Idempotents of M_2 (Z_{10}[x])

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ABSTRACT
The aim of this paper is to study idempotents in the matrix ring $M_2(Z_{10}[x])$

1 INTRODUCTION

Idempotents in rings play a very important role in the study of rings.The term "idempotents" was introduced by Benjamin Peirce in the context of elements of Algebra that remain invariant when raised to positive integer power and the literally means,"the same power",from idem+potence (same+power).

In Boolean algebra,the main subject of study are rings in which elements are idempotents under both addition and multiplication.In the case of polynomial rings,Kanwar,Leroy and Matczuk showed that for an abelian ring R,idempotents in the polynomial ring R[x] over R are precisely idempotents in R([7],Lemma 1).In fact,a ring is reduced if and only if the unit group of R[x] is same as the unit group of R.

In this article ,we study idempotents in matrix ring $M_2(Z_{10}[x])$.Throughout,a ring will mean an associative ring with unity and for any positive integer n, Z_n will denote the ring of integers modulo n. For any ring R, E(R) will denote the set of all idempotents in R. For any positive integer n, Mn(R) will denote the ring of n×n matrices over a ring R and GL(n,R) will denote the general linear group (the group of all n × n invertible matices over ring R).

We will use standard definitions for determinant and trace of matrices over commutative ring. More precisely for a 2×2 matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over a commutative ring R , determinant of A is ad−bc and trace of A is a + d. Recall that the determinant of product of two matrices over a commutative ring is the product of the determinant of two matrices.

2 IDEMPOTENTS OF $M_2(Z_{10}[x])$

We now give some results that will be useful in our study. We begin with the following proposition that may also be of independent interest.

Proposition 2.1. Let R be any ring with unity and $a = \sum^{n} a_i x^i$ is an element in R[x] such that $a^2 - a \in R$. If any of the following conditions hold:

1. R has no non-zero nilpotent elements,
2. $a_0 a_i = a_i a_0$ for $1 \le i \le n$ and $2a_0 - 1$ is a unit in R, then $a \in R$.

Proof.

If R has no non-zero nilpotent elements and $a^2 - a \in R$, then it is easy to see that $a_i = 0$ for $1 \le i \le n$. The proof, in the second case, is similar to the proof of Lemma 1 in [7]. We give a brief outline for the sake of completeness. If a $\notin$ R and a_i $(i > 0)$ is the first non-zero coefficient in a, then $a^2 - a \in R$ gives $2a_0a_i - a_i = 0$. But then $a_i = 0$ as $2a_0 - 1$is a unit in R, a contradiction. Thus $a \in R$.

In particular, we have the following corollary.

Corollary 2.2. [7, Lemma 1] If R is a commutative ring, then E(R[x]) = E(R).

Corollary 2.3. If R is a ring with no non-zero nilpotent elements, then E(R[x])= E(R).

Theorem 2.4. Any non-trivial idempotent in $M_2(Z_6[x])$ is of one of the following forms:

1. $\begin{bmatrix} a(x) & b(x) \\ c(x) & 1-a(x) \end{bmatrix}$,where a(x){1-a(x)}=b(x)c(x)
2. $\begin{bmatrix} 5a(x) & 5b(x) \\ 5c(x) & 5-5a(x) \end{bmatrix}$,where a(x){5-5a(x)}-5b(x)c(x)=2f(x)

3. $\begin{bmatrix} 2a(x) & 2b(x) \\ 2c(x) & 6-2a(x) \end{bmatrix}$,where 2a(x){6-2a(x)}-4b(x)c(x)=0
4. $\begin{bmatrix} 2a(x)+1 & 2b(x) \\ 2c(x) & 5-2a(x) \end{bmatrix}$,where {2a(x)+1}{5-2a(x)}-4b(x)c(x)=5
5. $\begin{bmatrix} 1 & 5b(x) \\ 5c(x) & 6 \end{bmatrix}$,where 6-5b(x)c(x)=6
6. $\begin{bmatrix} 6 & 5b(x) \\ 5c(x) & 1 \end{bmatrix}$,where 6-5b(x)c(x)=6
7. $\begin{bmatrix} 2 & 5b(x) \\ 5c(x) & 7 \end{bmatrix}$,where 4-5b(x)c(x)=6
8. $\begin{bmatrix} 7 & 5b(x) \\ 5c(x) & 2 \end{bmatrix}$,where 4-5b(x)c(x)=6
9. $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$ and $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$,where a(x),b(x) ,c(x),f(x) are polynomial in $Z_{10}[x]$,not necessarily non-zero.

proof:
Since the idempotents in $Z_{10}[x]$ are precisely the idempotents in Z_{10}. Therefore, the idempotents in $Z_{10}[x]$ are 0, 1, 5, and 6. Now let A= $\begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix}$ be a non-trivial idempotent of $M_2(Z_{10}[x])$. For convenience, we will write a, b, c, d for a(x), b(x), c(x), d(x) respectively. Since A is an idempotent, we have $a^2 + bc = a$, $b(a + d) = b$, $c(a + d) = c$, and $bc + d^2 = d$. Also since determinant of A is an idempotent in Z_{10}, so the determinant of A is 0 or 1 or 5 or 6.

If determinant of A is 1 then A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, a trivial idempotent in $M_2(Z_{10}[x])$. Hence,the determinant of A is 0 or 5 or 6. Also, trace of A is in Z_{10}, that is, $a + d \in Z_{10}$.

Case 1: Determinant of A is 0. This means $ad - bc = 0$. Since A is an idempotent, therefore, $a^2 + bc + bc + d^2 = a^2 + 2bc + d^2 = a^2 + 2ad + d^2 = a + d$.Thus, a+d is an idempotent in $Z_{10}[x]$. Thus a + d is either 0 or 1 or 5 or 6.

if a+d =0, then we get A to be a zero matrix, which is a trivial idempotent in $M_2(Z_{10}[x])$.

If a + d = 1 then d = 1−a and hence ad−bc = 0 gives $a^2 + bc = a$,(a+d)b=b Also (a + d)c = c, and $bc + d^2 = 1-a$. Thus, A^2= $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$.Thus in this case , matrix A= $\begin{bmatrix} a(x) & b(x) \\ c(x) & 1-a(x) \end{bmatrix}$,where a(x), b(x), c(x)$\in Z_{10}[x]$ such that a(x){1−a(x)} = b(x)c(x).

If a + d = 5 then d = 5 − a and hence ad − bc = 0 gives $a^2 + bc = 5a$ and so, 4a=0 . Also,(a+d)b=b implies 4b=0 and (a+d)c=c implies 4c=0. Therefore, a = $5a_0(x)$, b = $5b_0(x)$, and c = $5c_0(x)$, where $a_0(x)$, $b_0(x)$ and $c_0(x)$ are polynomials in $Z_{10}[x]$. Now since ad − bc = 0, we get $5a_0(x)\{5 - 5a_0(x)\} = 5b_0(x)c_0(x)$, which is equivalent $a_0(x)\{5-5a_0(x)\}-b_0(x)c_0(x) = 2f(x)$ for some polynomial $f(x) \in Z_{10}[x]$.

Hence, A = $\begin{bmatrix} 5a(x) & 5b(x) \\ 5c(x) & 5-5a(x) \end{bmatrix}$, where a(x), b(x), c(x) $\in Z_{10}[x]$ such that a(x){5−5a(x)}−b (x)c(x) = 2f(x) for some f(x) $\in Z_{10}[x]$.

If a + d = 6 then d = 6 − a and hence ad − bc = 0 gives $a^2 + bc = 6a$. Thus,5a=0.Also,(a+d)b=b gives 5b=0 and (a+d)c=c gives 5c=0.Therefore,a=$2a_0(x)$,b=$2b_0(x)$ and c=$2c_0(x)$ and d=6-$2a_0(x)$,where $a_0(x)$,$b_0(x)$ and $c_0(x)$ are polynomials in $Z_{10}[x]$.Now, since ad-bc=0,we get $2a_0(x)\{6-2a_0(x)\}=4b_0(x)c_0(x)$.Hence, idempotent matrix is A= $\begin{bmatrix} 2a(x) & 2b(x) \\ 2c(x) & 6-2a(x) \end{bmatrix}$,where 2a(x){6-2a(x)}=4b(x)c(x).

Next, we consider the case when determinant of A is 5.

Case2: Determinant of A is 5. This means ad−bc = 5, that is, 2ad-2bc=0. Since A is an idempotent, therefore, $a^2 + bc + bc + d^2 = a^2 + 2bc + d^2 = a^2 + 2ad + d^2 = a + d$. It means a + d is an idempotent in $Z_{10}[x]$. Thus, a+ d is either 0 or 1 or 5 or 6.

If a+d=0,then ad-bc=5 gives a^2+bc=5 and so,a=5.Also ,b(a+d)=b gives b=0andc(a+d)=c ,givesc=0.

Also,a+d=0 gives d=5. Hence, idempotent matrix A=$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$.

If a+d=1,then d=1-a,so,ad-bc=5 gives a^2+bc=a-5 .Also a^2+bc=a,so we have a=a-5,which can not be possible in $Z_{10}[x]$.So,if determinant of idempotent matrix A in $Z_{10}[x]$ is 5,then its trace can't be 1.

If a+d=5,then d=5-a,so ,ad-bc=5 implies 4a=5,which is again not possible in $Z_{10}[x]$.So, this case also fails.

If a+d=6,then ad-bc=5 imples 5a=5.so a can be 1,3,5,7,9 in Z_{10} i.e,a=2a(x)+1 for some polynomial a(x)€$Z_{10}[x]$. Also b(a+d)=b gives 5b=0 and c(a+d)=c gives 5c=0.Therefore,a=2a(x)+1,b=2b(x),c=2c(x), d=6-{2a(x) +1}, where a(x),b(x),c(x) are polynomials in $Z_{10}[x]$.Since ad-bc=5,so {2(a(x)+1}{5-2a(x)}-4ab(x)c(x)=5.Hence,matrix

A=$\begin{bmatrix} 2a(x)+1 & 2b(x) \\ 2c(x) & 5-2a(x) \end{bmatrix}$, where a(x),b(x),c(x) are polynomials in $Z_{10}[x]$ such that {2(a(x)+1}{5-2a(x)}- 4b(x)c(x)=5.

Case3: when determinant of A is 6. This means ad−bc = 6 ,so we get $a^2+bc+bc+d^2=a^2+2(ad-6)+d^2=(a+d)^2-2$.from This, we conclude that trace of matrix A is an idempotent iff a+d=2 or 4 or 7 or 9.

If a+d=2,then ad-bc=6 gives a=6.And (a+d)b=b gives b=0 and (a+d)c=c gives c=0.Also,a=6 gives d=6.Hence, matrix A=$\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$.

If a+d=4,then ad-bc=6 gives a=2.And (a+d)b=b gives 3b=0,which implies b=0 in $Z_{10}[x]$.Also,c (a+d)=c gives c=0 in $Z_{10}[x]$.Hence,matrix A=$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, whose determinant is 4 while, in this case ad-bc=6.So ,this case is not possible.

If a+d=7,then ad-bc=6 gives 6a=6 i.e,a=1 or 6 in $Z_{10}[x]$.Also,(a+d)b=b and (a+d)c=c gives 6b=0 and 6c=0 respectively. Therefore,b=5b(x) and c=5c(x),where b(x) and c(x) are polynomial in $Z_{10}[x]$.

Since ad-bc=6 so, 6-5b(x)c(x)=6.Hence,matrix

A= $\begin{bmatrix} 1 & 5b(x) \\ 5c(x) & 6 \end{bmatrix}$ or A=$\begin{bmatrix} 6 & 5b(x) \\ 5c(x) & 1 \end{bmatrix}$,where 6-5b(x)c(x)=6, where,b(x),c(x)€$Z_{10}[x]$.

If a+d=9,then ad-bc=6 implies 8a=6 i.e,a=2 or 7 in $Z_{10}[x]$.Also (a+d)b=b and (a+d)c=c gives 8b=0 and 8c=0 respectively.Hence, b=5b(x) and c=5c(x) ,where b(x) and c(x) are polynomials in $Z_{10}[x]$.Since ad-bc=6,so 4-5b(x)c(x)=6.Hence,matrix A =$\begin{bmatrix} 2 & 5b(x) \\ 5c(x) & 7 \end{bmatrix}$ or $\begin{bmatrix} 7 & 5b(x) \\ 5c(x) & 2 \end{bmatrix}$,where where b(x) and c(x) are polynomials in $Z_{10}[x]$.Since ad-bc=6,so 4-5b(x)c(x)=6.

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