Quaternion Quasi-Normal Matrices

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ABSTRACT

In hopes that it will be useful to a wide audience, a long list of conditions on an $n \times n$ quaternion matrix A, equivalent to its being quaternion quasi-normal, is presented. In most cases, a description of why the condition is equivalent to quasi-normality is given.

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INTRODUCTION

Our purpose is to present a list of conditions on an $n \times n$ quaternion matrix A, each of which is equivalent to A, being normal. Presumably many of these are known, and no attempt at history or comprehensive literature survey is intended. We have often noted that knowledge of even some of the simpler conditions would have been useful to various authors and since we know of no similar list, our hope is that this will be generally useful to the community. For this purpose, we define A to be quaternion quasi-normalif and only if

$$AA^{CT} = A^T A^C$$

Though many conditions we have listed are similar, the list could be expanded much further by including variations on the statement of commutativity, etc. Also, we have refrained from going beyond characterizations of the quasi-normal of a single matrix and not included results about sums or products of quaternion quasi-normality matrices, etc. However, a number of papers dealing with quaternion quasi-normal matrices.

The condition of quasi-normality is a strong one, but, as it includes the Hermitian, Unitary and skew-Hermitian matrices, it is an important one which often appears as the appropriate level of generality in highly algebraic work and for numerical results dealing with perturbation analysis.

Main part of the paper

Definition: A quasi-normal matrix A is defined to be a quaternion quasi-normal matrix such that $AA^{CT} = A^T A^C$

Theorem:1 If A is quaternion quasi-normal then A^C is quaternion quasi-normal for any conjugate.

Proof:

Given A is quaternion quasi-normal. We have to prove A^C is quaternion quasi-normal, $A^C(A^C)^{CT} = A^C(A^{CT})^C = (AA^{CT})^C = (A^C)^T(A^C)^C$. Hence A^C is quaternion quasi-normal.

Theorem:2 If A is quaternion quasi-normal then A^{CT} is quaternion quasi-normal.

Proof:

Given A is quaternion quasi-normal. We have to prove A^{CT} is quaternion quasi-normal, $A^{CT}(A^{CT})^{CT} = (A^T A^C)^{CT} = (A^{CT})^T (A^{CT})^C$. Hence A^{CT} is quaternion quasi-normal.

Theorem:3 If A is quaternion quasi-normal then A^T is quaternion quasi-normal.

Proof:

Given A is quaternion quasi-normal. We have to prove A^T is quaternion quasi-normal, $A^T(A^T)^{CT} = A^T(A^{CT})^T = (A^T)^T(A^T)^C$. Hence A^T is quaternion quasi-normal.

Theorem:4 If the sum of quaternion quasi-normal matrices A and B are quaternion quasi-normal then $AB^{CT} + BA^{CT} = (B^{CT}A + A^{CT}B)^T$

Proof:

Let A and B are quaternion quasi-normal.

$$(A+B)(A+B)^{CT} = (A+B)(A^{CT}+B^{CT})$$

$$= A^T A^C + B^T B^C + AB^{CT} + BA^{CT}$$

$$(A+B)^T(A+B)^C = (A^T+B^T)(A^C+B^C)$$

Since $(A+B)(A+B)^{CT} = (A+B)^T(A+B)^C$ then

$$AB^{CT} + BA^{CT} = A^T B^C + B^T A^C.$$

Therefore $AB^{CT} + BA^{CT} = (B^{CT}A + A^{CT}B)^T$

Remark:1 The sum of quaternion quasi-normal matrices need not be quaternion quasi-normal matrix.

Theorem:5 If A and B are quaternion quasi-normal then the product of AB is also a quaternion quasi-normal

Proof:

Given A and B are quaternion quasi-normal matrix. $(AB)(AB)^{CT} = ABB^{CT}A^{CT} = (AB)^T(AB)^C$. Therefore AB is also a quaternion quasi-normal.

Remark:2 $P = P_0 + P_1 i$ and $P^C = (P_0 + P_1 i)^C = P_0^C - P_1^C i$. If $P = P^C$ then $P_0 = P_0^C$ and $P_1 = -P_1^C$

Remark:3 If P is quaternion hermitian then P_0 and P_1 are hermitian and skew-hermitian where $P = P_0 + P_1 i$ i.e., $P^{CT} = P_0^{CT} - P_1^{CT} i$

Remark:4 If P is skew-quaternion hermitian then P_0 and P_1 are skew-hermitian respectively, where $P = -P^{CT} = -P_0^{CT} + P_1^{CT} i$

Remark:5 From Remark: 1,2 , if and only if P is either quaternion hermitian Q is skew-hermitian, P is sum of hermitian and skew-hermitian matrices.

Theorem: 6 P is double representation of quaternion matrices of the form $P = P_0 + P_1 i$ iff P_0 and P_1 are normal.

Proof:

Given P is double representation, $P = P_0 + P_1 i$, $P * P^{CT} = P^{CT} * P$ or $P_0 P_0^{CT} - P_1 P_1^{CT} i = P_0^{CT} P_0 - P_1^{CT} P_1 i$ equating these part $P_0 P_0^{CT} = P_0^{CT} P_0$ and $P_1 P_1^{CT} = P_1^{CT} P_1$. Thus P_0 and P_1 are normal matrices. Conversely, assume that P_0 and P_1 are normal. $P_0 P_0^{CT} = P_0^{CT} P_0$ and $P_1 P_1^{CT} = P_1^{CT} P_1 \Rightarrow$ $P_1 P_1^{CT}(-i) = P_1^{CT} P_1(-i)$. $P_0 P_0^{CT} + P_1 P_1^{CT} i = P_0^{CT} P_0 + P_1^{CT} P_1 i$, $(P_0 + P_1 i) * (P_0^{CT} + P_1^{CT}) = (P_0^{CT} + P_1^{CT} i) * (P_0 + P_1 i) \Rightarrow P * P^{CT} = P^{CT} * P \Rightarrow P$ is normal.

Theorem: 7 If P is quaternion quasi-normal matrix of the form $P = P_0 + P_1 i$ where P_0 and P_1 are quasi-normal then P^C is quasi-normal.

Proof:

Given P is quaternion quasi-normal matrix. $P^C * (P^C)^{CT} = (P_0^C - P_1^C i)*(P_0^C - P_1^C i)^{CT}$ or $(P_0 P_0^{CT})^C + P_1(-P_1^{CT}) i \Rightarrow (P_0^C)^T (P_0^C)^C + (P_1^C)^T (P_1^C)^C i$. Therefore $P^C * (P^C)^{CT} = (P^C)^T * (P^C)^C$. Hence P^C is quasi-normal.

Theorem: 8 If A is quaternion quasi-normal matrix then $p(A)$ is quaternion quasi-normal matrix for any polynomial of degree n.

Proof:

Let $p(A) = \alpha_0 + \alpha_1 A + \alpha_2 A^2 + \ldots\ldots\ldots + \alpha_n A^n$, $p(AA^{CT}) = \alpha_0 + \alpha_1 (AA^{CT}) + \alpha_2 (AA^{CT})^2 + \ldots\ldots\ldots + \alpha_n (AA^{CT})^n \Rightarrow p(AA^{CT}) = p(A^T A^C)$ therefore $p(A)$ is quaternion quasi-normal matrix.

Theorem:9 If A is quaternion quasi-normal matrix then A^{-1} is quaternion quasi-normal for invertible A.

Proof:

Let A is quaternion quasi-normal matrix then $AA^{CT} = A^T A^C$, $A^{-1}(A^{-1})^{CT} = (A^{CT} A)^{-1}$ $= (A^T)^{-1}(A^C)^{-1} = (A^{-1})^T (A^{-1})^C$. Therefore, A^{-1} is quaternion quasi-normal for invertible A.

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