A New Class of Regular Weakly Closed Sets and Functions Using Grills

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ABSTRACT. The purpose of this paper is to introduce and study a new class of regular weakly closed sets and functions in a topological space X, defined in terms of a grill $\mathcal{G}$ on X. Explicit characterization of such sets along with certain other properties of them are obtained. As applications, some characterizations of regular and normal spaces are achieved by use of the introduced class of sets.

Key words and phrases: Grill, rw-closed, topology $\tau_{\mathcal{G}}$, operator Φ, $\mathcal{G}$-rw-closed sets, $\mathcal{G}$-rw-continuity, $\mathcal{G}$-rw-regular, $\mathcal{G}$-rw-normal.

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1. Introduction and Preliminaries

It is found from literature that during recent years many topologists are interested in the study of generalized types of closed sets. For instance, a certain form of generalized closed sets was initiated by Levine [9], whereas the notion of regular weakly closed sets was studied by Wali [16]. Following the trend, we have introduced and investigated a kind of generalized closed sets, the definition being formulated in terms of grills. The concept of grill was first introduced by Choquet [4] in 1947. From subsequent investigations it is revealed that grills can be used as an extremely useful device for investigation of a number of topological problems, like extension of spaces, theory of proximity spaces and so on (see for instance, [2], [3], [14] for details). The definition of grill goes as follows.

Definition 1.1. [4] A nonempty collection $\mathcal{G}$ of nonempty subsets of a topological space X is called a grill if
(i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$, and (ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Let $\mathcal{G}$ be a grill on a topological space (X, τ). In [11] an operator $\Phi : P(X) \to P(X)$ (where $P(X)$ stands for the power set of X) was defined by $\Phi(A) = \{x \in X : U \cap A \in \mathcal{G}$ for all open set U containing $x\}$. It was also shown in the same paper that the map $\Psi : P(X) \to P(X)$, given by $\Psi(A) = A \cup \Phi(A)$ for all $A \in P(X)$). Corresponding to a grill $\mathcal{G}$, on a topological space (X, τ) there exists a unique topology

$\tau_{\mathcal{G}}$ (say) on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X \backslash U) = X \backslash U\}$ where for any $A \subseteq X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}\text{-}cl(A)$. Thus a subset A of X is $\tau_{\mathcal{G}}$-closed (resp. $\tau_{\mathcal{G}}$-dense in itself) if $\Psi(A) = A$ or equivalently if $\Phi(A) \subseteq A$ (resp. $A \subseteq \Phi(A)$).

In the next section we introduce and study a new class of generalized closed sets, termed $\mathcal{G}\text{-}rw$-closed, in terms of a given grill $\mathcal{G}$ on the ambient space, the definition having a close bearing to the above operator Φ. This class of $\mathcal{G}\text{-}rw$-closed sets will be seen to properly contain the class of rw-closed sets as introduced in [16]. An explicit form of such a $\mathcal{G}\text{-}rw$-closed set is also obtained. In section 3, we introduce and investigate the notion of regular weakly continuous functions in grill topological spaces. Also, we investigate its relationship with other functions. As applications, some formulations of certain separation axioms in terms of $\mathcal{G}\text{-}rw$-closed sets and associated concepts will be established in section 4.

Throughout the paper, by a space X we always mean a topological space (X, τ) with no separation properties assumed. If $A \subseteq X$, we shall adopt the usual notations $int(A)$ and $cl(A)$ respectively for the interior and closure of A in (X, τ). Again, $\tau_{\mathcal{G}}\text{-}cl(A)$ and $\tau_{\mathcal{G}}\text{-}int(A)$ will respectively denote the closure and interior of A in $(X, \tau_{\mathcal{G}})$. Similarly, whenever we say that a subset A of a space X is open (or closed), it will mean that A is open (resp. closed) in (X, τ). For open and closed sets with respect to any other topology on X, e.g. $\tau_{\mathcal{G}}$, we shall write '$\tau_{\mathcal{G}}$'-open and '$\tau_{\mathcal{G}}$'-closed. The collection of all open neighbourhoods of a point x in (X, τ) will be denoted by $\tau(x)$. A subset A of a space (X, τ) is said to be regular open [13] (regular closed [13]) if $A = intcl(A)$ (resp. $A = clint(A)$). A subset A of a space (X, τ) is said to be regular semiopen [1] if there is a regular open set U such that $U \subseteq A \subseteq cl(U)$.

We now append a few definitions and results that will be frequently used in the sequel.

Definition 1.2. A subset A of a space (X, τ) is said to be rw-closed [16] ($\mathcal{G}\text{-}g$-closed [5]) if $cl(A) \subseteq U$ (resp. $\Phi(A) \subseteq U$) whenever $A \subseteq U$ and U is regular semiopen (resp. open) in X.

The complement of a rw-closed ($\mathcal{G}\text{-}g$-closed) set is called a rw-open (resp. $\mathcal{G}\text{-}g$-open) set in X.

Theorem 1.1. [11] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for any $A, B \subseteq X$ the following hold:
(a) $A \subseteq B \Rightarrow \Phi(A) \subseteq \Phi(B)$.
(b) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$.
(c) $\Phi(\Phi(A)) \subseteq \Phi(A) = cl(\Phi(A)) \subseteq cl(A)$, and hence $\Phi(A)$ is closed in (X, τ), for all $A \subseteq X$.

Definition 1.3. [15] A subset A of a topological space X is said to be θ-closed if $A = \theta cl(A)$ where $\theta cl(A)$ is defined as $\theta cl(A) = \{x \in X / cl(U) \cap A \neq \phi$ for every $U \in \tau$ and $x \in U\}$.

Definition 1.4. [15] A subset A of X is said to be θ-open if $X \backslash A$ is θ-closed.

Definition 1.5. [15] A subset A of a topological space X is said to be δ-closed if $A = \delta cl(A)$ where $\delta cl(A)$ is defined as $\delta cl(A) = \{x \in X / intcl(U) \cap A \neq \phi$ for every $U \in \tau$ and $x \in U\}$.

Definition 1.6. [15] A subset A of X is said to be δ-open if $X \backslash A$ is δclosed.

Definition 1.7. [7] A subset A of a topological space X is said to be θg-closed if $\theta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 1.8. [6] A subset A of a topological space X is said to be δg-closed if $\delta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 1.9. [6, 7] A subset A of X is said to be θg-open (δg-open) if $X \backslash A$ is θg-closed (δg-closed) in X.

Definition 1.10. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be
    (1) θ-continuous [8] if $f^{-1}(V)$ is θ-closed set of (X, τ) for every closed set V of (Y, σ).
    (2) δ-continuous if $f^{-1}(V)$ is δ-closed set of (X, τ) for every closed set V of (Y, σ).
    (3) θg-continuous if $f^{-1}(V)$ is θg-closed set of (X, τ) for every closed set V of (Y, σ).
    (4) δg-continuous if $f^{-1}(V)$ is δg-closed set of (X, τ) for every closed set V of (Y, σ).
    (5) rw-continuous [16] if $f^{-1}(V)$ is $\mathcal{G}$-rw-closed set of (X, τ) for every closed set V of (Y, σ).

Definition 1.11. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be
    (1) θ-closed [10] if $f(F)$ is θ-closed set of (Y, σ) for every closed set F of (X, τ).
    (2) δ-closed [10] if $f(F)$ is δ-closed set of (Y, σ) for every closed set F of (X, τ).
    (3) θg-closed if $f(F)$ is θg-closed set of (Y, σ) for every closed set F of (X, τ).
    (4) δg-closed if $f(F)$ is δg-closed set of (Y, σ) for every closed set F of (X, τ).
    (5) rw-closed [16] if $f(F)$ is $\mathcal{G}$-rw-closed set of (Y, σ) for every closed set F of (X, τ).

Definition 1.12. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $\mathcal{G}g$-continuous if $f^{-1}(V)$ is $\mathcal{G}g$-closed set of (X, τ) for every closed set V of (Y, σ).

Definition 1.13. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is said to be $\mathcal{G}g$-closed if $f(F)$ is $\mathcal{G}g$-closed set of (Y, σ) for every closed set F of (X, τ).

2. rw-**Closed Sets With Respect to a Grill**

We begin by introducing a new class of regular weakly closed sets in terms of grills as follows

Definition 2.1. Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then a subset A of X is said to be rw-closed with respect to the grill $\mathcal{G}$ ($\mathcal{G}$-rw-closed, for short) if $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semiopen in X.

A subset A of X is said to be $\mathcal{G}$-rw-open if $X \backslash A$ is $\mathcal{G}$-rw-closed.

Proposition 2.1. For a topological space (X, τ) and a grill $\mathcal{G}$ on X, we obtain as follows.
(a) Every closed set in X is $\mathcal{G}$-rw-closed.
(b) For any subset A in X, $\Phi(A)$ is $\mathcal{G}$-rw-closed.
(c) Every $\tau_\mathcal{G}$-closed set is $\mathcal{G}$-rw-closed.
(d) Any non member of $\mathcal{G}$ is $\mathcal{G}$-rw-closed.
(e) Every w-closed set is $\mathcal{G}$-rw-closed.
(f) Every rw-closed set is $\mathcal{G}$-rw-closed.
(g) Every θ-closed set in X is $\mathcal{G}$-rw-closed.
(h) Every δ-closed set in X is $\mathcal{G}$-rw-closed.

Proof. (a) Let A be a closed set then $cl(A) = A$. Let U be any regular semiopen set in $X \ni A \subseteq U$. Then $\Phi(A) \subseteq cl(A) = A \subseteq U$ [by Theorem 1.1] $\Rightarrow \Phi(A) \subseteq U \Rightarrow A$ is $\mathcal{G}$-rw-closed.

(b) Let A be a subset in X. Then $\Phi(\Phi(A)) \subseteq \Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semiopen in $X \Rightarrow \Phi(A)$ is $\mathcal{G}$-rw-closed.

(c) Let A be a $\tau_\mathcal{G}$-closed set then $\tau_\mathcal{G}$-$cl(A) = A \Rightarrow A \cup \Phi(A) = A \Rightarrow \Phi(A) \subseteq A$. Therefore $\Phi(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open in $X \Rightarrow A$ is $\mathcal{G}$-rw-closed.

(d) Let $A \notin \mathcal{G}$ then $\Phi(A) = \phi \Rightarrow A$ is $\mathcal{G}$-rw-closed.

(e) Let A be a w-closed set and U be any regular semiopen set in $X \ni A \subseteq U$ then $cl(A) \subseteq U$, since every regular semiopen set is semiopen in X. Then $\Phi(A) \subseteq cl(A) \subseteq U \Rightarrow A$ is $\mathcal{G}$-rw-closed. Thus every w-closed set is $\mathcal{G}$-rw-closed.

(f) Let A be a rw-closed set and U be any regular semiopen set in $X \ni A \subseteq U$ then $cl(A) \subseteq U$, by Theorem 1.1 $\Phi(A) \subseteq cl(A) \subseteq U \Rightarrow A$ is $\mathcal{G}$-rw-closed. Thus every rw-closed set is $\mathcal{G}$-rw-closed.

(g) Let A be a θ-closed then $A = \theta cl(A)$. Let U be a regular semiopen set in X such that $A \subseteq U$ then by Theorem 1.1, $\Phi(A) \subseteq cl(A) \subseteq \theta cl(A) = A \subseteq U$. Thus A is $\mathcal{G}$-rw-closed.

(h) Let A be a δ-closed then $A = \delta cl(A)$. Let U be a regular semiopen set in X such that $A \subseteq U$ then by Theorem 1.1, $\Phi(A) \subseteq cl(A) \subseteq \delta cl(A) = A \subseteq U$. Thus A is $\mathcal{G}$-rw-closed. $\qquad \square$

The converse of the above proposition is not true in general as seen from the following examples.

Example 2.1. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then (X, τ) is a topological space and $\mathcal{G}$ is a grill on X. Then it is easy to verify that
(a) $\{a, b\}$ is not closed but is $\mathcal{G}$-rw-closed.
(b) $\{a, b\}$ is not $\tau_\mathcal{G}$-closed but is $\mathcal{G}$-rw-closed.
(c) $\{c, d\}$ is not a grill but is $\mathcal{G}$-rw-closed.
(d) $\{a, b\}$ is not w-closed but is $\mathcal{G}$-rw-closed.
(e) $\{b\}$ is not rw-closed but is $\mathcal{G}$-rw-closed.

Example 2.2. Let $X = \{a, b, c\}, \tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\},$ $\{a, b\}, \{b, c\}, X\}$. Then (X, τ) is a topological space and $\mathcal{G}$ is a grill on X. Then it is easy to verify that $\{a, b\}$ is not θ-closed (resp. δ-closed) but is $\mathcal{G}$-rw-closed.

Remark 2.1. The following Examples shows that the concept of $\mathcal{G}$-rw-closed sets and $\mathcal{G}$-g-closed sets, θg-closed sets, δg-closed sets are indepentent of each other.

Example 2.3. In the Example 2.1, the set $\{a, d\}$ is $\mathcal{G}$-g-closed set, θg-closed set, δg-closed set but not a $\mathcal{G}$-rw-closed set. Also the set $\{a, b\}$ is $\mathcal{G}$-rw-closed set but not a $\mathcal{G}$-g-closed set, not a θg-closed set, not a δg-closed set in X.

Remark 2.2. From the above discussions and known results we have the following implications. Here
$A \rightarrow B$ means A implies B, but not conversely and
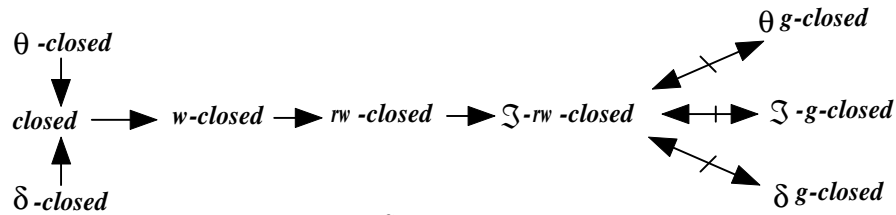$A \nrightarrow B$ means A and B are independent of each other.

fig-1

Corresponding to any nonempty subset A of X, a typical grill $[A]$ on X was defined in [12] in the following manner.

Definition 2.2. Let X be a space and $(\phi \neq)A \subseteq X$. Then $[A] = \{B \subseteq X : A \cap B \neq \phi\}$ is a grill on X, called the principal grill generated by A.

Proposition 2.2. In the case of principal grill $[X]$ generated by X, it is known [12] that $\tau = \tau_{[X]}$ so that any $[X]$-rw-closed set becomes simply a rw-closed set and vice-versa.

In what follows in this section, we derive certain characterizations and properties of $\mathcal{G}$-rw-closed sets.

Theorem 2.1. Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for a subset A of X, the following are equivalent:
(a) A is $\mathcal{G}$-rw-closed.
(b) $\tau_{\mathcal{G}}$-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular semi-open.
(c) For all $x \in \tau_{\mathcal{G}}$-$cl(A)$, $cl(\{x\}) \cap A \neq \phi$.
(d) $\tau_{\mathcal{G}}$-$cl(A) \backslash A$ contains no non-empty closed set of (X, τ).
(e) $\Phi(A) \backslash A$ contains no non-empty closed set of (X, τ).

Proof. (a) $\Rightarrow$ (b): Suppose A is $\mathcal{G}$-rw-closed set and $A \subseteq U$ where U is regular semi-open in X. Then $\Phi(A) \subseteq U \Rightarrow A \cup \Phi(A) \subseteq U \Rightarrow \tau_{\mathcal{G}}$-$cl(A) \subseteq U$.

(b) $\Rightarrow$ (c): Suppose $x \in \tau_{\mathcal{G}}$-$cl(A)$. If $cl(\{x\}) \cap A = \phi$, then $A \subseteq X \backslash cl(\{x\})$ and using (b), $\tau_{\mathcal{G}}$-$cl(A) \subseteq X \backslash cl(\{x\})$ which is a contradiction to our assumption that $x \in \tau_{\mathcal{G}}$-$cl(A)$. Therefore $cl(\{x\}) \cap A \neq \phi$.

(c) $\Rightarrow$ (d): Suppose F is a closed set of (X, τ) contained in $\tau_{\mathcal{G}}$-$cl(A)\backslash A$ and $x \in F$. Since $F \cap A = \phi$, we have $cl(\{x\}) \cap A = \phi$. Again since $x \in \tau_{\mathcal{G}}$-$cl(A)$, by (c) we have $cl(\{x\}) \cap A \neq \phi$, a contradiction. This proves (d).

(d) $\Rightarrow$ (e): It follows from the fact that $\Phi(A)\backslash A = \tau_{\mathcal{G}}$-$cl(A)\backslash A$.

(e) $\Rightarrow$ (a): Suppose that $A \subseteq U$ and U is regular semiopen in (X, τ). Since $\Phi(A)$ is closed (by Theorem 1.1) and $\Phi(A) \cap (X\backslash U) \subseteq \Phi(A)\backslash A$ holds, $\Phi(A) \cap (X\backslash U)$ is a closed set in (X, τ) contained in $\Phi(A)\backslash A$. Then by (e), $\Phi(A) \cap (X\backslash U) = \phi$ which gives $\Phi(A) \subseteq U$. Hence A is $\mathcal{G}$-rw-closed. $\qquad\square$

Corollary 2.1. Let (X, τ) be a T_1-space and $\mathcal{G}$ be a grill on X. Then every $\mathcal{G}$-rw-closed set is $\tau_{\mathcal{G}}$-closed.

Proof. Follows from Theorem 2.1 ((a) $\Rightarrow$ (c)). $\qquad\square$

Corollary 2.2. Let (X, τ) be a T_1-space and $\mathcal{G}$ be a grill on X. Then $A(\subseteq X)$ is $\mathcal{G}$-rw-closed iff A is $\tau_{\mathcal{G}}$-closed.

Corollary 2.3. Let $\mathcal{G}$ be grill on a space (X, τ) and A be a $\mathcal{G}$-rw-closed set. Then the following are equivalent
(a) A is $\tau_{\mathcal{G}}$-closed.
(b) $\tau_{\mathcal{G}}$-$cl(A)\backslash A$ is closed in (X, τ).
(c) $\Phi(A)\backslash A$ is closed in (X, τ).

Proof. $(a) \Rightarrow (b)$ Let A be $\tau_{\mathcal{G}}$-closed then $\tau_{\mathcal{G}}$-$cl(A)\backslash A = \phi$ and so $\tau_{\mathcal{G}}$-$cl(A)\backslash A$ is a closed set.
$(b) \Rightarrow (c)$ It is clear, since $\tau_{\mathcal{G}}$-$cl(A)\backslash A = \Phi(A)\backslash A$.
$(c) \Rightarrow (a)$ Let $\Phi(A)\backslash A$ be closed in (X, τ) and A is $\mathcal{G}$-rw-closed, then by Theorem 2.1, $\Phi(A)\backslash A = \phi$ and so A is $\tau_{\mathcal{G}}$-closed. $\qquad\square$

Lemma 2.1. [11] Let (X, τ) be a space and $\mathcal{G}$ be a grill on X. If $A(\subseteq X)$ is $\tau_{\mathcal{G}}$-dense in itself, then $\Phi(A) = cl(\Phi(A)) = \tau_{\mathcal{G}}$-$cl(A) = cl(A)$.

Theorem 2.2. Let $\mathcal{G}$ be a grill on a space (X, τ). If $A(\subseteq X)$ is $\tau_{\mathcal{G}}$-dense in itself and $\mathcal{G}$-rw-closed, then A is rw-closed.

Proof. Follows at once from Lemma 2.1 $\qquad\square$

Corollary 2.4. For a grill $\mathcal{G}$ on a space (X, τ), let $A(\subseteq X)$ be $\tau_{\mathcal{G}}$-dense in itself. Then A is $\mathcal{G}$-rw-closed iff it is rw-closed.

Proof. Follows from Proposition 2.1(f) and Theorem 2.2. $\qquad\square$

Theorem 2.3. For any grill $\mathcal{G}$ on a space (X, τ) the following are equivalent
(a) Every subset of X is $\mathcal{G}$-rw-closed.
(b) Every regular semiopen subset of (X, τ) is $\tau_{\mathcal{G}}$-closed.

Proof. (a) $\Rightarrow$ (b) Let A be regular semiopen in (X, τ) then by (a), A is $\mathcal{G}$-rw-closed so that $\Phi(A) \subseteq A \Rightarrow A$ is $\tau_{\mathcal{G}}$-closed.

(b) $\Rightarrow$ (a) Let $A \subseteq X$ and U be regular semiopen in (X, τ) such that $A \subseteq U$. Then by (b), $\Phi(U) \subseteq U$. Again $A \subseteq U \Rightarrow \Phi(A) \subseteq \Phi(U)$ (by Theorem 1.1) $\subseteq U \Rightarrow A$ is $\mathcal{G}$-rw-closed. $\qquad\square$

Proposition 2.3. For any subset A of a space (X, τ) and a grill $\mathcal{G}$ on X, the following are equivalent
(a) A is $\mathcal{G}$-rw-closed.
(b) $A \cup (X \backslash \Phi(A))$ is $\mathcal{G}$-rw-closed.
(b) $\Phi(A) \backslash A$ is $\mathcal{G}$-rw-open.

Proof. $(a) \Rightarrow (b)$: Let $A \cup (X \backslash \Phi(A)) \subseteq U$, where U is regular semiopen in X. Then $X \backslash U \subseteq X \backslash (A \cup (X \backslash \Phi(A))) = \Phi(A) \backslash A$. Since A is $\mathcal{G}$-rw-closed, by Theorem 2.1, we have $X \backslash U = \phi$, i.e., $X = U$. Since X is the only regular semiopen set containing $A \cup (X \backslash \Phi(A))$, $A \cup (X \backslash \Phi(A))$ is $\mathcal{G}$-rw-closed.
    $(b) \Rightarrow (a)$: Suppose $F \subseteq \Phi(A) \backslash A$ where F is regular semi closed in (X, τ). Then $A \cup (X \backslash \Phi(A)) \subseteq X \backslash F$ and so by (b), $\Phi(A \cup (X \backslash \Phi(A))) \subseteq X \backslash F \Rightarrow \Phi(A) \cup \Phi(X \backslash \Phi(A)) \subseteq X \backslash F \Rightarrow F \subseteq X \backslash \Phi(A)$. Again, since $F \subseteq \Phi(A)$ we have $F = \phi$. Hence by Theorem 2.1, A is $\mathcal{G}$-rw-closed.
    $(b) \Rightarrow (c)$: Follows from the fact that $X \backslash (\Phi(A) \backslash A) = A \cup (X \backslash \Phi(A))$. $\qquad\square$

Theorem 2.4. Let (X, τ) be a space, $\mathcal{G}$ be a grill on X and A, B be subsets of X such that $A \subseteq B \subseteq \tau_{\mathcal{G}}$-$cl(A)$. If A is $\mathcal{G}$-rw-closed, then B is $\mathcal{G}$-rw-closed.

Proof. Suppose $B \subseteq U$, where U is regular semiopen in X. Since A is $\mathcal{G}$-rw-closed, $\Phi(A) \subseteq U \Rightarrow \tau_{\mathcal{G}}$-$cl(A) \subseteq U$. Now, $A \subseteq B \subseteq \tau_{\mathcal{G}}$-$cl(A) \Rightarrow \tau_{\mathcal{G}}$-$cl(A) \subseteq \tau_{\mathcal{G}}$-$cl(B) \subseteq \tau_{\mathcal{G}}$-$cl(A)$. Thus $\tau_{\mathcal{G}}$-$cl(B) \subseteq U$ and hence B is $\mathcal{G}$-rw-closed. $\qquad\square$

Corollary 2.5. $\tau_{\mathcal{G}}$-closure of every $\mathcal{G}$-rw-closed set is $\mathcal{G}$-rw-closed.

Theorem 2.5. Let $\mathcal{G}$ be a grill on a space (X, τ) and A, B be subsets of X such that $A \subseteq B \subseteq \Phi(A)$. If A is $\mathcal{G}$-rw-closed. Then A and B are rw-closed.

Proof. $A \subseteq B \subseteq \Phi(A) \Rightarrow A \subseteq B \subseteq \tau_{\mathcal{G}}$-$cl(A)$, and hence by Theorem 2.4, B is $\mathcal{G}$-rw-closed. Again, $A \subseteq B \subseteq \Phi(A) \Rightarrow \Phi(A) \subseteq \Phi(B) \subseteq \Phi(\Phi(A)) \subseteq \Phi(A)$ (by Theorem 1.1) $\Rightarrow \Phi(A) = \Phi(B)$. Thus A and B are $\tau_{\mathcal{G}}$-dense in itself and hence by Theorem 2.2, A and B are rw-closed. $\qquad\square$

Theorem 2.6. Let $\mathcal{G}$ be a grill on a space (X, τ). Then a subset A of X is $\mathcal{G}$-rw-open iff $F \subseteq \tau_{\mathcal{G}}$-$int(A)$ whenever $F \subseteq A$ and F is closed.

Proof. Let A be $\mathcal{G}$-rw-open and $F \subseteq A$, where F is closed in (X, τ). Then $X \backslash A \subseteq X \backslash F \Rightarrow \Phi(X \backslash A) \subseteq X \backslash F \Rightarrow \tau_{\mathcal{G}}$-$cl(X \backslash A) \subseteq X \backslash F \Rightarrow F \subseteq \tau_{\mathcal{G}}$-$int(A)$.
    Conversely, $X \backslash A \subseteq U$ where U is open in $(X, \tau) \Rightarrow X \backslash U \subseteq \tau_{\mathcal{G}}$-$int(A) \Rightarrow \tau_{\mathcal{G}}$-$cl(X \backslash A) \subseteq U$. Thus $X \backslash A$ is $\mathcal{G}$-rw-closed and hence A is $\mathcal{G}$-rw-open. $\qquad\square$

3. $\mathcal{G}$-rw-Continuous and $\mathcal{G}$-rw-Closed Functions

Definition 3.1. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $\mathcal{G}$-rw-continuous (resp. rw-continuous [16]) if $f^{-1}(V)$ is $\mathcal{G}$-rw-open (resp. rw-open) for each $V \in \sigma$.

Remark 3.1.   (i) Every continuous function (resp. w-continuous) is rw-continuous, but the converse is false as is shown in Examples 3.2.3 and 3.2.6 in [16].

(ii) Every rw-continuous function is $\mathcal{G}$-continuous, but the converse is false as is shown in Example 3.1.

But the reverses of the above implications are false as is shown below.

Example 3.1. In the Example 2.1, we define a function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ as follows: $f(a) = c$, $f(b) = d$, $f(c) = a$ and $f(d) = b$. Then it is easy to see that f is $\mathcal{G}$-rw-continuous but not rw-continuous (in fact, $A = \{d\} \in \tau^c$ and $f^{-1}(\{d\}) = \{b\}$ is not rw-closed).

Remark 3.2. Every θ-continuous (resp. δ-continuous) is $\mathcal{G}$-rw-continuous, but the converse is false as is shown in Example 3.2.

Example 3.2. Let $X = \{a,\ b,\ c\}$, $\tau = \{X,\ \phi,\ \{a\},\ \{b\},\ \{a,\ b\},\ \{b,\ c\}\}$ and $\mathcal{G} = \{\{a\},\ \{c\},\ \{a,\ c\},\ \{a,\ b\},\ \{b,\ c\},\ X\}$. Then the identity function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ is $\mathcal{G}$-rw-continuous but not θ-continuous (resp. δ-continuous) in fact, $A = \{c\} \in \tau^c$ and $f^{-1}(\{c\}) = \{c\}$ is not θ-closed (resp. δ-closed).

Remark 3.3. In the following Examples (3.3 and 3.4) show that $\mathcal{G}$-rw-continuous function and $\mathcal{G}$-g-continuous, θg-continuous, δg-continuous functions are indepentent.

Example 3.3. In the Example 2.1, we define a function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ as follows: $f(a) = c$, $f(b) = b$, $f(c) = a$ and $f(d) = d$. Then the inverse image of every closed set in Y is $\mathcal{G}$-g-closed, θg-closed, δg-closed in X and hence f is $\mathcal{G}$-g-continuous, θg-continuous, δg-continuous. But f is not $\mathcal{G}$-rw-continuous as the inverse image of the closed set $\{c, d\}$ in X is $\{a, d\}$ in X which is not $\mathcal{G}$-rw-closed.

Example 3.4. In the Example 2.1, we define a function $f : (X, \tau, \mathcal{G}) \to (X, \tau)$ as follows: $f(a) = c$, $f(b) = d$, $f(c) = a$ and $f(d) = b$. Then the inverse image of every closed set in Y is $\mathcal{G}$-rw-closed in X and hence f is $\mathcal{G}$-rw-continuous. Let $\{c, d\}$ is closed set in Y, $f^{-1}(\{c, d\}) = \{a, b\}$ is not $\mathcal{G}$-g-closed, θg-closed, δg-closed in X. Thus f is not $\mathcal{G}$-g-continuous, θg-continuous, δg-continuous.

Remark 3.4. From the above discussions and known results we have the following implications. Here
$A \to B$ means A implies B, but not conversely and
$A \nleftrightarrow B$ means A and B are independent of each other.

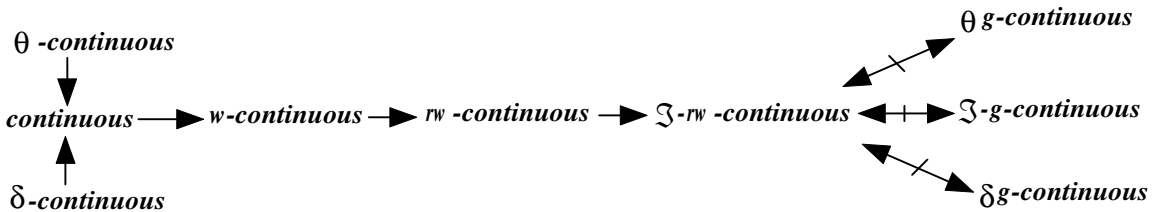*fig-2*

Theorem 3.1. For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following are equivalent:

(a) f is $\mathcal{G}$-*rw*-continuous.
(b) The inverse image of each closed subset of Y is $\mathcal{G}$-*rw*closed.
(c) For each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists a $\mathcal{G}$-*rw*-open set U containing x such that $f(U) \subseteq V$.

Proof. (a) $\Leftrightarrow$ (b): It is clear.
(a) $\Rightarrow$ (c): Let $V \in \sigma$ and $f(x) \in V (x \in X)$. Then by (a), $f^{-1}(V)$ is a $\mathcal{G}$-*rw*-open set containing x. Taking $f^{-1}(V) = U$, we have $x \in U$ and $f(U) \subseteq V$.
(c) $\Rightarrow$ (a): Let V be any open set in Y and $x \in f^{-1}(V)$. Then $f(x) \in V \in \sigma$ and hence by (c), there exists a $\mathcal{G}$-*rw*-open set U containing x such that $f(U) \subseteq V$. Now $x \in U \subseteq \Psi(int(U)) \subseteq \Psi(int(f^{-1}(V)))$. This shows that $f^{-1}(V) \subseteq \Psi(int(f^{-1}(V)))$. Thus f is $\mathcal{G}$-*rw*-continuous. $\square$

Theorem 3.2. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is $\mathcal{G}$-*rw*-continuous iff the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$, for each $x \in X$, is $\mathcal{G}$-*rw*-continuous.

Proof. Suppose that f is $\mathcal{G}$-*rw*-continuous. Let $x \in X$ and W be any open set in $X \times Y$ containing $g(x)$. Then there exist $U \in \tau$ and $V \in \sigma$ such that $g(x) = (x, f(x)) \in U \times V \subseteq W$. Since f is $\mathcal{G}$-*rw*-continuous, there exists a $\mathcal{G}$-*rw*-open set G of X containing x such that $f(G) \subseteq V$, $G \cap U$ is $\mathcal{G}$-*rw*-open and $g(G \cap U) \subseteq U \times V \subseteq W$. This shows that g is $\mathcal{G}$-*rw*-continuous.

Conversely, suppose that g is $\mathcal{G}$-*rw*-continuous. Let $x \in X$ and V be any open set in Y containing $f(x)$. Then $X \times V$ is open in $X \times Y$ and by $\mathcal{G}$-*rw*-continuity of g, there exists a $\mathcal{G}$-*rw*-open set U containing x such that $g(U) \subseteq X \times V$. Thus we have $f(U) \subseteq V$ and hence f is $\mathcal{G}$-*rw*-continuous. $\square$

Definition 3.2. *Let (X, τ) be a topological space and $(Y, \sigma, \mathcal{G})$ a grill topological space. A function $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ is said to be $\mathcal{G}$-rw-open (resp. $\mathcal{G}$-rw-closed) if for each $U \in \tau$ (resp. closed set U in (X, τ)), $f(U)$ is $\mathcal{G}$-rw-open (resp. $\mathcal{G}$-rw-closed) in $(Y, \sigma, \mathcal{G})$.*

Remark 3.5. (a) Every closed (resp. *w*-closed) function is *rw*-closed, but the converse is false as is shown in Examples 3.4.3 and 3.4.4 [16].
(b) Every *rw*-closed function is $\mathcal{G}$-*rw*-closed, but the converse is false as is shown in Example 3.5.

Example 3.5. Let $X = \{a, b, c, d\}$, $\tau = \{X, \phi, \{a, c, d\}\}$, $\sigma = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma, \mathcal{G})$ is $\mathcal{G}$-*rw*-closed, but not *rw*-closed.

Remark 3.6. Every θ-closed (resp. δ-closed) function is $\mathcal{G}$-*rw*-closed, but the converse is false as is shown in Example 3.6.

Example 3.6. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\mathcal{G} = \{\{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \tau, \mathcal{G})$ is $\mathcal{G}$-*rw*-closed but it is not θ-closed (resp. δ-closed).

Remark 3.7. In the following Examples (3.7 and 3.8) show that $\mathcal{G}$-*rw*-closed function and $\mathcal{G}$-*g*-closed, θg-closed, δg-closed functions are indepentent.

**Example 3.7.** In the Example 2.1, we define a function $f : (X, \tau) \to (X, \tau, \mathcal{G})$ as follows: $f(a) = c$, $f(b) = b$, $f(c) = a$ and $f(d) = d$. Then this function is $\mathcal{G}$-*g*-closed, θg-closed, δg-closed but not $\mathcal{G}$-*rw*-closed, as the image of the closed set $\{c, d\}$ in X is $\{a, d\}$ which is not $\mathcal{G}$-*rw*-closed in X.

Example 3.8. In the Example 2.1, we define a function $f : (X, \tau) \to (X, \tau, \mathcal{G})$ as follows: $f(a) = c$, $f(b) = d$, $f(c) = a$ and $f(d) = b$. Then the image of every closed set in X is $\mathcal{G}$-*rw*-closed in X and hence f is $\mathcal{G}$-*rw*-closed function. Let $\{c, d\}$ is closed set in X, $f(\{c, d\}) = \{a, b\}$ is not $\mathcal{G}$-*g*-closed, θg-closed, δg-closed in X. Thus f is not $\mathcal{G}$-*g*-closed, θg-closed, δg-closed functions.

Remark 3.8. From the above discussions and known results we have the following implications. Here
$A \to B$ means A implies B, but not conversely and
$A \nleftrightarrow B$ means A and B are independent of each other.

θ -closed function

closed function ⟶ **w-closed function** ⟶ **rw -closed function** ⟶ **ℑ -rw -closed function**

δ-closed function   **fig-3**

θ g-closed function   **ℑ-g-closed function**
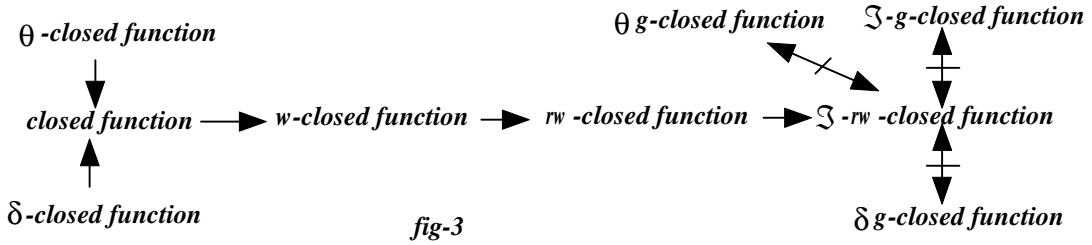
δ g-closed function

Theorem 3.3. Let $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ be a $\mathcal{G}$-*rw*-open function. If V is any subset of Y and F is a closed subset of X containing $f^{-1}(V)$, then there exists a $\mathcal{G}$-*rw*-open set H in $(Y, \sigma, \mathcal{G})$ containing V such that $f^{-1}(H) \subseteq F$.

Proof. Suppose that f is $\mathcal{G}$-*rw*-open. Let V be any subset of Y and F be a closed subset of X containing $f^{-1}(V)$. Then $X \backslash F$ is open in (X, τ) and hence by $\mathcal{G}$-*rw*-openness of f, $f(X \backslash F)$ is $\mathcal{G}$-open. Thus $H = Y \backslash f(X \backslash F)$ is $\mathcal{G}$-*rw*-closed and consequently $f^{-1}(V) \subseteq F$ implies that $V \subseteq H$. Further we obtain that $f^{-1}(H) \subseteq F$. □

Theorem 3.4. For any bijection $f : (X, \tau) \to (Y, \sigma, \mathcal{G})$ the following are equivalent:
(a) $f^{-1} : (Y, \sigma, \mathcal{G}) \to (X, \tau)$ is $\mathcal{G}$-*rw*-continuous.
(b) f is $\mathcal{G}$-*rw*-open.
(c) f is $\mathcal{G}$-*rw*-closed.

Proof. Obvious. □

4. **Some Characterizations of Regular and Normal Spaces**

As already proposed, this section is meant for deriving certain applications of the study in the last section; some characterizations of regular and normal spaces are achieved here in terms of the introduced concept of $\mathcal{G}$-*rw*-closed sets.

Theorem 4.1. Let X be a normal space and $\mathcal{G}$ be a grill on X then for each pair of disjoint closed sets F and K, there exist disjoint $\mathcal{G}$-rw-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Proof. It is obvious, since every open set is $\mathcal{G}$-rw-open. □

Theorem 4.2. Let X be a normal space and $\mathcal{G}$-be a grill on X then for each closed set F and any open set V containing F, there exist a $\mathcal{G}$-rw-open set U such that $F \subseteq U \subseteq \tau_{\mathcal{G}}\text{-}cl(U) \subseteq V$.

Proof. Let F be a closed set and V an open set in (X, τ) such that $F \subseteq V$. Then F and $X \backslash V$ are disjoint closed sets. by Theorem 4.1, there exist disjoint $\mathcal{G}$-rw-open sets U and W such that $F \subseteq U$ and $X \backslash V \subseteq W$. Since W is $\mathcal{G}$-rw-open and $X \backslash V \subseteq W$ where $X \backslash V$ is closed, by Theorem 2.6 $X \backslash V \subseteq \tau_{\mathcal{G}}\text{-}int(W)$. So $X\ V \subseteq \tau_{\mathcal{G}}\text{-}int(W) \subseteq V$. Again, $U \cap W = \phi \Rightarrow U \cap \tau_{\mathcal{G}}\text{-}int(W) = \phi$. Hence $\tau_{\mathcal{G}}\text{-}cl(U) \subseteq X \backslash \tau_{\mathcal{G}}\text{-}int(W) \subseteq V$. Thus $F \subseteq U \subseteq \tau_{\mathcal{G}}\text{-}cl(U) \subseteq V$ where U is a $\mathcal{G}$-rw-open set. □

The following theorems gives characterizations of a normal space in terms of rw-open sets which are the consequence of Theorems 4.1, 4.2 and Proposition 2.2 if one takes $\mathcal{G} = [X]$.

Theorem 4.3. Let X be a normal space and $\mathcal{G}$ be a grill on X then for each pair of disjoint closed sets F and K, there exist disjoint rw-open sets U and V such that $F \subseteq U$ and $K \subseteq V$.

Theorem 4.4. Let X be a normal space and $\mathcal{G}$ be a grill on X then for each closed set F any open set V containing F, there exist a rw-open set U such that $F \subseteq U \subseteq \tau_{\mathcal{G}}\text{-}cl(U) \subseteq V$.

Theorem 4.5. Let X is regular and $\mathcal{G}$ be a grill on a space (X, τ). Then for each closed set F and each $x \in X \backslash F$, there exist disjoint $\mathcal{G}$-rw-open sets U and V such that $x \in U$ and $F \subseteq V$.

Proof. The proof is immediate. □

Theorem 4.6. Let X be a regular space and $\mathcal{G}$ be a grill on a space (X, τ). Then for each regular semiopen set V of (X, τ) and each point $x \in V$ there exist a $\mathcal{G}$-rw-open set U such that $x \in U \subseteq \tau_{\mathcal{G}}\text{-}cl(U) \subseteq V$.

Proof. Let V be any regular semipen in (X, τ) containing a point x of X. Then by Theorem 4.5, there exist disjoint $\mathcal{G}$-rw-open sets U and W such that $x \in U$ and $X \backslash V \subseteq W$. Now, $U \cap W = \phi$ implies $\tau_{\mathcal{G}}\text{-}cl(U) \subseteq X \backslash W \subseteq V$. Thus $x \in U \subseteq \tau_{\mathcal{G}}\text{-}cl(U) \subseteq V$. □

The following theorems gives characterizations of a regular space in terms of rw-open sets which are the consequence of Theorems 4.5, 4.6 and Proposition 2.2 if one takes $\mathcal{G} = [X]$.

Theorem 4.7. Let X be a regular and $\mathcal{G}$ be a grill on a space (X, τ). Then for each closed set F and each $x \in X \backslash F$, there exist disjoint rw-open sets U and V such that $x \in U$ and $F \subseteq V$.

Theorem 4.8. Let X be a regular space and $\mathcal{G}$ be a grill on a space (X, τ). Then for each regular semiopen set V of (X, τ) and each point $x \in V$ there exist a rw-open set U such that $x \in U \subseteq \tau_{\mathcal{G}}$-$cl(U) \subseteq V$.

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