K - Class estimators - a Review

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Abstract — *In the estimation of linear equations various estimators have been developed over the years. An important family of estimators which includes many interesting estimators is $k-$ class estimators based on the pioneering work of Previous researchers. In this paper an attempt has been made to review the work done on the properties of $k-$ class estimators, both asymptotic and small sample properties. The case of small disturbances and system estimators has also been covered. In the end scope for their investigations is briefly discussed.*

Keywords — *Estimation, Econometric relation, Least Squares, Asymptotic and small sample, Small disturbance.*

An important family of estimators that encompasses many interesting estimators is k -class estimators based on the pioneering work and studies of [11] and [3].

This family for the coefficient vector δ is defined as

$$\hat{\delta}_k = \left[A_1'\left(I_T - k\,\overline{P}_X\right)A_1\right]^{-1} A_1'\left(I_T - k\,\overline{P}_X\right)y$$

When $\overline{P}_X \leq I_T - X\left(X'X\right)^{-1}X'$ and k is the scalar characterizing the estimators.

Hence, if $k=0$, we obtain the OLS estimators $\hat{\delta}_{OLS}$ while for $k=1$, we get 2SLS estimators $\hat{\delta}_{2SLS}$. Similarly if we take k to be stochastic and substitute $k=\lambda$, we get LIML estimator where λ is given by

$$\lambda = \min_{\beta} \frac{\left(y - Y_1\beta\right)' \overline{P}_{X1}\left(y - Y_1\beta\right)}{\left(y - Y_1\beta\right)' \overline{P}_X\left(y - Y_1\beta\right)} \qquad [11]$$

plotted the k -class estimates in the Girshick-Haavelmo model for the values of k ranging between 0 and 1.5. [2] established that the k-class estimator can be interpreted as an instrumental variable estimator. Maeshiro[6] considered a structural equation containing only one explanatory jointly dependent variable and explained that from a given member of k -class, how the other k -class estimators can be generated. Oi[34] proved that the k-class estimator can be derived as classical least squares estimator in a transformed structural equation, and obtained a mathematical relationship connecting k-class and two-stage least square estimators. He also showed that k -class estimator is a weighted average of ordinary least squares and two-stage least squares estimators. A simple derivation of the identity between k-class and two-stage least squares can be worked out following [31].

The k -class estimator is consistent when

$$p\lim(k-1)=0$$

If

$$p\lim\sqrt{T}(k-1)=0$$

The k -class estimator has the same asymptotic second order moment matrix as the two stage least squares estimator. It is shown that all the k-class estimators with characterizing scalar k that includes LIML estimator too have identical asymptotic properties and therefore the search for an optimal k on the basis of them is futile.

The large sample approach was pioneered by Nagar[3] who worked out the bias vector analytical methods, and mean square error matrix of the asymptotic distribution of consistent k -class estimators with fixed k as the number of observations grow large.

Kadane[14]added the small disturbance approach and obtained the bias vector and mean squared error matrix of the asymptotic distribution of k-class estimators when disturbances are small. Later Sawa[25] envisaged a structural equation with merely one explanatory jointly dependent variable and no lagged endogenous variables. He derived exact expressions for first two moments of k-class estimators with $0 \leq k \leq 1$ under normality of disturbances. As reference [33] provided a simple, derivation of his results. They also obtained results when k is negative.

For estimating the disturbance variance σ^2, a general family of estimators stemming from k-class is proposed by Brown *et.al.*[10]. An interesting subset of that family is defined as follows

$$\hat{\sigma}^2 = \hat{u}'\left[\frac{\nu}{\alpha}P_X + \frac{1-\nu}{\alpha}\left(I_T - P_X\right)\right]\hat{u}_k$$

Where, $\hat{u}_k = y - A_1\hat{\delta}_k$

$$P_X = X\left(X'X\right)^{-1}X'$$

ν and α being the scalars characterizing the estimator.

Brown *et al.*[10] studied the small disturbance approximations for bias and mean squared error. Also, Srivastava [29] worked out the exact moments of estimator for a structural equation containing merely one explanatory endogenous variables and no lagged endogenous variables in the model.

Various special cases of $\hat{\sigma}^2$ are in wide spread use. But still the estimation of covariance's of disturbances has not received sufficient attention. The only work in this regard was done by [30]

For finite sample properties of various estimation methods in a system of simultaneous equation several pioneering works includes Basmann [19, 20, 21], Kabe[7, 8] and Bergstrom [1]. They derived the exact finite-sample density function of the two stage or ordinary least square estimators in certain specific systems including at least three equations. other studies by Richardson [9], Sawa [23, 24], Mariano[22] and Takeuchi [16] also considered the finite sample problem and analysed the exact finite sample properties of the ordinary and two stage least squares and or limited information maximum likelihood estimators.

Nagar [3] studied the small sample properties of the general k-class estimators of simultaneous equations. He obtained two members of the family of k-class estimators, one to be unbiased and the other found to possesses a minimum second order moment around the true parameter value. He also analyzed the bias and moment matrix of the general k-class estimators of the coefficients of a single equation which is a part of simultaneous equations. He gave theorems on bias and moment matrix which follows certain assumptions. He obtained the bias of the 2SLS estimators and also gave the moment matrix of the 2SLS estimators and the 'best' value of k in a certain sense.

Quandt [18], computed with the help of sampling experiments on estimates for alternative values of k, with emphasis on direct least-squares $(k = 0)$ and two stage-least squares $(k = 1)$.

He showed that two stage least squares estimates are not unambiguously better than direct least squares estimates in small sample situations. However, the estimates are relatively poor when there are high multi-collinearity among exogenous variables, with two stage least squares being relatively more affected. Also, the distribution of two stage least squares has higher density than direct least squares in some neighborhood of the true value, but it also has thicker tails.

The k class estimates obtained by Quandt [18] may be considered rational alternatives to both two-stage least squares and direct least squares.

Cragg [13] investigated these results in addition to giving more experience in the use of simultaneous-equation estimators. He investigated six estimates of the structural coefficients viz, ordinary least squares (OLS), Two stage-least square (2SLS), Nagar's unbiased k-class estimator (UBK), Limited information maximum likelihood (LIML), Three-stage least squares (3SLS), Full information maximum likelihood (FIML).

The first four are the members of the k-class estimators and estimates one equation at a time. The other two methods are full-model methods, estimating all the coefficients simultaneously. Ordinary least squares is biased and inconsistent. 2SLS, UBK and LIML are consistent and all have the same asymptotic distribution. 3SLS and FIML are consistent and efficient and asymptotically the same. For finite sample the different estimators are distinct.

Several studies have either investigated the small sample properties of some specific member of k-class estimator in specific equation system or have investigated the approximate properties of the k-class estimators. However, no work has been done on exact sample properties of the simultaneous equations.

Kadiyala[17] studied this problem. He considered that the OLS which corresponds to $k=0$ minimizes the residual sum of squares but it possesses an undesirable property that it is inconsistent. The other two popularly used k-class estimators are two stage least squares (TSLE) and the limited information single equation estimator (LISE), corresponding to $k=1$ are consistent and have the same asymptotic covariance matrices. There is no exact sample criterion to prefer one over the other, or to prefer any consistent k-class estimator over another consistent k-class estimator.

In estimation of parameters of econometric relations the finite sample distribution of the estimators are generally unknown. The finite sample distribution of the estimators like two stage least squares or k-class estimators is not known except for some special cases. In this regard approach given by Nagar [3, 5] and others have been adopted. In this approach the sampling error of an estimator is expressed as the sum of an infinite series of random variables, successive terms of which are of decreasing order of sample size. Therefore, the sample properties of the estimators under consideration can be approximated by the properties of the first few terms of the infinite series.

Srinivasan [26] criticized the Nagar approach saying it can yield an estimate for finite sample bias that differs from the true finite sample bias and ay suggest that the bias is infinite. It may also give infinite valued expressions for bias. However the results by Nagar cannot be termed as unbiased but further investigations are required which has not been done. It is said that Nagar approach had given inadequate attention to the difference between probability limit of a sequence of random variables and probability limit of sequence of their expected values.

To examine the small sample properties of k class estimators Monte-Carlo studies have been used with the assumption that disturbances are normally distributed. Estimators like 3SLS and k class do not depend on the normality of the disturbances and their asymptotic disturbance has also been obtained without assuming normality of errors (Theil [12]).

The estimators studied are OLS, 2SLS, 3SLS and Full information maximum likelihood are asymptotically efficient. It may be noted that likelihood estimator and 2SLS are members of k class for $k=0$, and $k=1$ respectively. Also the main findings of the study is that the small sample ranking of least square, 2SLS, 3SLS and FIML estimators, except in a few instances, are invariant to the form of the error distribution. The small properties are seems, by and large to accord with the asymptotic and finite sample properties of the above estimators.

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