Equivariant Estimation of the Parameter of a Location Model Based on General Progressive Type II Right Censored Sample

Leo Alexander, T
Associate Professor, Department of Statistics, Loyola College

Abstract: In this paper, by assuming that a general progressive Type II right censored sample is available, we obtain Minimum Risk Equvariant (MRE) estimator for the parameter of the exponential model in three situations. These generalize the results of Chandrasekar et.al. (2002) for progressive Type II right censored sample . The paper is organized as follows : Section 2 deals with the problem of equivariant estimation under Squared error loss function. Section 3 discusses the problem of equivariant estimation under Absolute error loss function. In the last Section, we consider the problem of equivariant estimation of the parameter under Linex loss function (Varian,1975).

Keywords and Phrases: Absolute error loss, Exponential distribution, Minimum risk equivariant estimation, Linex loss, Squared error loss and Type II general progressive censoring.

1. Summary

Progressive Type II right censored sampling is an important method of obtaining data in life-testing studies. As pointed out by Aggarwala and Balakrishnan (1998), the scheme of progressive censoring enables us to use live units, removed early, in other tests. Balakrishnan and Sandhu (1996), by assuming a general progressive Type II right censored sample, derived the BLUE's for the parameters of one-and two-parameter exponential distributions. For the later, they also derived MLE's and shown that they are simply the BLUE's, adjusted for their bias.

Let us consider the following general progressive Type II right censoring scheme (Balakrishnan and Sandhu, 1996) : Suppose N randomly selected units were placed on a life test; the failure times of the first r units to fail were not observed ; at the time of the (r+1)-th failure, R_{r+1} number of surviving units are withdrawn from the test randomly, and so on; at the time of the (r+i)-th failure, R_{r+i} number of surviving units are randomly withdrawn from the test ; finally, at the time of the n-th failure, the remaining $R_n = N - n - R_{r+1} - R_{r+2} - \dots - R_{n-1}$ are withdrawn from the test. Suppose $X_{r+1:N} \le X_{r+2:N} \le \dots \le X_{n:N}$ are the life-times of the completely observed units to fail, and $R_{r+1}, R_{r+2}, \dots, R_n$ are the number of units withdrawn from the test at these failure times, respectively. It follows that $N = n + \sum_{i=r+1}^{n} Ri$. If the failure times are from a continuous population with the pdf f and the distribution function F, then the joint pdf of $(X_{r+1:N}, X_{r+2:N}, \dots, X_{n:N})$ is given by

$$g_\theta(x_{r+1},\dots,x_n)=\mathbf{c}\{F_\theta(x_{r+1})\}^r \prod_{i=r+1}^{n} f_\theta(x_i)\{1-F_\theta(x_i)\}^{R_i} \tag{1.1}$$

where $\mathbf{c} = \binom{N}{r}(N-r)\prod_{j=r+2}^{n}\left(N-\sum_{i=r+1}^{j-1}R_i - j + 1\right)$ In this paper, by assuming that such a general progressive type II censored sample is available from exponential distribution, we derive MRE estimator of the location parameter ξ with respect to squared error loss function, absolute error loss function and linex loss function.

Exponential location model

In this case the common pdf of the failure times is taken to be

$$f_\theta(x) = \begin{cases} e^{-(x-\xi)} & ; x > \xi \; ; \xi \varepsilon \mathbf{R} \, , \\ 0 & , \text{otherwise.} \end{cases}$$

Thus (1.1) reduces to

$$g_\xi\,(x_{r+1}\,, \ldots, x_n) = \mathbf{c}\,\{1 - \exp\,(x_{r+1} - \xi)\}^r$$

$$\times \exp\,\{ - \sum_{i=r+1}^{n} (R_i + 1)(x_i - \xi)\}\,,$$

$$x_{r+1} \geq \xi \quad ; \quad \xi \in \mathbf{R}\,.$$

Here $X_{i:N}$, i = r+1, r+2, ..., n are the order statistics from a sample of size N from $f_\xi(x)$. Note that the above pdf belongs to a location family. We are interested in deriving the MRE estimator of ξ by considering three loss functions. Following Lehmann and Casella (1998), the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X}) = \delta_0(\mathbf{X}) - v^*(\mathbf{Y})\,,$$

where δ_0 is a location equivariant estimator and $v(\mathbf{y}) = v^*(\mathbf{y})$ minimizes

$$E_0\,[\,\rho\,\{\,\delta_0(\mathbf{X}) - v(\mathbf{y})\,\}\,|\,\mathbf{y}\,]\,,$$

where E_0 denotes E_ξ when ξ=0. In this case, take $Y_i = X_{i:N} - X_{n:N}$, i = r+1,r+2,...,n–1, $\mathbf{Y} = (Y_{r+1}, Y_{r+2}, .., Y_{n-1})$ and ρ is an invariant loss function.

2. Squared error loss function

If the loss is squared error then

$$v^* = E_0\,(\,\delta_0\,|\,\mathbf{y}\,)\,.$$

Take $\delta_0(\mathbf{X}) = X_{r+1:N}$. The pdf of $X_{r+1:N}$ is given by

$$g_\xi\,(x_{r+1}) = \frac{1}{B(N-r,\, r+1)}\{1 - \exp\,[-(x_{r+1} - \xi)]\}^r$$

$$\times \exp\,\{-(N-r)(x_{r+1} - \xi)\}\,,$$

$$x_{r+1} \geq \xi \quad ; \quad \xi \in \mathbf{R}\,.$$

Clearly δ_0 is an equivariant estimator and also a sufficient statistic. It may be noted that the distribution of $Y=\exp\{-(X_{r+1}-\xi)\}$ is Beta of first kind with parameters (N – r, r + 1) and denoted by B_1 (N-r, r+1). From Sukhatme (1937), it follows that the random variables $X_{r+1:N}$, $X_{r+2:N} - X_{r+1:N}$, ..., $X_{n:N} - X_{n-1:N}$ are independent. Thus $\delta_0 = X_{r+1:N}$ is independent of $\mathbf{Y}$. Now $v^* = E_0\,(\delta_0)$

$$= \frac{1}{B(N-r,\, r+1)} \int_0^\infty x\,(1 - e^{-x})^r\, e^{-(N-r)x}\, dx\,.$$

Take $u = e^{-x}$, then

$$v^* = \frac{1}{B(N-r,\, r+1)} \int_0^1 (-\log u)(1-u)^r\, u^{(N-r-1)} du$$

$$= E_o\,(-\log U)\,,$$

where $U \sim B_1$ (N –r, r+1). Therefore the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X}) = X_{r+1:N} - E_0\,(-\log U)\,.$$

3. Absolute error loss function

If the loss is absolute error then v_0 = median of the conditional distribution of $\delta_0(\mathbf{X})$ given $\mathbf{Y} = \mathbf{y}$. Take $\delta_0(\mathbf{X}) = X_{r+1:N}$. Since $X_{r+1:N}$ is independent of $\mathbf{Y}$, v_0 is a solution of

$$\frac{1}{B(N-r,\ r+1)}\int_0^{v_0}(1-e^{-x})^r\, e^{-(N-r)x}\,dx = 1/2 .$$

If $u = e^{-x}$, then the above integral reduces to

$$\frac{1}{B(N-r,\ r+1)}\int_{e^{-v_0}}^{1}(1-u)^r u^{(N-r-1)}du = 1/2 .$$

This implies

$$\frac{1}{B(N-r,\ r+1)}\left\{\int_0^1 (1-u)^r u^{(N-r-1)}du - \int_0^{e^{-v_0}}(1-u)^r u^{(N-r-1)}du\right\} = 1/2 .$$

That is $$\frac{1}{B(N-r,\ r+1)}\int_0^{e^{-v_0}}(1-u)^r u^{(N-r-1)}du = 1/2$$

In particular for $N = 3$, $r = 1$, we get $v_0 = \log 2$.

Hence MRE estimation of ξ is

$$\delta^*(\mathbf{X}) = X_{2:3} - \log 2 .$$

4. Linex loss function

Consider the location invariant Linex loss function (Varian, 1975).

$L(\xi;\delta) = e^{a(\delta-\xi)} - a(\delta-\xi) - 1,\ a \in \mathbf{R} - \{0\}$.

Take $\delta_0(\mathbf{X}) = X_{r+1+:N}$. In order to find

$$v^* = E_0\{\rho(\delta_0 - v) \mid \mathbf{y}\},$$

consider ,

$$\begin{aligned} R(\delta \mid \mathbf{y}) &= E_0\{(e^{a(\delta_0 - v)} - a(\delta_0 - v) - 1) \mid \mathbf{y}\} \\ &= e^{-av} E_0(e^{a\delta_0} \mid \mathbf{y}) - a E_0(\delta_0 \mid \mathbf{y}) \\ &\quad + av - 1 \end{aligned}$$

$= e^{-av} E_0(e^{a\delta_0}) - a E_0(\delta_0) + av - 1$,

since δ_0 is independent of $\mathbf{Y}$.

Now $R(\delta \mid \mathbf{y}) = e^{-av}\frac{\Gamma(N-a-r)\Gamma(N+1)}{\Gamma(N-r)\Gamma(N-a+1)} - aE_0(-\log U) + av - 1$ since $U \sim B_1(N-r, r+1)$ and

$$E_0\left(e^{aX_{r+1:N}}\right) = \frac{1}{B(N-r,\ r+1)}\int_0^\infty e^{ax}(1-e^{-x})^r e^{-(N-r)x}dx = \frac{1}{B(N-r,\ r+1)}\int_0^\infty (1-e^{-x})^r e^{-(N-r-a)x}dx$$

Take $u = e^{-x}$ so that

$$E_0\left(e^{aX_{r+1:N}}\right) = \frac{1}{B(N-r,\ r+1)}\int_0^1 (1-u)^r u^{(N-r-a)-1}du = \frac{\Gamma(N-a-r)\ \Gamma(N+1)}{\Gamma(N-r)\ \Gamma(N-a+1)} .$$

Therefore

$$v^* = -\frac{1}{a}\log\left(\frac{\Gamma(N-r)\ \Gamma(N-a+1)}{\Gamma(N-a-r)\ \Gamma(N+1)}\right)$$ Hence the MRE estimator of ξ is given by

$$\delta^*(\mathbf{X}) = X_{r+1:N} + \frac{1}{a}\log\left(\frac{\Gamma(N-r)\ \Gamma(N-a+1)}{\Gamma(N-a-r)\ \Gamma(N+1)}\right)$$

Note that if r = 0, then the above estimator reduces to

$\delta^{*}(\mathbf{X}) = X_{1:N} + 1/a \log (1 - a/N)$

which is the one obtained under progressive Type II right censored sample case (Leo Alexander , 2000).

Acknowledgement

The author thank Dr. B. Chandrasekar, Department of Statistics, Loyola College, Chennai for his valuable guidance and suggestions.

References

1. Aggarwala, R. and Balakrishnan, N. (1998). Some properties of progressive censored order statistics from arbitrary and uniform distributions with applications to inference and simulation, J. Statist. Planning. Infer. 70, 35-49.
2. Balakrishnan, N. and Sandhu,R.A. (1996). Best linear unbiased and maximum likelihood estimation for exponential distributions under general progressive Type II censored samples, **Sankhya, Series B, 58, 1-9.**
3. Chandrasekar, B., Leo Alexander, T. and Balakrishnan, N. (2002). Equivariant estimation for the parameters of an exponential model based on Type II progressively censored samples. **Commun.Stat.Theory & Methods,31(10),1675-1686.**
4. Lehmann, E.L. and Casella, G. (1998). **Theory of Point Estimation,** Second edition, Springer – Verlag, New York.
5. Leo Alexander, T. (2000). Contributions to simultaneous equivariant estimation based on censored samples, Ph.D Thesis, University of Madras, India.
6. Sukhatme, T.V. (1937). Tests of Significance for samples of the chi-squared population with 2 degrees of freedom. Annals of Eugenics, 8, 52-56.
7. Varian, H.R. (1975). A Bayesian Approch to Real Estate Assessment In : **Studies in Bayesian Econometric and Statistics** in Honor of Leonard J. Savage, eds. S.E. Frienberg and A.Zellner. North Holland, Amsterdam, 195-208.