- \quad \frac{\gamma_5}{\gamma_6}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}\int_0^a\int_0^a f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n'\Delta m'\Bigg|$$

$$\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_5}{\gamma_6}sup_{t_2\in\mathbb{T}_2}\Bigg(\frac{a^2t_2^2}{2}-a^2\sigma(a)t_2+\frac{a^2\sigma(a)^2}{2}\Bigg). \quad (3.29)$$

Now

$$\Bigg|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_0^p f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m$$

$$- \quad \frac{\gamma_3}{\gamma_4}\int_0^a\int_0^m\int_0^n\int_0^p f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg|$$

$$\begin{aligned}
\leq\quad & \frac{\gamma_1}{\gamma_1}\int_0^a\int_0^m\int_0^n\int_0^p |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\\
-\quad & f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m\\
\leq\quad & \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_0^p |x_1-z_1|\Delta s\Delta p\Delta n\Delta m\\
\leq\quad & sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_1}{\gamma_2}\frac{a^4}{24}
\end{aligned}$$

which implies that

$$sup_{t_2\in\mathbb{T}_2}\Bigg|\frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_0^p f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'$$

$$- \quad \frac{\gamma_3}{\gamma_4}\int_{\sigma(a)}^{t_2}\int_0^a\int_0^n\int_0^p f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m'\Bigg|$$

$$\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{6}\Bigg(t_2-\sigma(a)\Bigg). \quad (3.30)$$

Now

$$\Bigg|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m$$

$$- \quad \frac{\gamma_2}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_1^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\Bigg|$$

$$\begin{aligned}
&\leq \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} |f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta}) \\
&- f_2(s,z_2,z_1^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m \\
&\leq \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} |x_2-z_2|\Delta s\Delta p\Delta n\Delta m \\
&\leq sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_1}{\gamma_2}\left(\rho^3(b)-\sigma(a)\right)\frac{a^3}{6}
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,x_2,x_2^{\Delta},x_2^{\Delta\Delta},x_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\right. \\
&- \left.\frac{\gamma_2}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_{\sigma(a)}^{\rho^3(b)} f_2(s,z_2,z_1^{\Delta},z_2^{\Delta\Delta},z_2^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\right| \\
&\leq sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_1}{\gamma_2}\left(\rho^3(b)-\sigma(a)\right)\frac{a^3}{6}. \qquad (3.31)
\end{aligned}$$

And finally,

$$\begin{aligned}
&\left|\frac{\gamma_3}{\gamma_4}\int_0^a\int_0^m\int_0^n\int_0^a f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\right. \\
&- \left.\frac{\gamma_3}{\gamma_4}\int_0^a\int_0^m\int_0^n\int_0^a f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\right| \\
&\leq \frac{\gamma_1}{\gamma_1}\int_0^a\int_0^m\int_0^n\int_0^a |f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta}) \\
&- f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})|\Delta s\Delta p\Delta n\Delta m \\
&\leq \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m\int_0^n\int_0^a |x_1-z_1|\Delta s\Delta p\Delta n\Delta m \\
&\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_1}{\gamma_2}\frac{a^4}{6}
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left|\frac{\gamma_3}{\gamma_4}\int_0^a\int_0^m\int_0^n\int_0^a f_1(s,x_1,x_1^{\Delta},x_1^{\Delta\Delta},x_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\right. \\
&- \left.\frac{\gamma_3}{\gamma_4}\int_0^a\int_0^m\int_0^n\int_0^a f_1(s,z_1,z_1^{\Delta},z_1^{\Delta\Delta},z_1^{\Delta\Delta\Delta})\Delta s\Delta p\Delta n\Delta m\right| \\
&\leq sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_1}{\gamma_2}\frac{a^4}{6}. \qquad (3.32)
\end{aligned}$$

From the equations (3.19)-(3.32) we can conclude that

$\|T(x_1,x_2)-T(z_1,z_2)\|$

$$
\begin{aligned}
\leq\ & sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\Bigg(sup_{t_1\in\mathbb{T}_1}\frac{{t_1}^4}{24}+sup_{t_1\in\mathbb{T}_1}a\frac{t_1^3}{6}\\
+\ & \frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{6}\Bigg(t_2-\sigma(a)\Bigg)\\
+\ & \frac{\gamma_5}{\gamma_6}sup_{t_2\in\mathbb{T}_2}\Bigg(\frac{a^2t_2^2}{4}-\frac{a^2\sigma(a)t_2}{2}+\frac{a^2\sigma(a)^2}{4}\Bigg)\\
+\ & \frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{2}\Bigg(t_2-\sigma(a)\Bigg)\\
+\ & \frac{\gamma_5}{\gamma_6}sup_{t_2\in\mathbb{T}_2}\Bigg(\frac{a^2t_2^2}{2}-a^2\sigma(a)t_2+\frac{a^2\sigma(a)^2}{2}\Bigg)\\
+\ & \frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{6}\Bigg(t_2-\sigma(a)\Bigg)+\frac{\gamma_1}{\gamma_2}\frac{a^4}{6}\Bigg)\\
+\ & sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\Bigg(sup_{t_1\in\mathbb{T}_1}\frac{t_1^3}{6}\frac{\gamma_8}{\gamma_7}\Bigg(\rho^3(b)-\sigma(a)\Bigg)\\
+\ & sup_{t_2\in\mathbb{T}_2}\Bigg(\frac{t_2^4}{24}-\frac{\sigma(a)t_2^3}{6}+\frac{\sigma(a)^2t_2^2}{4}-\frac{\sigma(a)^3t_2}{6}+\frac{\sigma(a)^4}{24}\Bigg)\\
+\ & sup_{t_2\in\mathbb{T}_2}\Bigg(\rho^3(b)-\sigma(a)\Bigg)\Bigg(\frac{t_2^3}{6}-\frac{\sigma(a)t_2^2}{2}+\frac{\sigma(a)^2t_2}{2}-\frac{\sigma(a)^3}{6}\Bigg)\\
+\ & \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}sup_{t_2\in\mathbb{T}_2}\frac{a^2}{2}\Bigg(\rho^3(b)-\sigma(a)\Bigg)\Bigg(t_2-\sigma(a)\Bigg)\\
+\ & \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}a\Bigg(\rho^3(b)-\sigma(a)\Bigg)sup_{t_2\in\mathbb{T}_2}\Bigg(\frac{t_2^2}{2}-\sigma(a)t_2+\frac{\sigma(a)^2}{2}\Bigg)\\
+\ & \frac{\gamma_1}{\gamma_2}\Bigg(\rho^3(b)-\sigma(a)\Bigg)\frac{a^3}{6}\Bigg)
\end{aligned}
$$

$$\Rightarrow \|T(x_1,x_2)-T(z_1,z_2)\|\leq K_1\Bigg(sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\Bigg)+K_2\Bigg(sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\Bigg)$$

where

$$\begin{aligned}
K_1 &= sup_{t_1\in\mathbb{T}_1}\frac{{t_1}^4}{24}+sup_{t_1\in\mathbb{T}_1}a\frac{t_1^3}{6}\\
&+ \frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{6}\Big(t_2-\sigma(a)\Big)\\
&+ \frac{\gamma_5}{\gamma_6}sup_{t_2\in\mathbb{T}_2}\Big(\frac{a^2t_2^2}{4}-\frac{a^2\sigma(a)t_2}{2}+\frac{a^2\sigma(a)^2}{4}\Big)\\
&+ \frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{2}\Big(t_2-\sigma(a)\Big)\\
&+ \frac{\gamma_5}{\gamma_6}sup_{t_2\in\mathbb{T}_2}\Big(\frac{a^2t_2^2}{2}-a^2\sigma(a)t_2+\frac{a^2\sigma(a)^2}{2}\Big)\\
&+ \frac{\gamma_3}{\gamma_4}sup_{t_2\in\mathbb{T}_2}\frac{a^3}{6}\Big(t_2-\sigma(a)\Big)+\frac{\gamma_1}{\gamma_2}\frac{a^4}{6},\\
K_2 &= sup_{t_1\in\mathbb{T}_1}\frac{t_1^3}{6}\frac{\gamma_8}{\gamma_7}\Big(\rho^3(b)-\sigma(a)\Big)\\
&+ sup_{t_2\in\mathbb{T}_2}\Big(\frac{t_2^4}{24}-\frac{\sigma(a)t_2^3}{6}+\frac{\sigma(a)^2t_2^2}{4}-\frac{\sigma(a)^3t_2}{6}+\frac{\sigma(a)^4}{24}\Big)\\
&+ sup_{t_2\in\mathbb{T}_2}\Big(\rho^3(b)-\sigma(a)\Big)\Big(\frac{t_2^3}{6}-\frac{\sigma(a)t_2^2}{2}+\frac{\sigma(a)^2t_2}{2}-\frac{\sigma(a)^3}{6}\Big)\\
&+ \frac{\gamma_3}{\gamma_4}\frac{\gamma_8}{\gamma_7}sup_{t_2\in\mathbb{T}_2}\frac{a^2}{2}\Big(\rho^3(b)-\sigma(a)\Big)\Big(t_2-\sigma(a)\Big)\\
&+ \frac{\gamma_5}{\gamma_6}\frac{\gamma_8}{\gamma_7}a\Big(\rho^3(b)-\sigma(a)\Big)sup_{t_2\in\mathbb{T}_2}\Big(\frac{t_2^2}{2}-\sigma(a)t_2+\frac{\sigma(a)^2}{2}\Big)\\
&+ \frac{\gamma_1}{\gamma_2}\Big(\rho^3(b)-\sigma(a)\Big)\frac{a^3}{6}
\end{aligned}$$

Let $K = max\{K_1, K_2\}$. Then $K < 1$(since (3.15)-(3.16)). We now have

$\|T(x_1, x_2) - T(z_1, z_2)\|$

$$\begin{aligned}
&\le K\Big(sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|+sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\Big)\\
&= K\|(x_1-z_1, x_2-z_2)\|\\
&= K\|(x_1,x_2)-(z_1,z_2)\|.
\end{aligned}$$

Since $K < 1$, by Banach Contraction mapping theorem (2.5) we have a unique fixed point for $(y_1, y_2) = T(y_1, y_2)$. Hence a unique solution exists for the BVP-SIP. □

Remark 3.7. The above theorems can be proved for BVPs for the Interface II and Interface III with suitable changes in the notations.

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