Some Common Fixed Point Theorems Using Implicit Relations in Menger Spaces

Vishal Gupta
Department of Mathematics,
M.M. University, Mullana,
Ambala, Haryana, India.

Balbir Singh
Department of Mathematics
B.M.I.E.T, Sonipat, Haryana, India.

and
Sanjay Kumar
Department of Mathematics,
D.C.R. University of Science and Technology,
Murthal, Sonipat, Haryana, India.

Abstract: In this paper, we prove common fixed point theorems for pairs of weakly compatible mappings along with E.A. and (CLR_S) properties using implicit relations in Menger spaces.

Mathematics Subject Classification (2000): 47H10, 54H25.

Key Words: Menger space, weakly compatible mappings, t-norm of Hadzic- type, E.A. and CLR properties.

1. INTRODUCTION

The theory of probabilistic metric spaces is an important part of stochastic Analysis, and so it is of interest to develop the fixed point theory in such spaces. The first result from the fixed point theory in probabilistic metric spaces is obtained by Sehgal and Bharucha- Reid [16]. Since then many fixed points theorems for single valued and multi valued mappings in probabilistic metric spaces have been proved in [2]-[5].

A probabilistic metric space is an ordered pair (X, F), where X is an arbitrary set and F is a mapping from X^2 into the set of distribution functions. The distribution function $F_{x,y}(t)$ will denote the value of $F_{x,y}$ at the real number t. The function $F_{x,y}$ are assumed to satisfy the following conditions:

(i) $F_{x,y}(0) = 0$ for all $x, y \in X$

(ii) $F_{x,y}(t) = 1$ for all $t > 0$ iff $x = y$

(iii) For distinct points $x, y \in X$, $F_{x,y}(t) \neq 1$ for $t > 0$

(iv) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t > 0$

(v) If $F_{x,y}(t_1) = 1$, $F_{y,,z}(t_2) = 1$, then $F_{x,z}(t_1 + t_2) = 1$ for all $x, y, z \in X$ and $t_1, t_2, > 0$.

In 2003, Ren and Wang [18] gave the notion of n-th order t- norm as follows:

Definition 1.1. A mappings $\Delta : \prod_{i=1}^{n}[0,1] \to [0,1]$ is called a n-th order t-norm if following conditions are satisfied:

(i) $\Delta(0,0,\ldots,0) = 0$, $\Delta(a,1,1,\ldots1) = a$ for all $a \in [0,1]$

(ii) $\Delta(a_1, a_2, a_3, \ldots, a_n) = \Delta(a_2, a_1, a_3, \ldots, a_n) = \Delta(a_2, a_3, a_1, \ldots, a_n)$

$= \quad = \Delta(a_2, a_3, a_4, \ldots, a_n, a_1)$

(iii) $a_i \geq b_i$, $i = 1,2,3\ldots,n$ implies $\Delta(a_1, a_2, a_3, \ldots, a_n) \geq \Delta(b_1, b_2, b_3, \ldots, b_n)$

(iv) $\Delta(\Delta(a_1, a_2, a_3, \ldots, a_n), b_2, b_3, \ldots, b_n)$

$= \Delta(a_1, \Delta(a_2, a_3, \ldots, a_n, b_2), b_3, \ldots b_n)$

$= \Delta(a_1, a_2, \Delta(a_3, a_4, \ldots, a_n, b_2, b_3), b_4, \ldots, b_n)$

$= \ldots$

$= \Delta(a_1, a_2, a_3, \ldots, a_{n-1}, \Delta(a_n, b_2, b_3, \ldots, b_n))$.

For $n = 2$, we have a binary t- norm, which is commonly known as t- norm.

Basics examples of t-norm are the Lukasiewicz t- norm Δ_L, $\Delta_L(a, b) = \max(a+b-1, 0)$, t-norm Δ_P, $\Delta_P(a, b) = ab$ and t- norm Δ_M, $\Delta_M(a, b) = \min\{a, b\}$.

Definition 1.2.([3]) Let Δ be a t-norm and let $\Delta_n : [0, 1] \to [0, 1]$ ($n \in \mathbb{N}$) be defined by

$$\Delta_1(x) = \Delta(x, x),\ \Delta_{n+1}(x) = \Delta(\Delta_n(x), x) \quad (n \in \mathbb{N},\ x \in [0,1]).$$

We say that the t-norm Δ is of Hadzic- type if the family $\{\Delta_n(x); n \in \mathbb{N}\}$ is equicontinuous at $x=1$.The family $\{\Delta_n(x), n \in \mathbb{N}\}$ is equicontinuous at $x=1$ if for every $\lambda \in (0, 1)$, there $\delta(\lambda) \in (0,1)$ such that

$$x > 1 - \delta(\lambda) \text{ implies } \Delta_n(x) > 1 - \lambda \text{ for all } (n \in \mathbb{N}).$$

A trivial example of t-norm of Hadzic- type is $\Delta = \Delta_M$.

Remark 1.3. Every t-norm Δ_M is of Hadzic- type but converse need not be true see [4]

There is a nice characterization of continuous t-norm of Hadzic - type :

(i) If there exists a strictly increasing sequence $\{b_n\}_{n\in\mathbb{N}}$ in $[0,1]$ such that $\lim_{n\to\infty} b_n = 1$ and $\Delta(b_n, b_n) = b_n$ for all $n \in \mathbb{N}$, then Δ is of Hadzic – type.

(ii) If Δ is continuous and Δ is of Hadzic- type, then there exists a sequence $\{b_n\}_{n\in\mathbb{N}}$ as in (i).

Definition 1.4.([4]) If Δ is a t-norm and $(x_1, x_2, x_3, \ldots, x_n) \in [0,1]^n$ ($n \in \mathbb{N}$), then $\Delta_{i=1}^{n} x_i$ is defined recurrently by 1, if $n=1$ and $\Delta_{i=1}^{n} x_i = \Delta(\Delta_{i=1}^{n-1} x_i, x_n)$ for all $n \geq 2$. If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $[0,1]$, then $\Delta_{i=1}^{\infty} x_i$ is defined as $\lim_{n\to\infty} \Delta_{i=1}^{n} x_i$ (this limit always exists) and $\Delta_{i=1}^{\infty} x_i$ as $\Delta_{i=1}^{\infty} x_{n+i}$.

Definition 1.5. Let X be any non-empty set and D the set of all left-continuous distribution functions. A triplet (X, F, Δ) is said to be a Menger space if the probabilistic metric space (X, F) satisfies the following condition:

(vi) $F_{x,z}(t) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2))$,

where $t, t_2 > 0$, $t_1 + t_2 = t$ and $x, y, z, \in X$ and Δ is the t- norm.

Definition 1.6. A sequence $\{x_n\}$ in a Menger space (X, F, Δ) is said to be

(i) convergent with limit x if $\lim_{n\to\infty} F_{x_n,x}(t)= 1$ for all $t > 0$.

(ii) Cauchy sequence in X if given $\in > 0, \lambda > 0$, there exists a positive integer $N_{\epsilon,\lambda}$ such that

$$F_{x_n,x_m}(\in) > 1-\lambda \text{ for all } m, n > N_{\epsilon,\lambda}.$$

(iii) Complete if every Cauchy sequence in X is convergent in X.

Definition 1.7.([12]) Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

Definition 1.8 Two self-mapping f and g of a Menger space (X, F, Δ) are said to be weakly commuting if $F(fgx, gfx, t) \geq F(fx, gx,t)$, for each $x \in X$ and for each $t>0$.

Definition 1.9.([14]) Let f and g mapping from a Menger space (X, F, Δ) into itself. A pair of map {f, g} is said to be compatible if $\lim_{n\to\infty} F(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

Definition 1.10. Let f and g self- mapping from on Menger space (X, F, Δ).The mappings f and g are said to be non-compatible if $\lim_{n\to\infty} F(fgx_n, gfx_n, t) \neq 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u$ for some $u \in X$ and for all $t > 0$.

In 2007, Kohli et.al [13] introduced the notion of variants of R-weak commutative maps as follows:

Definition 1.11. A pair of self- mappings (f, g) of a Menger space (X, F, Δ) is said to be

(i) Weakly commuting if $F(fgx, gfx,t) \geq F(fx, gx,t)$

(ii) R- Weakly commuting if there exists some $R > 0$ such that
$F(fgx, gfx,t) \geq F(fx,gx, t/R)$

(iii) R- Weakly commuting mappings of the type (i) if there exists some $R > 0$ such that
$F(gfx, ffx,t) \geq F(gx,fx, t/R)$

(iv) R- Weakly commuting mappings of the type (ii) if there exists some $R > 0$ such that
$F(fgx, ggx,t) \geq F(fx,gx, t/R)$

(v) R- Weakly commuting mappings of the type (iii) if there exists some $R > 0$ such that
$F(ffx, ggx,t) \geq F(fx,gx, t/R)$, for all $x \in X$ and $t > 0$.

In our further discussion, we adopt the terminology from the paper of Imdad et.al.[8] .

We rename R-Weakly commuting mappings of the type (i), R-Weakly commuting mappings of the type (ii) and R-Weakly commuting mappings of the type (iii) by R-Weakly commuting mappings of the type (A_g), R-Weakly commuting mappings of the type (A_f) and R-Weakly commuting mappings of the type (P) respectively.

One can notice that definition 1.11.(iii) and 1.11.(iv) was inspired by Imdad et.al. [8] from the paper of Pathak et. al. [15], whereas definition 1.11.(v) was introduce by Imdad et.al. [8].

In 2002, Aamri and Moutawakil [1] generalized the notion of non compatible mapping to E.A. property. It was pointed out in [1] that property E.A. buys containment of ranges without any continuity requirements besides minimizes the commutativity conditions of the maps at their points of coincidence. Moreover, E.A. property allows replacing the completeness requirement of the space with a more natural condition of closeness of the range. Recently, common fixed point theorems in probabilistic metric spaces/fuzzy metric spaces using E.A. property along with weak compatibility have been recently obtained in ([8],[10]).

Definition 1.12.([1]) Let f and g be two self-maps of a metric (X, d), then they are said to satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u, \text{ for some } u \in X.$$

Now in a similar mode, we can state E.A. property in Menger space as follows:

Definition 1.13. A pair of self-mapping (f, g) of Menger spaces (X, F, Δ) is said to hold E.A. property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} F_{f_{x_n}, g_{x_n}}(t)=1 \text{ for all } t > 0.$$

Example.1.14. Let X = [0, ∞) be the usual metric space. Define f, g: X→ X by $fx=\frac{x}{4}$ and $gx=\frac{3X}{4}$ for all $x \in X$. Consider the sequence $\{x_n\}= \frac{1}{n}$. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n=0$, then f and g satisfy the E.A. property.

Although E.A property is generalization of the concept of non compatible maps, yet it requires either completeness of the whole space or any of the range space or continuity of maps. Recently, the new notion of CLR property (common limit range property) was given by Sintunavarat and Kuman [19] that does not impose such conditions. Their importance of CLR property ensures that one does not require the closeness of range subspaces.

Definition 1.15.([19]) Two maps f and g on Menger spaces X are satisfy the common limit in the range of g (CLRg) property if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$, for some $x \in X$.

Example.1.16. Let X = [0, ∞) be the usual metric space. Define f, g: X→ X by fx=x+1 and gx=2x for all $x \in X$. Consider the sequence $\{x_n\}= 1+ \frac{1}{n}$. Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n=2=$ g1, therefore f and g satisfy the (CLRg) property.

Now we state a Lemma which is useful in our study:

LEMMA 1.17.([14]) Let (X, F, Δ) be a Menger space. If there exists $q\in(0, 1)$ such that $F(x, y, qt) \geq F(x, y, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

Implicit relations

Let Ψ be set of all continuous function $\phi(t_1,t_2,t_3,t_4,t_5,t_6): \mathbb{R}^6 \to \mathbb{R}$ is a non-increasing in 6-th co-ordinate variable and satisfying the following conditions:

(i) $\phi(u, 1, v, 1, v, \Delta(u, v)) \geq 1$ or $\phi(u, v, v, u, 1, \Delta(u, v)) \geq 1$ implies that $u \geq v$,

(ii) $\phi(u, 1, 1, u, 1, u) \geq 1$ implies that $u \geq 1$,

(iii) $\phi(u, v, 1, 1, v, u) \geq 1$ implies that $u \geq v$.

Example 1.18. Define $\phi(t_1,t_2,t_3,t_4,t_5,t_6) = 15t_1 - 13t_2 + 5t_3 - 7t_4 + t_5 - t_6$, then $\phi \in \Psi$.

2. Weakly compatible maps

Theorem 2.1. Let (X, F, Δ) be a complete Menger space with continuous t-norm of Hadzic - type. Let A, B, S and T be self mappings on X satisfying the following conditions:

(2.1) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(2.2) the pairs (A, S) and (B, T) are weakly compatible,

(2.3) there exists $q \in (0, 1)$ such that for every $x, y \in X$, $t > 0$ and $\phi \in \Psi$,

$$\phi(F(Ax, By, qt), F(Sx, Ty, t), F(Ax, Sx, t), F(By, Ty, qt), F(Ax, Ty, t), F(By, Sx, (q+1)t)) \geq 1,$$

(2.4) One of the subsets A(X), B(X), S(X) and T(X) is a closed subset of X.

Assume that there exists $x_0, x_1 \in X$ such that for $y_1 = Ax_0 = Tx_1$, $y_2 = Bx_1$ and $\mu \in (q, 1)$

$$\lim_{n\to\infty} \Delta_{i=n}^{\infty} F(y_1, y_2, \tfrac{1}{\mu^i}) = 1.$$

Then A, B, S, and T have a unique common fixed point in X.

Proof. Since $B(X) \subset S(X)$, there exists $x_1, x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ of X such that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \text{ and}$$

$$y_{2n} = Sx_{2n} = Bx_{2n-1} \quad \text{for } n = 1,2,\ldots$$

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (2.3), we have that for all $t > 0$

$$\begin{aligned}
1 \leq\ & \phi(F(Ax_{2n}, Bx_{2n+1}, qt), F(Sx_{2n}, Tx_{2n+1}, t), F(Ax_{2n}, Sx_{2n}, t), \\
& F(Bx_{2n+1}, Tx_{2n+1}, qt), F(Ax_{2n}, Tx_{2n+1}, t), \\
& F(Bx_{2n+1}, Sx_{2n}, (q+1)t)) \\
=\ & \phi(F(y_{2n+1}, y_{2n+2}, qt), F(y_{2n}, y_{2n+1}, t), F(y_{2n+1}, y_{2n}, t), \\
& F(y_{2n+2}, y_{2n+1}, qt), F(y_{2n+1}, y_{2n+1}, t), \\
& F(y_{2n+2}, y_{2n}, (q+1)t)) \\
\leq\ & \phi(F(y_{2n+1}, y_{2n+2}, qt), F(y_{2n}, y_{2n+1}, t), F(y_{2n+1}, y_{2n}, t), \\
& F(y_{2n+2}, y_{2n+1}, qt), F(y_{2n+1}, y_{2n+1}, t), \\
& \Delta(F(y_{2n+2}, y_{2n+1}, qt), F(y_{2n+1}, y_{2n}, t))),
\end{aligned}$$

since the function ϕ is non-increasing in the 6-th coordinate variable.

Using properties of implicit relations Ψ, we get

$$F(y_{2n+1}, y_{2n+2}, qt) \geq F(y_{2n}, y_{2n+1}, t).$$

Again, putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (2.3), we have for all $t > 0$

$$\begin{aligned}1 \leq \phi\,(&F(Ax_{2n+1}, Bx_{2n+2}, qt), F(Sx_{2n+1}, Tx_{2n+2}, t), F(Ax_{2n+1}, Sx_{2n+1}, t),\\ &F(Bx_{2n+2}, Tx_{2n+2}, qt), F(Ax_{2n+1}, Tx_{2n+2}, t),\\ &F(Bx_{2n+2}, Sx_{2n+1}, (q+1)t))\\ = \phi(&F(y_{2n+2}, y_{2n+3}, qt), F(y_{2n+1}, y_{2n+2}, t), F(y_{2n+2}, y_{2n+1}, t),\\ &F(y_{2n+3}, y_{2n+2}, qt), F(y_{2n+2}, y_{2n+2}, t),\\ &F(y_{2n+3}, y_{2n+1}, (q+1)t))\\ \leq \phi\,(&F(y_{2n+2}, y_{2n+3}, qt), F(y_{2n+1}, y_{2n+2}, t), F(y_{2n+2}, y_{2n+1}, t),\\ &F(y_{2n+3}, y_{2n+2}, qt), F(y_{2n+2}, y_{2n+2}, t),\\ &\Delta\,(F(y_{2n+3}, y_{2n+2}, qt), F(y_{2n+2}, y_{2n+1}, t))),\end{aligned}$$

Hence we get

$$F(y_{2n+2}, y_{2n+3}, qt) \geq F(y_{2n+1}, y_{2n+2}, t).$$

Thus for any $n \in \mathbb{N}$, we have

$$F(y_{n+1}, y_n, qt) \geq F(y_n, y_{n-1}, t)$$

i.e.,

$$\begin{aligned}F(y_{n+1}, y_n, t) &\geq F(y_n, y_{n-1}, \tfrac{t}{q})\\ &\geq F(y_{n-1}, y_{n-2}, \tfrac{t}{q^2})\\ &\;\;\cdots\\ &\geq F(y_1, y_2, \tfrac{t}{q^{n-1}})\end{aligned}$$

Thus for all $t > 0$ and $n = 1, 2, 3\ldots$

$$F(y_n, y_{n+1}, qt) \geq F(y_1, y_2, \tfrac{t}{q^{n-1}}). \qquad (2.5)$$

Now, we show that $\{y_n\}$ is a Cauchy sequence in X.

Let $\sigma = \frac{q}{\mu}$. Since $0 < \sigma < 1$, the series $\sum_{i=1}^{\infty} \sigma^i$ is convergent and there exists $m_0 \in \mathbb{N}$ such that $\sum_{i=m_0}^{\infty} \sigma^i < 1$. Hence for every $m > m_0+1$ and every $s \in \mathbb{N}$

$$t > t\sum_{i=m_0}^{\infty} \sigma^i > t\sum_{i=m-1}^{m+s-1} \sigma^i.$$

Now

$$\begin{aligned}&F(y_{m+s+1}, y_m, t)\\ &\geq F(y_{m+s+1}, y_m, t\textstyle\sum_{i=m-1}^{m+s-1} \sigma^i)\\ &\geq F(y_{m+s+1}, y_m, t\sigma^{m-1} + t\sigma^{m-1+1} + t\sigma^{m-1+2} \ldots + t\sigma^{m-1+s})\end{aligned}$$

$\geq F(y_{m+s+1}, y_m, t\sigma^{m-1+1} + t\sigma^{m-1+2} \ldots + t\sigma^{m-1+s} + t\sigma^{m-1})$

$\geq \underbrace{\Delta}_{1}(F(y_{m+s+1}, y_{m+1}, t\sigma^{m-1+1} + t\sigma^{m-1+2} \ldots + t\sigma^{m-1+s}),$

$F(y_{m+1}, y_m, t\sigma^{m-1}))$

$\geq \underbrace{\Delta(\Delta}_{2}(F(y_{m+s+1}, y_{m+2}, t\sigma^{m-1+2} + \ldots + t\sigma^{m-1+s}),$

$F(y_{m+2}, y_{m+1}, t\sigma^{m}), F(y_{m+1}, y_m, t\sigma^{m-1})))$

$\geq \underbrace{\Delta(\Delta(\Delta}_{3-\text{times}}(F(y_{m+s+1}, y_{m+3}, t\sigma^{m-1+3} + \ldots + t\sigma^{m-1+s}),$

$F(y_{m+3}, y_{m+2}, t\sigma^{m+1}), F(y_{m+2}, y_{m+1}, t\sigma^{m}), F(y_{m+1}, y_m, t\sigma^{m-1}))))$

$\geq \underbrace{\Delta(\Delta(\Delta(\Delta}_{4-\text{times}}(F(y_{m+s+1}, y_{m+4}, t\sigma^{m-1+4} + \ldots + t\sigma^{m-1+s}), F(y_{m+4}, y_{m+3}, t\sigma^{m+2})$

$, F(y_{m+3}, y_{m+2}, t\sigma^{m+1}), F(y_{m+2}, y_{m+1}, t\sigma^{m}), F(y_{m+1}, y_m, t\sigma^{m-1})))))$

$\geq \underbrace{\Delta(\Delta(\Delta(\Delta(\Delta}_{5-\text{times}}(F(y_{m+s+1}, y_{m+5}, t\sigma^{m-1+5} + \ldots + t\sigma^{m-1+s}), F(y_{m+5}, y_{m+4}, t\sigma^{m+3})$

$, F(y_{m+4}, y_{m+3}, t\sigma^{m+2}), F(y_{m+3}, y_{m+2}, t\sigma^{m+1}), F(y_{m+2}, y_{m+1}, t\sigma^{m}),$

$, F(y_{m+1}, y_m, t\sigma^{m-1}))))))$

$\geq \underbrace{\Delta(\Delta(\Delta(\Delta(\Delta(\Delta}_{6-\text{times}}(F(y_{m+s+1}, y_{m+6}, t\sigma^{m-1+6} + \ldots + t\sigma^{m-1+s}), F(y_{m+6}, y_{m+5}, t\sigma^{m+4})$

$, F(y_{m+5}, y_{m+4}, t\sigma^{m+3}), F(y_{m+4}, y_{m+3}, t\sigma^{m+2}), F(y_{m+3}, y_{m+2}, t\sigma^{m+1}),$

$F(y_{m+2}, y_{m+1}, t\sigma^{m}), F(y_{m+1}, y_m, t\sigma^{m-1})))))))$

$\geq \quad \ldots$

.

$\geq \underbrace{\Delta(\Delta(\ldots(\Delta}_{s-\text{times}}(F(y_{m+s+1}, y_{m+s}, t\sigma^{m-1+s}), F(y_{m+s}, y_{m+s-1}, t\sigma^{m+s-2})$

$\ldots, F(y_{m+1}, y_m, t\sigma^{m-1}))) \ \ldots))$

$\geq \underbrace{\Delta(\Delta(\ldots(\Delta}_{s-\text{times}}(F(y_1, y_2, t\frac{\sigma^{m-1+s}}{q^{m-1+s}}), F(y_1, y_2, t\frac{\sigma^{m-2+s}}{q^{m-2+s}}), \ldots, F(y_1, y_2, t\frac{\sigma^{m-1}}{q^{m-1}}))) \ldots))$

$\geq \underbrace{\Delta(\Delta(\ldots(\Delta}_{s-\text{times}}(F(y_1, y_2, \frac{t}{\mu^{m-1+s}}), F(y_1, y_2, \frac{t}{\mu^{m-2+s}}), \ldots, F(y_1, y_2, \frac{t}{\mu^{m-1}}))) \ldots))$

$\geq \Delta_{i=m-1}^{m+s-1} F(y_1, y_2, \frac{t}{\mu^i})$

$\geq \Delta_{i=m-1}^{\infty} F(y_1, y_2, \frac{t}{\mu^i}).$

It is obvious that

$\lim_{n\to\infty} \Delta_{i=n}^{\infty} F(y_1, y_2, \frac{1}{\mu^i}) = 1$, implies $\lim_{n\to\infty} \Delta_{i=n}^{\infty} F(y_1, y_2, \frac{t}{\mu^i}) = 1$ for every $t > 0$.

Now for every $t > 0$ and every $\lambda \in (0, 1)$, there exists $m_1(t, \lambda)$ such that

$F(y_{m+s+1}, y_m, t) > 1-\lambda$ for every $m \geq m_1(t, \lambda)$ and every $s \in \mathbb{N}$. Hence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, therefore, there exists a point z in X such that $\lim_{n\to\infty} y_n = z$ and this gives

$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n-1} = \lim_{n\to\infty} Ax_{2n-2} = \lim_{n\to\infty} Bx_{2n-1} = z$, for all $n \in \mathbb{N}$. Without loss of generality, we assume that S(X) is a closed subset of X. Then z = Su for some $u \in X$. Subsequently, we have

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z = Su.$$

Next, we claim that Au = Su.

For this purpose, if we put x= u and y= x_n in (2.3), then this gives

$$1 \leq \phi\,(F(Au, Bx_n, qt), F(Su, Tx_n, t), F(Au, Su, t),$$
$$F(Bx_n, Tx_n, qt), F(Au, Tx_n, t), F(Bx_n, Su, (q+1)t))$$
$$\leq \phi(F(Au, Bx_n, qt), F(Su, Tx_n, t), F(Au, Su, t),$$
$$F(Bx_n, Tx_n, qt), F(Au, Tx_n, t),$$
$$\Delta(F(Bx_n, Au, qt), F(Au, Su, t))).$$

Taking limit as $n \to \infty$, we have

$$1 \leq \phi(F(Au, z, qt), F(z, z, t), F(Au, z, t), F(z, z, qt),$$
$$F(Au, z, t), \Delta(F(Au, z, qt), F(Au, z, t)).$$

Hence we have $F(Au, z, qt) \geq F(Au, z, t)$ for all $t > 0$, by Lemma 1.17, we have Au= Su =z. Since $A(X) \subset T(X)$, therefore there exists a point $v \in X$ such that Au= z=Tv.

Next, we claim that Tv = Bv.

Putting x = u and y= v in (2.3), we have

$$1 \leq \phi(F(Au, Bv, qt), F(Su, Tv, t), F(Au, Su, t),$$
$$F(Bv, Tv, qt), F(Au, Tv, t), F(Bv, Su, (q+1)t))$$
$$= \phi(F(Au, Bv, qt), F(z, z, t), F(z, z, t),$$
$$F(Bv, Tv, qt), F(z, z, t), F(Bv, Su, (q+1)t))$$
$$\leq \phi(F(Au, Bv, qt), 1, 1, F(Bv, Tv, qt), 1,$$
$$\Delta(F(Bv, Au, qt), F(Au, Su, t)),$$

Therefore, we obtain that

$$\phi(F(Au, Bv, qt), 1, 1, F(Bv, Au, qt), 1, F(Bv, Au, qt)) \geq 1,$$

by $\phi \in \boldsymbol{\Psi}$, we have $F(Bv, Tv, qt) \geq 1$ for all $t > 0$ implies that Tv = Bv. Thus Au = Su = Tv = Bv = z. Since the pairs (A, S) and (B, T) are weakly compatible and u and v are their coincidence points respectively, we obtain Az = A (Su) = S(Au) =Sz and Bz= B(Tv) = T(Bv) = Tz.

Now, we prove that z is a common fixed point of A,B, S and T.

For this purpose, putting $x = z$, $y = v$ in (2.3), we get

$$1 \leq \phi\,(F(Az, Bv, qt), F(Sz, Tv, t), F(Az, Sz, t),$$
$$F(Bv, Tv, qt), F(Az, Tv, t), F(Bv, Sz, (q+1)t))$$
$$\leq \phi\,(F(Az, Bv, qt), F(Sz, Tv, t), F(Az, Sz, t), F(Bv, Tv, qt)$$
$$, F(Az, Tv, t), \Delta(F(Bv, Az, qt), F(Az, Sz, t)).$$

Again we note that

$$\phi\,(F(Az, Bv, qt), F(Sz, Tv, t), 1, 1,$$
$$F(Az, Tv, t), F(Bv, Az, qt)) \geq 1,$$

by $\phi \in \boldsymbol{\Psi}$, we have $F(Az, Bv, qt)) \geq F(Sz, Tv, t)$, for all $t > 0$, by Lemma 1.17, we get $Az = Bv$. Hence $Az = Bv = z$. Hence $z = Az = Sz$ and z is a common fixed point of A and S. One can prove that $Bv=z$ is also a common fixed point of B and T.

Finally, in order to prove the uniqueness, suppose that w ($z \neq w$) be another fixed point of A, B, S and T. Then, for all $t > 0$, we have

$$1 \leq \phi(F(Az, Bw, qt), F(Sz, Tw, t), F(Az, Sz, t),$$
$$F(Bw, Tw, qt), F(Az, Tw, t), F(Bw, Sz, (q+1)t))$$
$$\leq \phi(F(Az, Bw, qt), F(Sz, Tw, t), F(Az, Sz, t),$$
$$F(Bw, Tw, qt), F(Az, Tw, t),$$
$$\Delta(F(Bw, Az, qt), F(Az, Az, t)).$$

Therefore we have

$$\phi(F(Az, Bw, qt), F(Sz, Tw, t), 1, 1,$$
$$F(Az, Tw, t), F(Bw, Az, qt)) \geq 1.$$

Hence we have $F(Az, Bw, qt)) \geq F(Sz, Tw, t)$ for all $t > 0$, by Lemma1.17, we get $Az = Bw$. Hence $z = w$. This completes the proof.

3. E.A. property and weakly compatible maps.

Now, we prove a fixed point theorem for weakly compatible maps with E.A. property.

Theorem 3.1. Let (X, F, Δ) be complete Menger space with continuous t-norm of Hadzic type. Let A, B, S and T be self mapping on X satisfying (2.1)-(2.4) and the following condition:

(3.1) the pairs (A, S) or (B, T) satisfy E.A. property.

Then A, B, S and T have a unique common fixed point in X.

Proof. Without loss of generality, we assume that the pair (B, T) satisfies the E.A. property. Then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in X$. Since $B(X) \subset S(X)$, there exists a sequence $\{y_n\}$ in X such that

$Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sy_n = z$. Also $A(X) \subset T(X)$, there exists a sequence $\{y'_n\}$ in X such that $Ay'_n = Tx_n$. Hence $\lim_{n\to\infty} Ay'_n = z$.

Suppose that S(X) is a closed subset of X. Then z = Su for some u ∈X. Subsequently, we have

$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ay'_n = \lim_{n\to\infty} Sy_n = z = Su,$

for some u ∈X.

Next, we claim that Au = Su.

For this purpose, if we put x= u and y= x_n in (2.3), then this gives

$$1 \le \phi\,(F(Au, Bx_n, qt), F(Su, Tx_n, t), F(Au, Su, t),$$
$$F(Bx_n, Tx_n, qt), F(Au, Tx_n, t), F(Bx_n, Su, (q+1)t))$$
$$\le \phi(F(Au, Bx_n, qt), F(Su, Tx_n, t), F(Au, Su, t),$$
$$F(Bx_n, Tx_n, qt), F(Au, Tx_n, t),$$
$$\Delta(F(Bx_n, Au, qt), F(Au, Su, t))$$

since the function ϕ is non-increasing in the 6-th coordinate variable.

Taking limit as $n \to \infty$, we have

$$\phi(F(Au, z, qt), F(z, z, t), F(Au, z, t), F(z, z, qt),$$
$$F(Au, z, t), \Delta(F(Au, z, qt), F(Au, z, t)) \ge 1.$$

Thus we obtain

$$\phi(F(Au, z, qt), 1, F(Au, z, t), 1, F(Au, z, t),$$
$$\Delta(F(Au, z, qt), F(Au, z, t)) \ge 1,$$

by $\phi \in \boldsymbol{\Psi}$, we have $F(Au, z, qt) \ge F(Au, z, t)$ for all $t > 0$, by Lemma 1.17, we have Au= Su =z. Since $A(X) \subset T(X)$, therefore there exists a point $v \in X$ such that Au= z=Tv.

Next, we claim that Tv = Bv.

Putting x = u and y= v in (2.3), we have

$$1 \le \phi(F(Au, Bv, qt), F(Su, Tv, t), F(Au, Su, t),$$
$$F(Bv, Tv, qt), F(Au, Tv, t), F(Bv, Su, (q+1)t))$$
$$= \phi(F(Au, Bv, qt), F(z, z, t), F(z, z, t),$$
$$F(Bv, Tv, qt), F(z, z, t), F(Bv, Su, (q+1)t)).$$
$$\le \phi(F(Au, Bv, qt), 1, 1, F(Bv, Tv, qt), 1,$$
$$\Delta(F(Bv, Au, qt), F(Au, Su, t))).$$

Therefore we have

$$\phi(F(Au, Bv, qt), 1, 1, F(Bv, Tv, qt), 1, F(Bv, Au, qt)) \ge 1.$$

Hence we get $F(Bv, Tv, qt)) \ge 1$ for all $t > 0$ implies that Tv = Bv. Thus Au = Su = Tv = Bv = z. Since the pairs (A, S) and (B, T) are weakly compatible and u and v are their coincidence points respectively, we obtain $Az = A(Su) = S(Au) = Sz$ and $Bz = B(Tv) = T(Bv) = Tz$.

Now, we prove that z is a common fixed point of A,B, S and T.

For this purpose, if put x = z and y = v in (2.3), then this gives

$$1 \leq \phi\,(F(Az, Bv, qt), F(Sz, Tv, t), F(Az, Sz, t),$$
$$F(Bv, Tv, qt), F(Az, Tv, t), F(Bv, Sz, (q+1)t))$$
$$\leq \phi\,(F(Az, Bv, qt), F(Sz, Tv, t), F(Az, Sz, t),$$
$$F(Bv, Tv, qt), F(Az, Tv, t),$$
$$\Delta(F(Bv, Az, qt), F(Az, Sz, t))).$$

Therefore we have

$$\phi(F(Az, Bv, qt), F(Sz, Tv, t), 1, 1, F(Az, Tv, t), F(Bv, Az, qt)) \geq 1.$$

Hence we get $F(Az, Bv, qt)) \geq 1$ for all $t > 0$ implies that $Az = Bv$. Hence $Az = Bv = z$. Therefore $z = Az = Sz$ and z is a common fixed point of A and S. One can prove that $Bv=z$ is also a common fixed point of B and T.

Finally, in order to prove the uniqueness, let w $(z \neq w)$ be another fixed point of A, B, S and T.Then, for all $t > 0$, we have

$$1 \leq \phi\,(F(Az, Bw, qt), F(Sz, Tw, t), F(Az, Sz, t),$$
$$F(Bw, Tw, qt), F(Az, Tw, t), F(Bw, Sz, (q+1)t))$$
$$\leq \phi\,(F(Az, Bw, qt), F(Sz, Tw, t), F(Az, Sz, t),$$
$$F(Bw, Tw, qt), F(Az, Tw, t),$$
$$\Delta(F(Bw, Az, qt), F(Az, Az, t)).$$

Therefore we have

$$\phi\,(F(Az, Bw, qt), F(Sz, Tw, t), 1, 1, F(Az, Tw, t), F(Bw, Az, qt)) \geq 1,$$

Hence we get $F(Az, Bw, qt) \geq 1$ for all $t > 0$ implies that $Az = Bw$ i.e., $z = w$. This completes the proof.

4. (CLR) property and weakly compatible maps

Now we prove a result for weakly compatible maps along with (CLR_S) property.

Theorem 4.1. Let (X, F, Δ) be Menger space with continuous t-norm of Hadzic type. Let A, B, S and T be self mapping on X satisfying (2.1), (2.2), (2.3) and the following conditions:

(4.1) pairs (A, S) or (B, T) satisfy (CLR_S) property,

(4.2) One of the subspaces A(X), B(X), S(X) or T(X) is a closed subspace.

Then A, B, S, and T have a unique common fixed point in X.

Proof. If the pair (A,S) satisfies the (CLR_S) property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, where $z \in S(X)$. Therefore there exists a point $u \in X$ such that $Su = z$. Since $T(X)$ is a closed subset of X and $A(X) \subset T(X)$, so for each $\{x_n\}$ in X, there corresponds a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore, $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z$, where $z \in S(X)$.

Thus we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z$.

Now, we are required to show that $\lim_{n\to\infty} By_n = z$.

Putting $x = x_n$ and $y = y_n$ in (2.3), we get

$$\phi\,(F(Ax_n, By_n, qt), F(Sx_n,Ty_n,t),F(Ax_n,Sx_n, t), F(By_n,Ty_n,qt)$$
$$,F(Ax_n,Ty_n,t), F(By_n,Sx_n,(q+1)t)) \geq 1.$$

We assume that $\lim_{n\to\infty} By_n = l \neq z$ for $t > 0$. Then taking limit as $n \to \infty$, we have

$$1 \leq \phi(F(z, l, qt), F(z, z, t),F(z, z, t),$$
$$F(l, z ,qt) ,F(z ,z ,t), F(l, z ,(q+1)t))$$
$$= \phi\,(F(z, l, qt), 1,1, F(l,z ,qt) ,1, \Delta(F(l,z ,qt), F(z, z ,t)),$$

since the function ϕ is non-increasing in the 6-th coordinate variable. Therefore, we have

$$\phi(F(z, l, qt), 1,1, F(l,z ,qt) ,1, (F(l,z ,qt)) \geq 1,$$

by $\phi \in \boldsymbol{\Psi}$, we get $F(z, l, qt) \geq 1$ implies that $z = l$, then hence $\lim_{n\to\infty} By_n = z$. Therefore

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z = Su, \text{ for some } u \in X.$$

Using the Theorem 2.1 and implicit relations $\boldsymbol{\Psi}$, we can easily prove that z is a unique common fixed point of A, B, S and T. This completes the proof.

Acknowledgement: One of the authors (S. Kumar) would like to acknowledge UGC for providing financial grant Major Research Project under Ref.(39-41/2010(SR).

REFERENCES

[1] M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 27 (2002), 181-188.

[2] O. Hadzic, Some theorems on the fixed point in probabilistic metric and random normed spaces, Bull. Unione Mat. Ital. (6), 1-B (1982), 381-391.

[3] O. Hadzic. A fixed point theorem in menger spaces. Publ.Inst. Math. Beograd, 20;107:112,1979.

[4] O. Hadzic and E.Pap. Fixed point theory in probabilistic metric spaces. Kluwer Academic Publishers, Dordrechi, 2001.

[5] T.L., Hicks, Fixed point theory in probabilistic metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod. Mat. Fak. Ser. Mat. 13 (1983), 63-72.

[6] K. Iseki, Fixed point theorems in 2-metric spaces, Math. Sem. Notes, Kobe Uni., 3 (1975), 133-136.

[7] M. Imdad and J. Ali, Jungck common fixed point theorems in fuzzy metric spaces ,

Mathematical Communication 11(2006), 153-163.

[8] M. Imdad and J. Ali, Jungck's common fixed point theorem and E.A. property, Acta Mathematica Sinica 24 (2008), 87-94.

[9] M. Imdad, B. D. Pant, S. Chauhan, Fixed point theorems in Menger spaces using The (CLR_{ST}) property and applications, Journal of Nonlinear Analysis and Optimization: Theory and Applications (in press).

[10] G. Jungck and B.E.Rhoades, Fixed points for set valued functions without continuity, Indian Journal of pure and applied mathematics, vol. 29, no. 3, pp. 227-238,1998.

[11] G. Jungck, Common fixed point for noncontinuous nonself mappings on non metric Spaces, Far East J. Math.Sci. 4(2) (1996), 199-212.

[12] S.Jain , B. Singh, A fixed point theorem in Menger space through weak compatibility ,J. Math. Anal.Appl. 301(2005),439.

[13] J. K. Kohli and S. Vashistha, Common Fixed Point Theorems in Probabilistic Metric Spaces, Acta Mathematica Hungarica, Vol. 115, No. 1-2, 2007, pp. 37-47.

[14] S.N. Mishra, Common fixed points of compatible mappings in probabilistic metric Spaces, Math. Japon. 36(1991),283.

[15] H. K. Pathak, Y.J. Cho and S.M. Kang, Remarks on R-weakly commuting mappings and common fixed point theorems, Bull. Korean Math. Soc.34(1997),247-257.

[16] V.M. Sehgal, A. Bharucha-Reid, T., Fixed points of contraction mappings in probabilistic metric spaces, Math. Systems Theory 6 (1972), 97-102.

[17] Shi-Sen Chang, Nan-Jing Huan, On the generalized 2-metric spaces andprobabilistic 2-metric spaces with applications to fixed point theory, Math.Japon. 34 (1989), 885-900.

[18] Y.Shi, L.Ren and X.Wang, The extension of fixed point theorems for set valued mappings, J. Appl.Math. Computing, 13 (2003), 277-286.

[19] W. Sintunavarat and P. Kumam, Common fixed point theorems for a pair of weakly Compatible mappings in fuzzy metric spaces, Journal of Applied Mathematics, vol. 2011, Article ID 637958, 14 pages, 2011.

[20] W. Sintunavarat and P. Kumam, Common fixed point theorems for R- weakly commuting in fuzzy metric spaces, Annali dell’Universita di Ferrara, DOI: 10.1007/s11565-012-0150-z (in press).

[21] Zeng Wenzhi, Probabilistic 2- metric spaces, J. Math. Research Expo. 2 (1987), 241-245.