Common $n-$tupled fixed point results for multivalued hybrid pair of mappings under new condition

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Abstract. We introduce the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X^n \to 2^X$. We also give some common $n-$tupled fixed point results for this hybrid pair of mappings. It is to be noted that in our results neither condition of continuity is necessary for any mapping nor the completeness of space is necessary involved there in. Finally an example is also given to validate our results. We extend and generalize several known results.

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1. Introduction and Preliminaries

Multivalued fixed point theory has wide application potential in various fields, in particular game theory and mathematical economics. Thus, it is natural to extend the known fixed point results for single-valued mappings to the setting of multivalued mappings. The results of multivalued nonexpansive mappings are much more complicated than the corresponding results of singlevalued nonexpansive mappings and hence many problems remain unsolved in it. The study of fixed points for multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [22].

Let (X, d) be a metric space. We denote $CB(X)$ the family of all nonempty closed and bounded subsets of X and $CL(X)$ the set of all nonempty closed subsets of X. For $A, B \in CB(X)$ and $x \in X$, we denote $D(x, A) = \inf\{d(x, y) : a \in A\}$. Let H be the Hausdorff metric induced by the metric d on X, that is,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

for every $A, B \in CB(X)$ .It is clear that for $A, B \in CB(X)$, and $a \in A$, we have $d(a, B) \leq H(A, B)$.

The existence of fixed points for various multivalued contractive mappings has been studied by many authors under different conditions. For details, we refer the reader to [3, 6, 7, 9, 14, 18, 21, 23, 24] and the references therein.

Samet and Vetro [25] introduced the concept of coupled fixed point for multivalued mapping and later several authors namely Hussain and Alotaibi [16] and Aydi et. al.[4] proved coupled coincidence point theorems in partially ordered metric spaces. Deshpande et al. in [12] introduced triple fixed, triple coincidence and triple common fixed points for multivalued maps. Imdad, Soliman, Choudhury and Das [17] introduced the concept of $n-$tupled fixed point, $n-$tupled coincidence point and proved some $n-$tupled coincidence point and n-tupled fixed point results for single valued mapping. These concepts was extended by Deshpande and Handa [10] to multivalued mappings and obtained $n-$tupled coincidence points and common $n-$tupled fixed point theorems involving hybrid pair of mappings under generalized Mizoguchi-Takahashi contraction. In [10] Deshpande and Handa introduced the following for multivalued mappings:

Definition 1.1.[10] Let X be a nonempty set, $F : X^r \to 2^X$(a collection of all nonempty subsets of X) and g be a self-mapping on X: An element $\left(x^1, ..., x^r\right) \in X^r$ is called

(1) an r-tupled fixed point of F if $x^1 \in F\left(x^1, ..., x^r\right), ..., x^r \in \left(x^r,\ x^1,\ ...,\ x^{r-1}\right)$.

(2) an $r-$tupled coincidence point of hybrid pair $\{F, g\}$ if $g\left(x^1\right) \in F\left(x^1,\ ...,\ x^r\right)$, ..., $g\left(x^r\right) \in \left(x^r,\ x^1,\ ...,\ x^{r-1}\right)$.

(3) a common r-tupled fixed point of hybrid pair $\{F,\ g\}$ if $x^1 = g\left(x^1\right) \in F\left(x^1,\ ...,\ x^r\right), ..., x^r = g\left(x^r\right) \in F\left(x^r,\ x^1,\ .\ .\ .,\ x^{r-1}\right)$.

We denote the set of $r-$tupled coincidence points of mappings F and g by $C\{F, g\}$. Note that if $\left(x^1,\ ...,\ x^r\right) \in C\{F, g\}$ then $\left(x^2,\ ...,\ x^r,\ x^1\right), ..., \left(x^r,\ x^1,\ ...,\ x^{r-1}\right)$ are also in $C\{F,\ g\}$.

Definition 1.2.[10] Let $F : X^r \to 2^X$ be a multivalued mapping and g be a self mapping on X. The hybrid pair $\{F, g\}$ is called $w-$compatible if $g\left(F\left(x^1,\ x^2,\ ...,\ x^r\right)\right) \subseteq F\left(g\left(x^1\right),\ g\left(x^2\right),\ ...,\ g\left(x^r\right)\right)$ whenever $\left(x^1,\ x^2,\ ...,\ x^r\right) \in C\{F,\ g\}$.

Definition 1.3. [10] Let $F : X^r \to 2^X$ be a multivalued mapping and g be a self mapping on X. The mapping g is called $F-$weakly commuting at some point $\left(x^1, x^2, ..., x^r\right) \in X^r$ if $g^2\left(x^1\right) \in F\left(g\left(x^1\right), g\left(x^2\right), ..., g\left(x^r\right)\right), g^2\left(x^2\right) \in F\left(g\left(x^2\right), g\left(x^3\right), ..., g\left(x^r\right), g\left(x^1\right)\right), ..., g^2\left(x^r\right) \in F\left(g\left(x^r\right), g\left(x^1\right), ..., g\left(x^{r-1}\right)\right)$.

Aamri and ElMoutawakil [1] defined (EA) property for self-mappings which contained the class of non-compatible mappings. Kamran [19] extended the (EA) property for hybrid pair $g : X \to X$ and $F : X \to 2^X$. Abbas and Rhoades [2] extended the concept of occasionally weakly compatible mappings for hybrid pair $g : X \to X$ and $F : X \to 2^X$. Deshpande and Handa [11] introduced the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X \times X \to 2^X$.

In this paper, we introduce the concept of (EA) property and occasionally $w-$compatibility for hybrid pair $g : X \to X$ and $F : X^n \to 2^X$. We also give some common $n-$tupled fixed point results for this hybrid pair of mappings. It is to be noted that in our results neither condition of continuity is necessary for any mapping nor the completeness of space is necessary involved there in. We improve, extend and generalize the results of Bhaskar and Lakshmikantham [5], Ciric et al. [8], Ding et al. [13], Gordji et al. [15], Deshpande and Handa [11] and Lakshmikantham and Ciric [20]. Finally an example is also given to validate our results.

2. Main results

First we introduce the following

Definition 2.1. Mappings $g : X \to X$ and $F : X^r \to CB(X)$ are said to satisfy the (EA) property if there exist sequences $\{x_n^1\}, \{x_n^2\}, ..., \{x_n^r\}$ in X, some $t^1, t^2, ..., t^r$ in X and $A, B, ..., Z$ in $CB(X)$ such that

$$\begin{aligned} \lim_{n\to\infty} gx_n^1 &= t^1 \in A = \lim_{n\to\infty} F(x_n^1, x_n^2, ..., x_n^r), \\ \lim_{n\to\infty} gx_n^2 &= t^2 \in B = \lim_{n\to\infty} F(x_n^2, x_n^3, ..., x_n^r, x_n^1), \\ ..., \lim_{n\to\infty} gx_n^r &= t^r \in Z = \lim_{n\to\infty} F(x_n^r, x_n^1, ..., x_n^{r-1}). \end{aligned}$$

Definition 2.2. Mappings $F : X^r \to 2^X$ and $g : X \to X$ are said to be occasionally $w-$compatible if and only if there exists some point $(x^1, x^2, ..., x^r) \in X^r$ such that $gx^1 \in F(x^1, x^2, ..., x^r), gx^2 \in F(x^2, x^3, ..., x^r, x^1), ..., gx^r \in F(x^r, x^1, ..., x^{r-1})$, and $gF(x^1, .., x^r) \subseteq F(gx^1, ..., gx^r)$.

Following example shows that, occasionally $w-$compatibility is weaker condition than $w-$compatibility.

Example 2.1. Let $X = [0, +\infty)$ with usual metric. Define $g : X \to X$, $F : X^r \to CB(X)$, by

$$gx = \begin{cases} 0, & 0 \le x < 1, \\ (r+1)x, & 1 \le x < \infty, \end{cases} \quad \text{for some } r \in \{1, 2, ..., n\},$$

$$F(x^1, x^2, ..., x^r) = \begin{cases} [0, 1 + x^1 + x^2 + ... + x^r], & (x^1, ..., x^r) \ne (0, ..., 0), \\ \{x^1\}, & (x^1, ..., x^r) = (0, ..., 0). \end{cases}$$

It can be easily verified that $(0, ..., 0)$ and $(1, ..., 1)$ are $r-$tupled coincidence points of g and F, but $gF(0, ..., 0) \subseteq F(g0, ..., g0)$ and $gF(1, ..., 1) \subsetneq$

$F(g1, ..., g1)$. So g and F are not w-compatible. However, the pair $\{F, g\}$ is occasionally w-compatible.

Let Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying
(i_ψ) ψ is continuous and non-decreasing,
(ii_ψ) $\psi(t) = 0 \Leftrightarrow t = 0$,
(iii_ψ) $\limsup_{s\to 0+} \frac{s}{\psi(s)} < \infty$,
and Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying
(i_φ) φ is lower semi-continuous and non-decreasing,
(ii_φ) $\varphi(t) = 0 \Leftrightarrow t = 0$,
(iii_φ) for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exist $k \in (0, 1)$ and $n_0 \in \mathbb{N}$, such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$,
and Θ denote the set of all functions $\theta : [0, +\infty) \to [0, +\infty)$ satisfying
(i_θ) θ is continuous,
(ii_θ) $\theta(t) = 0 \Leftrightarrow t = 0$.

For simplicity, we define

$$
\begin{aligned}
&(I)\ M(x^1, x^2, .., x^r, y^1, y^2, ..., y^r) \\
= \ &\max \left\{ \begin{array}{c}
d(gx^1, gy^1), D(gx^1, F(x^1, ..., x^r)), D(gy^1, F(y^1, ..., y^r)), \\
d(gx^2, gy^2), D(gx^2, F(x^2, ..., x^r, x^1)), D(gy^2, F(y^2, ..., y^r, y^1)), \\
..., d(gx^r, gy^r), D(gx^r, F(x^r, x^1, ..., x^{r-1})), D(gy^r, F(y^r, y^1, ..., y^{r-1})), \\
\frac{D(gx^1,\ F(y^1,...,y^r))+D(gy^1,\ F(x^1,...,x^r))}{2}, \\
\frac{D(gx^2,F(y^2,...,y^r,y^1))+D(gy^2,F(x^2,...,x^r,x^1))}{2}, \\
..., \frac{D(gx^r,F(y^r,y^1,...,y^{r-1}))+D(gy^r,F(x^r,x^1,...,x^{r-1}))}{2}.
\end{array} \right\},
\end{aligned}
$$

$$
\begin{aligned}
&(II)\ N(x^1, x^2, .., x^r, y^1, y^2, ..., y^r) \\
= \ &\min \left\{ \begin{array}{c}
D(gx^1, F(x^1, ..., x^r)), D(gy^1, F(y^1, ..., y^r)), \\
D(gx^1, F(y^1, ..., y^r)), D(gy^1, F(x^1, ..., x^r)), \\
D(gx^2, F(x^2, ..., x^r, x^1)), D(gy^2, F(y^2, ..., y^r, y^1)), \\
D(gx^2, F(y^2, ..., y^r, y^1)), D(gy^2, F(x^2, ..., x^r, x^1)), \\
..., D(gx^r, F(x^r, x^1, ..., x^{r-1})), D(gy^r, F(y^r, y^1, ..., y^{r-1})), \\
D(gx^r, F(y^r, y^1, ..., y^{r-1})), D(gy^r, F(x^r, x^1, ..., x^{r-1})).
\end{array} \right\}.
\end{aligned}
$$

Theorem 2.1. Let (X, d) be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$, $\varphi \in \Phi$ and $\theta \in \Theta$ such that

$$
\begin{aligned}
& \psi\left(H(F(x^1, ..., x^r), F(y^1, ..., y^r))\right) \\
\leq \ & \psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right) - \\
& \varphi\left(\psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right)\right) + \\
& \theta\left(N(x^1, ..., x^r, y^1, .., y^r)\right), \qquad (2.1)
\end{aligned}
$$

for all $x^1, ..., x^r, y^1, ..., y^r \in X$. Furthermore, assume that $\{F, g\}$ satisfies the (EA) property. Then F and g have a $r-$tupled coincidence point. Moreover, F

and g have a common $r-$tupled fixed point, if one of the following conditions holds:

(a) F and g are $w-$compatible. $\lim_{n\to\infty} g^n x^1 = y^1$, $\lim_{n\to\infty} g^n x^2 = y^2$, ..., $\lim_{n\to\infty} g^n x^r = y^r$ for some $(x^1, ..., x^r) \in C\{F,\, g\}$ and for some $y^1, ..., y^r \in X$ and g is continuous at $y^1, y^2, ..., y^r$.

(b) g is $F-$weakly commuting for some $(x^1, ..., x^r) \in C\{F,\, g\}$ and gx^1, gx^2, ..., gx^r are fixed points of g, that is, $g^2x^1 = gx^1, g^2x^2 = gx^2, ..., g^2x^r = gx^r$.

(c) g is continuous at $x^1, ..., x^r$. $\lim_{n\to\infty} g^n y^1 = x^1$, $\lim_{n\to\infty} g^n y^2 = x^2$, ..., $\lim_{n\to\infty} g^n y^r = x^r$ for some $(x^1, ..., x^r) \in C\{F,\, g\}$ and for some $y^1, ..., y^r \in X$.

(d) $g(C\{F,\, g\})$ is a singleton subset of $C\{F,\, g\}$.

Proof. Since $\{F,\, g\}$ satisfies the (EA) property, therefore there exist sequences $\{x_n^1\}, \{x_n^2\}, ..., \{x_n^r\}$ in X, some $t^1, ..., t^r$ in X and $A, ..., Z$ in $CB(X)$ such that

$$\begin{aligned}
\lim_{n\to\infty} gx_n^1 &= t^1 \in A = \lim_{n\to\infty} F(x_n^1, x_n^2, ..., x_n^r), \\
\lim_{n\to\infty} gx_n^2 &= t^2 \in B = \lim_{n\to\infty} F(x_n^2, x_n^3, ..., x_n^r, x_n^1), \\
..., \lim_{n\to\infty} gx_n^r &= t^r \in Z = \lim_{n\to\infty} F(x_n^r, x_n^1, ..., x_n^{r-1}). \qquad (2.2)
\end{aligned}$$

Since $g(X)$ is a subset of X, then there exist $x^1, .., x^r \in X$, we have

$$t^1 = gx^1, ..., t^r = gx^r \qquad (2.3)$$

Now, by using condition (2.1) and (i_ψ), we get

$$\begin{aligned}
&\psi\left(H(F(x_n^1, ..., x_n^r), F(x^1, ..., x^r))\right) \\
&\quad \le \psi\left(M(x_n^1, ..., x_n^r, x^1, ..., x^r)\right) - \varphi\left(\psi\left(M(x_n^1, ..., x_n^r, x^1, ..., x^r)\right)\right) \\
&\qquad\qquad + \theta\left(N(x_n^1, ..., x_n^r, x^1, ..., x^r)\right),
\end{aligned}$$

where

$$\begin{aligned}
&(I)\ M(x_n^1, ..., x_n^r, x^1, ..., x^r) \\
= \ &\max\left\{\begin{array}{c}
d(gx_n^1, gx^1), D(gx_n^1, F(x_n^1, ..., x_n^r)), D(gx^1, F(x^1, ..., x^r)), \\
d(gx_n^2, gx^2), D(gx_n^2, F(x_n^2, ..., x_n^r, x_n^1)), D(gx^2, F(x^2, ..., x^r, x^1)), \\
..., d(gx_n^r, gx^r), D(gx_n^r, F(x_n^r, x_n^1, ..., x_n^{r-1})), D(gx^r, F(x^r, x^1, ..., x^{r-1})), \\
\frac{D(gx_n^1, F(x^1, ..., x^r)) + D(gx^1, F(x_n^1, ..., x_n^r))}{2}, \\
\frac{D(gx_n^2, F(x^2, ..., x^r, x^1)) + D(gx^2, F(x_n^2, ..., x_n^r, x_n^1))}{2}, \\
..., \frac{D(gx_n^r, F(x^r, x^1, ..., x^{r-1})) + D(gx^r, F(x_n^r, x_n^1, ..., x_n^{r-1}))}{2}.
\end{array}\right\}
\end{aligned}$$

and

$$(II)\ N(x_n^1,x_n^2,..,x_n^r,x^1,x^2,...,x^r)$$
$$=\min\left\{\begin{array}{c} D(gx_n^1,F(x_n^1,...,x_n^r)),D(gx^1,F(x^1,...,x^r)),\\ D(gx_n^1,F(x^1,...,x^r)),D(gx^1,F(x_n^1,...,x_n^r)),\\ D(gx_n^2,F(x_n^2,...,x_n^r,x_n^1)),D(gx^2,F(x^2,...,x^r,x^1)),\\ D(gx_n^2,F(x^2,...,x^r,x^1)),D(gx^2,F(x_n^2,...,x_n^r,x_n^1)),\\ ...,D(gx_n^r,F(x_n^r,x_n^1,...,x_n^{r-1})),D(gx^r,F(x^r,x^1,...,x^{r-1})),\\ D(gx_n^r,F(x^r,x^1,...,x^{r-1})),D(gx^r,F(x_n^r,x_n^1,...,x_n^{r-1})). \end{array}\right\}.$$

Letting $n\to\infty$ in the above inequality, by using (i_ψ), (i_φ), (i_θ), (ii_θ), (2.2), (2.3), $gx^1\in A$, $gx^2\in B,...,gx^r\in Z$,we get

$$\begin{aligned}&\psi\left(D(gx^1,F(x^1,...,x^r))\right)\\ &\leq\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1...,x^{r-1})\right\}\right)\\ &-\varphi\left(\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1,...,x^{r-1})\right\}\right)\right).\end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned}&\psi\left(D(gx^2,F(x^2,...,x^r,x^1))\right)\\ &\leq\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1...,x^{r-1})\right\}\right)\\ &-\varphi\left(\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1,...,x^{r-1})\right\}\right)\right)\end{aligned}$$

$$,...,$$

$$\begin{aligned}&\psi\left(D(gx^r,F(x^r,x^1...,x^{r-1}))\right)\\ &\leq\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1...,x^{r-1})\right\}\right)\\ &-\varphi\left(\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1,...,x^{r-1})\right\}\right)\right).\end{aligned}$$

Combining them, we get

$$\begin{aligned}&\max\psi\left(D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1,...,x^{r-1}))\right)\\ &\leq\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1...,x^{r-1})\right\}\right)\\ &-\varphi\left(\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1,...,x^{r-1})\right\}\right)\right).\end{aligned}$$

Since ψ is non-decreasing, therefore,

$$\begin{aligned}&\psi\left(\max\left(D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1,...,x^{r-1}))\right)\right)\\ &\leq\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1...,x^{r-1})\right\}\right)\\ &-\varphi\left(\psi\left(\max\left\{D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,D(gx^r,F(x^r,x^1,...,x^{r-1})\right\}\right)\right).\end{aligned}$$

which, by (ii_φ) and (ii_ψ), implies that

$$\max\left(\begin{array}{c} D(gx^1,F(x^1,...,x^r)),D(gx^2,F(x^2,...,x^r,x^1)),...,\\ D(gx^r,F(x^r,x^1,...,x^{r-1})) \end{array}\right)=0,$$

it follows that

$$gx^1 \in F(x^1, ..., x^r), ..., gx^r \in F\left(x^r, x^1, ..., x^{r-1}\right),$$

that is, $(x^1, .., x^r)$ is a $r-$tupled coincidence point of F and g. That is $C\{F,\, g\}$ is non empty.

Suppose now that (a) holds. Assume that for some $(x^1, .., x^r) \in C\{F,\, g\}$,

$$\lim_{n\to\infty} g^n x^1 = y^1,\ \lim_{n\to\infty} g^n x^2 = y^2, ..., \lim_{n\to\infty} g^n x^r = y^r, \tag{2.4}$$

where $y^1, ..., y^r \in X$. Since g is continuous at $y^1, ..., y^r$. We have, by (2.4), that $y^1, ..., y^r$ are fixed points of g, that is,

$$gy^1 = y^1, ..., gy^r = y^r. \tag{2.5}$$

As F and g are $w-$compatible, so

$$(g^n x^1, ..., g^n y^r) \in C\{F,\ g\},\ \text{for all } n \geq 1,$$

that is,

$$\begin{array}{rcl} g^n x^1 & \in & F(g^{n-1}x^1, ..., g^{n-1}x^r), \\ g^n x^2 & \in & F(g^{n-1}x^2, ..., g^{n-1}x^r, g^{n-1}x^1), ..., \\ g^n x^r & \in & F(g^{n-1}x^r,\ g^{n-1}x^1, ..., g^{n-1}x^{r-1}),\ \text{for all } n \geq 1. \end{array} \tag{2.6}$$

Now, by using (2.1), (2.6) and (i_ψ), we obtain

$$\begin{aligned} &\psi\left(D(g^n x^1, F(y^1, ..., y^r))\right) \\ &\leq \psi\left(H(F(g^{n-1}x^1, ..., g^{n-1}x^r), F(y^1, ..., y^r))\right) \\ &\leq \psi\left(M(g^{n-1}x^1, ..., g^{n-1}x^r, y^1, ..., y^r)\right) - \varphi\left(\psi\left(M(g^{n-1}x^1, ..., g^{n-1}x^r, y^1, ..., y^r)\right)\right) \\ &+ \theta\left(N(g^{n-1}x^1, ..., g^{n-1}x^r, y^1, ..., y^r)\right), \end{aligned}$$

where

$$\begin{aligned} &M(g^{n-1}x^1, ..., g^{n-1}x^r, y^1, ..., y^r) \\ &= \max\left\{ \begin{array}{c} d(g^n x^1, gy^1), D(g^n x^1, F(g^{n-1}x^1, ..., g^{n-1}x^r)), D(gy^1, F(y^1, ..., y^r)), \\ d(g^n x^2, gy^2), D(g^n x^2, F(g^{n-1}x^2, ..., g^{n-1}x^r, g^{n-1}x^1)), D(gy^2, F(y^2, ..., y^r, y^1)), \\ ..., d(g^n x^r, gy^r), D(g^n x^r, F(g^{n-1}x^r, g^{n-1}x^1, ..., g^{n-1}x^{r-1})), D(gy^r, F(y^r, y^1, ..., y^{r-1})), \\ \frac{D(g^n x^1,\ F(y^1, ..., y^r)) + D(gy^1,\ F(g^{n-1}x^1, ..., g^{n-1}x^r))}{2}, \\ \frac{D(g^n x^2, F(y^2, ..., y^r, y^1)) + D(gy^2, F(g^{n-1}x^2, ..., g^{n-1}x^r, g^{n-1}x^1))}{2}, \\ ..., \frac{D(g^n x^r, F(y^r, y^1, ..., y^{r-1})) + D(gy^r, F(g^{n-1}x^r, g^{n-1}x^1, ..., g^{n-1}x^{r-1}))}{2}. \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} d(g^n x^1, gy^1), d(g^n x^1, g^n x^1), D(gy^1, F(y^1, ..., y^r)), \\ d(g^n x^2, gy^2), d(g^n x^2, g^n x^2), D(gy^2, F(y^2, ..., y^r, y^1)), \\ ..., d(g^n x^r, gy^r), d(g^n x^r, g^n x^r), D(gy^r, F(y^r, y^1, ..., y^{r-1})), \\ \frac{D(g^n x^1,\ F(y^1, ..., y^r)) + D(gy^1,\ F(g^{n-1}x^1, ..., g^{n-1}x^r))}{2}, \\ \frac{D(g^n x^2, F(y^2, ..., y^r, y^1)) + D(gy^2, g^n)}{2}, \\ ..., \frac{D(g^n x^r, F(y^r, y^1, ..., y^{r-1})) + D(gy^r, g^n x^r)}{2}. \end{array} \right\} \end{aligned}$$

and

$$\begin{aligned}
&N(g^{n-1}x^1,...,g^{n-1}x^r,y^1,...,y^r)\\
&=\min\left\{\begin{array}{c}
D(g^nx^1,F(g^{n-1}x^1,...,g^{n-1}x^r)),D(gy^1,F(y^1,...,y^r)),\\
D(g^nx^1,F(y^1,...,y^r)),D(gy^1,F(g^{n-1}x^1,...,g^{n-1}x^r)),\\
D(g^nx^2,F(g^{n-1}x^2,...,g^{n-1}x^r,g^{n-1}x^1)),D(gy^2,F(y^2,...,y^r,y^1)),\\
D(g^nx^2,F(y^2,...,y^r,y^1)),D(gy^2,F(g^{n-1}x^2,...,g^{n-1}x^r,g^{n-1}x^1)),\\
...,D(g^nx^r,F(g^{n-1}x^r,g^{n-1}x^1,...,g^{n-1}x^{r-1})),D(gy^r,F(y^r,y^1,...,y^{r-1})),\\
D(g^nx^r,F(y^r,y^1,...,y^{r-1})),D(gy^r,F(g^{n-1}x^r,g^{n-1}x^1,...,g^{n-1}x^{r-1})).
\end{array}\right\}=0.
\end{aligned}$$

On taking limit as $n\to\infty$ in the above inequality, by using (i_ψ), (i_φ), (ii_θ), (2.4), (2.5) and (2.6), we get

$$\begin{aligned}
&\psi\left(D(gy^1,F(y^1,...,y^r))\right)\\
&\le\psi\left(\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}\right)\\
&-\varphi\psi\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\psi\left(D(gy^2,F(y^2,...,y^r,y^1))\right)\\
&\le\psi\left(\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}\right)\\
&-\varphi\psi\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}.
\end{aligned}$$

On continuing, we get

$$\begin{aligned}
&\psi\left(D(gy^r,F(y^r,y^1...,y^{r-1}))\right)\\
&\le\psi\left(\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}\right)\\
&-\varphi\psi\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}.
\end{aligned}$$

Combining them, we get

$$\begin{aligned}
&\max\left\{\psi\left(D(gy^1,F(y^1,...,y^r))\right),\psi\left(D(gy^2,F(y^2,...,y^r,y^1))\right),...,\psi\left(D(gy^r,F(y^r,y^1,...,y^{r-1}))\right)\right\}\\
&\le\psi\left(\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}\right)\\
&-\varphi\left(\psi\left(\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}\right)\right).
\end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned}
&\psi\left\{\max\left(D(gy^1,F(y^1,...,y^r))\right),\left(D(gy^2,F(y^2,...,y^r,y^1))\right),...,\left(D(gy^r,F(y^r,y^1,...,y^{r-1}))\right)\right\}\\
&\le\psi\left(\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}\right)\\
&-\varphi\left(\psi\left(\max\left\{D(gy^1,F(y^1,...,y^r)),D(gy^2,F(y^2,...,y^r,y^1)),...,D(gy^r,F(y^r,y^1,...,y^{r-1}))\right\}\right)\right),
\end{aligned}$$

which, by (ii_{φ}) and (ii_{ψ}), implies that

$$\max\left(D(gy^1, F(y^1,...,y^r))\right), \left(D(gy^2, F(y^2,...,y^r,y^1))\right), \\ ..., \left(D(gy^r, F(y^r,y^1,...,y^{r-1}))\right) = 0,$$

it follows that

$$gy^1 \in F(y^1,...,y^r), gy^2 \in F(y^2,...,y^r,y^1), ..., gy^r \in F(y^r,y^1,...,y^{r-1}). \quad (2.7)$$

Now, from (2.5) and (2.7), we have

$$y^1 = gy^1 \in F(y^1,...,y^r)\ , y^2 = gy^2 \in F(y^2,...,y^r,y^1), ..., y^r \\ = gy^r \in F(y^r,y^1,...,y^{r-1}),$$

that is, $(y^1,..,y^r)$ is a common $r-$tupled fixed point of F and g.

Suppose now that (b) holds. Assume that for some $(x^1,...,x^r) \in C\{F,g\}$, g is $F-$weakly commuting, that is $g^2x^1 \in F(gx^1,..,gx^r)$, $g^2x^2 \in F(gx^2,...,gx^r,gx^1)$, ..., $g^2x^r \in F(gx^r,\ gx^1,\ ...,\ gx^{r-1})$ and $g^2x^1 = gx^1$, ..., $g^2x^r = gx^r$. Thus $gx^1 = g^2x^1 \in F(gx^1,...,gx^r)$,..., $gx^r = g^2x^r \in F(gx^r,\ gx^1,\ ...,\ gx^{r-1})$, that is, $(gx^1,...,gx^r)$ is a common $r-$tupled fixed point of F and g.

Suppose now that (c) holds. Assume that for some $(x^1,...,x^r) \in C\{F,\ g\}$ and for some $y^1,...,y^r \in X$,

$$\lim_{n\to\infty} g^n y^1 = x^1, ..., \lim_{n\to\infty} g^n y^r = x^r. \quad (2.8)$$

Since g is continuous at $x^1,...,x^r$. Therefore, by (2.8), we obtain that $x^1,...,x^r$ are fixed points of g, that is,

$$gx^1 = x^1, ..., gx^r = x^r. \quad (2.9)$$

Since $(x^1,...,x^r) \in C\{F,\ g\}$. Therefore, by (2.9), we obtain

$$x^1 = gx^1 \in F(x^1,...,x^r), ..., x^r = gx^r \in F(x^r,x^1,...,x^{r-1}),$$

that is, $(x^1,...,x^r)$ is a common $r-$tupled fixed point of F and g.

Finally, suppose that (d) holds. Let $g(C\{F,\ g\}) = \{(x^1,...,x^1)\}$. Then $\{x^1\} = \{gx^1\} = F(x^1,...,x^1)$. Hence $(x^1,...,x^1)$ is a common $r-$tupled fixed point of F and g.

If we put $\theta(t) = 0$ in the Theorem 2.1, we get the following result:

Corollary 2.2. Let $(X,\ d)$ be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\begin{aligned} &\psi\left(H(F(x^1,...,x^r), F(y^1,...,y^r))\right) \quad (2.10) \\ \leq\ &\psi\left(M(x^1,...,x^r,y^1,...,y^r)\right) - \varphi\left(\psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)\right), \end{aligned}$$

for all $x^1, ..., x^r, y^1, ..., y^r \in X$. Furthermore, assume that $\{F,\, g\}$ satisfies the (EA) property. Then F and g have a $r-$tupled coincidence point. Moreover, F and g have a common $r-$tupled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

If we put $\varphi(t) = t - t\widetilde{\varphi}(t)$ for all $t \geq 0$ in Corollary 2.2, then we get the following result:

Corollary 2.3. Let $(X,\, d)$ be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$ and $\widetilde{\varphi} \in \Phi$ such that

$$\begin{aligned} &\psi\left(H(F(x^1, ..., x^r), F(y^1, ..., y^r))\right) \\ &\quad \leq \widetilde{\varphi}\left(\psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right)\right)\psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right), \end{aligned} \tag{2.11}$$

for all $x^1, ..., x^r, y^1, ..., y^r \in X$. Furthermore, assume that $\{F,\, g\}$ satisfies the (EA) property. Then F and g have a $r-$tupled coincidence point. Moreover, F and g have a common $r-$tupled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Corollary 2.3, then we get the following result:

Corollary 2.4. Let $(X,\, d)$ be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exists some $\widetilde{\varphi} \in \Phi$ such that

$$\begin{aligned} & H(F(x^1, ..., x^r), F(y^1, ..., y^r)) \\ \leq\ & \widetilde{\varphi}\left(2M(x^1, ..., x^r, y^1, ..., y^r)\right)2M(x^1, ..., x^r, y^1, ..., y^r), \end{aligned} \tag{2.12}$$

for all $x^1, ..., x^r, y^1, ..., y^r \in X$. Furthermore, assume that $\{F,\, g\}$ satisfies the (EA) property. Then F and g have a $r-$tupled coincidence point. Moreover, F and g have a common $r-$tupled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

If we put $\widetilde{\varphi}(t) = \frac{k}{2}$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.4, then we get the following result:

Corollary 2.5. Let $(X,\, d)$ be a metric space. Assume $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings satisfying

$$H(F(x^1, ..., x^r), F(y^1, ..., y^r)) \leq kM(x^1, ..., x^r, y^1, ..., y^r), \tag{2.13}$$

for all $x^1, ..., x^r, y^1, ..., y^r \in X$, where $0 < k < 1$. Furthermore, assume that $\{F,\, g\}$ satisfies the (EA) property. Then F and g have a $r-$tupled coincidence point. Moreover, F and g have a common $r-$tupled fixed point, if one of the conditions (a) to (d) of Theorem 2.1 holds.

Theorem 2.6. Let $(X,\, d)$ be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings. Suppose that there exist some $\psi \in \Psi$, $\varphi \in \Phi$ and

$\theta \in \Theta$ satisfying (2.1) and $\{F, g\}$ is occasionally $w-$compatible. Then F and g have a common $r-$tupled fixed point.

Proof. Since the pair $\{F, g\}$ is occasionally $w-$compatible, therefore there exists some point $(x^1, ..., x^r) \in X^r$ such that

$$gx^1 \in F(x^1, ..., x^r),\ gx^2 \in F(x^2, ..., x^r, x^1), ..., gx^r \in F\left(x^r, x^1, ..., x^{r-1}\right), \text{ and } gF(x^1, ..., x^r) \subseteq F(gx^1, ..., gx^r). \quad (2.14)$$

It follows that

$$g^2x^1 \in F(gx^1, ..., gx^r), ..., g^2x^r \in F(gx^r, gx^1, ..., gx^{r-1}). \quad (2.15)$$

Now, suppose $y^1 = gx^1, ..., y^r = gx^r$, then by (2.15), we get

$$gy^1 \in F(y^1, ..., y^r), ..., gy^r \in F(y^r, y^1, ..., y^{r-1}). \quad (2.16)$$

Thus, by condition (2.1), we have

$$\begin{aligned}&\psi\left(H(F(x^1, ..., x^r), F(y^1, ..., y^r))\right)\\ &\leq \psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right) - \varphi\left(\psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right)\right) + \theta\left(N(x^1, ..., x^r, y^1, ..., y^r)\right).\end{aligned}$$

which, by (2.14), (2.16), (i_ψ), (i_φ), (i_θ), (ii_θ), and triangle inequality, implies

$$\begin{aligned}&\psi\left(d(gx^1, gy^1)\right)\\ &\leq \psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right)\right).\end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned}&\psi\left(d(gx^2, gy^2)\right)\\ &\leq \psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right)\right),\end{aligned}$$

continuing in this way, we get

$$\begin{aligned}&\psi\left(d(gx^r, gy^r)\right)\\ &\leq \psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right)\right).\end{aligned}$$

Combining them, we get

$$\begin{aligned}&\max\left\{\psi\left(d(gx^1, gy^1)\right), ..., \psi\left(d(gx^r,\ gy^r)\right)\right\}\\ \leq\ &\psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right)\right).\end{aligned}$$

Since ψ is non-decreasing, therefore

$$\begin{aligned}&\psi\left(\max\left\{\left(d(gx^1, gy^1)\right), ..., \left(d(gx^r,\ gy^r)\right)\right\}\right)\\ &\leq \psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right) - \varphi\left(\psi\left(\max\left\{d(gx^1, gy^1), ..., d\left(gx^r, gy^r\right)\right\}\right)\right),\end{aligned}$$

which, by (ii_φ) and (ii_ψ), implies that

$$\max\left\{\left(d(gx^1, gy^1)\right), ..., (d(gx^r,\ gy^r))\right\} = 0,$$

it follows that $d(gx^1, gy^1) = 0, ..., d(gx^r,\ gy^r) = 0$. Hence

$$y^1 = gx^1 = gy^1, ..., y^r = gx^r = gy^r. \tag{2.17}$$

Thus, by (2.16) and (2.17), we get

$$y^1 = gy^1 \in F(y^1, ..., y^r), ...,\ y^r = gy^r \in F(y^r, y^1, ..., y^1),$$

that is, $(y^1, ..., y^r)$ is a common $r-$tupled fixed point of F and g.

If we put $\theta(t) = 0$ in the Theorem 2.6, we get the following result:

Corollary 2.7. Let (X, d) be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings satisfying (2.10) and $\{F, g\}$ is occasionally $w-$compatible. Then F and g have a common $r-$tupled fixed point.

If we put $\varphi(t) = t - t\widetilde{\varphi}(t)$ for all $t \geq 0$ in Corollary 2.7, then we get the following result:

Corollary 2.8. Let (X, d) be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings satisfying (2.11) and $\{F, g\}$ is occasionally $w-$compatible. Then F and g have a common $r-$tupled fixed point.

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Corollary 2.8, then we get the following result:

Corollary 2.9. Let (X, d) be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be two mappings satisfying (2.12) and $\{F, g\}$ is occasionally $w-$compatible. Then F and g have a common $r-$tupled fixed point.

If we put $\widetilde{\varphi}(t) = \frac{k}{2}$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.9, then we get the following result:

Corollary 2.10. Let (X, d) be a metric space, $F : X^r \to CB(X)$ and $g : X \to X$ be mappings satisfying (2.13) and $\{F,\ g\}$ is occasionally $w-$compatible. Then F and g have a common $r-$tupled fixed point.

Example 2.2. Suppose that $X = [0,\ 1]$, equipped with the metric $d : X \times X \to [0,\ +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y, \in X$. Let $F : X^r \to CB(X)$ be defined as

$$F(x^1, ..., x^r) = \begin{cases} \{0\}, & \text{for } x^1, ..., x^r = 1, \\ \left[0,\ \frac{x^1 + ... + x^r}{2r}\right], & \text{for } x^1, ..., x^r \in [0,\ 1). \end{cases}$$

and $g : X \to X$ be defined as

$$gx = \frac{x}{2} \text{ for all } x \in X.$$

Define $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \frac{t}{2}, \text{ for all } t \geq 0,$$

and $\varphi : [0, +\infty) \to [0, +\infty)$ by

$$\varphi(t) = \frac{t}{3}, \text{ for all } t \geq 0,$$

and $\theta : [0, +\infty) \to [0, +\infty)$ by

$$\theta(t) = \frac{t}{4}, \text{ for all } t \geq 0.$$

Now, for all $x^1, ..., x^r, y^1, ..., y^r \in X$ with $x^1, ..., x^r, y^1, ..., y^r \in [0, 1)$, we have
Case (a). If $x^1 + ... + x^r = y^1 + ... + y^r$, then

$$\begin{aligned}
&\psi\left(H(F(x^1, ..., x^r), F(y^1, ..., y^r))\right)\\
&= \frac{1}{2}H(F(x^1, ..., x^r), F(y^1, ..., y^r))\\
&= \frac{1}{4r}(y^1 + ... + y^r)\\
&\leq \frac{1}{2r}\max\left\{\frac{x^1}{2}, \frac{y^1}{2}\right\} + ... + \frac{1}{2r}\max\left\{\frac{x^r}{2}, \frac{y^r}{2}\right\}\\
&\leq \frac{1}{2r}d(gx^1, gy^1) + ... + \frac{1}{2r}d(gx^r, gy^r)\\
&\leq \frac{1}{r}M(x^1, ..., x^r, y^1, ..., y^r)\\
&\leq \psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right) - \varphi\left(\psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right)\right)\\
&\leq \psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right) - \varphi\left(\psi\left(M(x^1, ..., x^r, y^1, ..., y^r)\right)\right)\\
&+ \theta\left(N(x^1, ..., x^r, y^1, ..., y^r)\right).
\end{aligned}$$

Case (b). If $x^1 + ... + x^r \neq y^1 + ... + y^r$ with $x^1 + ... + x^r < y^1 + ... + y^r$, then

$$
\begin{aligned}
& \psi\left(H(F(x^1,...,x^r),F(y^1,...,y^r))\right) \\
= \ & \frac{1}{2}H(F(x^1,...,x^r),F(y^1,...,y^r)) \\
= \ & \frac{1}{4r}(y^1+...+y^r) \\
\leq \ & \frac{1}{2r}\max\left\{\frac{x^1}{2},\frac{y^1}{2}\right\}+...+\frac{1}{2r}\max\left\{\frac{x^r}{2},\frac{y^r}{2}\right\} \\
\leq \ & \frac{1}{2r}d(gx^1,gy^1)+...+\frac{1}{2r}d(gx^r,gy^r) \\
\leq \ & \frac{1}{r}M(x^1,...,x^r,y^1,...,y^r) \\
\leq \ & \psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)-\varphi\left(\psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)\right) \\
\leq \ & \psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)-\varphi\left(\psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)\right) \\
& +\theta\left(N(x^1,...,x^r,y^1,...,y^r)\right).
\end{aligned}
$$

Similarly, we obtain the same result for $y^1+...+y^r<x^1+...+x^r$. Thus the contractive condition (2.1) is satisfied for all $x^1,...,x^r,y^1,...,y^r\in X$ with $x^1,...,x^r,y^1,...,y^r\in[0,1)$. Again, for all $x^1,...,x^r,y^1,...,y^r\in X$ with $x^1,...,x^r\in[0,\ 1)$ and $y^1,...,y^r=1$, we have

$$
\begin{aligned}
& \psi\left(H(F(x^1,...,x^r),F(y^1,....,y^r))\right) \\
= \ & \frac{1}{2}H(F(x^1,...,x^r),F(y^1,...,y^r)) \\
= \ & \frac{1}{4r}(x^1+...+x^r) \\
\leq \ & \frac{1}{2r}\max\left\{\frac{x^1}{2},\frac{y^1}{2}\right\}+...+\frac{1}{2r}\max\left\{\frac{x^r}{2},\frac{y^r}{2}\right\} \\
\leq \ & \frac{1}{2r}d(gx^1,gy^1)+...+\frac{1}{2r}d(gx^r,gy^r) \\
\leq \ & \frac{1}{r}M(x^1,...,x^r,y^1,...,y^r) \\
\leq \ & \psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)-\varphi\left(\psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)\right) \\
\leq \ & \psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)-\varphi\left(\psi\left(M(x^1,...,x^r,y^1,...,y^r)\right)\right) \\
& +\theta\left(N(x^1,...,x^r,y^1,...,y^r)\right).
\end{aligned}
$$

Thus the contractive condition (2.1) is satisfied for all $x^1,...,x^r,y^1,...,y^r\in X$ with $x^1,...,x^r\in[0,\ 1)$ and $y^1,...,y^r=1$. Similarly, we can see that the contractive condition (2.1) is satisfied for all x^1, ..., x^r, y^1, ..., $y^1\in X$ with x^1, ..., x^r, y^1, ..., $y^r=1$. Hence, the hybrid pair $\{F, g\}$ satisfies the contractive condition (2.1), for all x^1, ..., x^r, y^1, ..., $y^r\in X$. In addition, all the other conditions of Theorem 2.1 and Theorem 2.6 are satisfied and $z=(0,...,0)$ is a common $r-$tripled fixed point of hybrid pair $\{F,\ g\}$. The function $F:X\to CB(X)$ involved in this example is not continuous at the point $(1,...,1)\in X^r$.

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