Deriving Shape Functions For 8-Noded Rectangular Serendipity Element in Horizontal Channel Geometry and Verified

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Abstract — *In this paper, I derived shape functions for 8-noded rectangular serendipity element in horizontal channel geometry and also I verified two verification conditions for shape functions. First verification condition is sum of all the shape functions is equal to one and second verification condition is each shape function has a value of one at its own node and zero at the other nodes. For computational purpose I used Mathematica 9 Software [2].*

Keywords — *Serendipity element, Horizontal Channel Geometry, Shape functions*.

I. INTRODUCTION

In between two parallel planes due to flow takes along the x-axis of the channel three dimensional plane changes to two dimensional yz- plane. Shape functions have great significance in finite element method. Once if we find shape functions for that particular geometry we can easily analyze heat and mass analysis. In geometry if nodes are located on the boundary only then we call it as serendipity element [1].

II. GEOMETRICAL DESCREPTION

We choose the Cartesian frame of reference 0(x,y,z), such that the imposed pressure gradient is along x-axis and $y = \pm h$ are the boundary planes shown in Fig.1. The flow occurs at low concentration difference so that the thermo-diffusion effects and the interfacial velocity due to mass diffusion can be neglected. In the absence of extraneous forces the flow is unidirectional along x-axis.

In view of the two dimensionality and symmetry of the flow w.r.t. the midplane of the channel we analyse the flow features in a domain in the upper half of the channel bounded by the impermeable wall lying between two parallel planes normal to the wall at unit distance apart. The finite element analysis with quadratic approximation functions is carried out using eight noded serendipity elements.

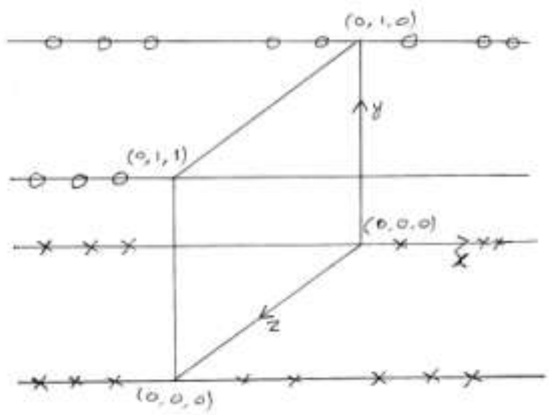

Fig.1: Schematic diagram of Horizontal Channel

For computational purpose we choose a serendipity element with (0,0),(0,1),(1,0) and (1,1) as its vertices.

The eight nodes of the element are shown in Fig.2.

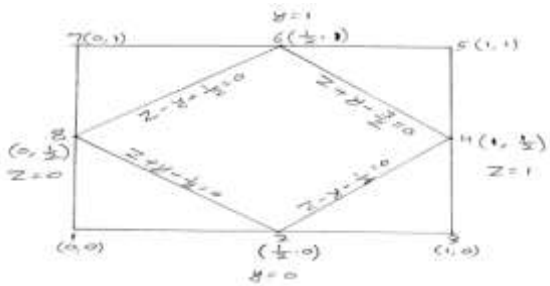

Fig.2: Eight noded Rectangular serendipity Element.

The element consists of eight nodes all of which are located on the boundary.

III. DERIVING SHAPE FUNCTIONS FOR 8-NODED RECTANGULAR SERENDIPITY ELEMENT IN HORIZONTAL CHANNEL GEOMETRY

Our task is to define shape functions N_i such that $N_i = 1$ at node i and 0 at all other nodes.

(I a) Shape function for Node 1
Corner node 1

z,y

At Node1(0,0)

$\therefore z=0, y=0$ at node 1

$$Node2\left(\overset{z_1,y_1}{\frac{1}{2},0}\right) \text{ and } Node8\left(\overset{z_2,y_2}{0,\frac{1}{2}}\right)$$

Two point formulat for Node2 and Node8

$$\frac{z-z_1}{z_2-z_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{z-\frac{1}{2}}{0-\frac{1}{2}}=\frac{y-0}{\frac{1}{2}-0}\Rightarrow\frac{z-\frac{1}{2}}{-\frac{1}{2}}=\frac{y}{\frac{1}{2}}\Rightarrow z-\frac{1}{2}=-\frac{1}{2}\left(\frac{y}{\frac{1}{2}}\right)$$

$$\Rightarrow z-\frac{1}{2}=-y\Rightarrow z+y-\frac{1}{2}=0$$

$N_1=1$ at node 1 and 0 at other nodes.

Thus N_1 has to vanish along the lines

z=1,y=1 and z+y-$\frac{1}{2}=0$

(For node 1 we should not take y=0 & z=0)

Consequently, N_1 is of the form

$$N_1=C(z-1)(y-1)\left(z+y-\frac{1}{2}\right) \quad (1)$$

Where C is some constant. The constant is determined from the condition $N_1=1$ at node 1.

we should find constant c value

*Node*1(0,0)

$z=0, y=0$ at node 1

$$N_1=C(0-1)(0-1)\left(0+0-\frac{1}{2}\right)=C(-1)(-1)\left(-\frac{1}{2}\right)$$

$$=C\left(-\frac{1}{2}\right) \quad (\because N_1=1)$$

$$1=C\left(-\frac{1}{2}\right)\Rightarrow C=\frac{2}{-1}\Rightarrow C=-2$$

$$(1)\Rightarrow N_1=-2(z-1)(y-1)\left(z+y-\frac{1}{2}\right) \quad (2)$$

(II a) Shape function for Node 3

Corner node 3

z,y

At Node3(1,0)

$\therefore z=1, y=0$ at node 3

$$Node2\left(\overset{z_1,y_1}{\frac{1}{2},0}\right) \text{ and } Node4\left(\overset{z_2,y_2}{1,\frac{1}{2}}\right)$$

Two point formula for Node2 and Node4

$$\frac{z-z_1}{z_2-z_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{z-\frac{1}{2}}{1-\frac{1}{2}}=\frac{y-0}{\frac{1}{2}-0}\Rightarrow\frac{z-\frac{1}{2}}{\frac{1}{2}}=\frac{y}{\frac{1}{2}}\Rightarrow z-\frac{1}{2}=-\frac{1}{2}\left(\frac{y}{\frac{1}{2}}\right)$$

$$\Rightarrow z-\frac{1}{2}=y\Rightarrow z-y-\frac{1}{2}=0$$

$N_3=1$ at node 3 and 0 at other nodes.

Thus N_3 has to vanish along the lines

z=0,y=1 and z-y-$\frac{1}{2}=0$

(For node 3 we should not take y=0 & z=1)

Consequently, N_3 is of the form

$$N_3=C(z-0)(y-1)\left(z+y-\frac{1}{2}\right) \quad (3)$$

Where C is some constant. The constant is determined from the condition $N_3=1$ at node 3.

we should find constant c value

*Node*3(1,0)

$z=1, y=0$ at node 3

$$N_3=C(1-0)(0-1)\left(1-0-\frac{1}{2}\right)$$

$$=C(1)(-1)\left(1-\frac{1}{2}\right)=C\left(-\frac{1}{2}\right) \quad (\because N_3=1)$$

$$1=C\left(-\frac{1}{2}\right)\Rightarrow C=\frac{2}{-1}\Rightarrow C=-2$$

$$(3)\Rightarrow N_3=-2z(y-1)\left(z-y-\frac{1}{2}\right) \quad (4)$$

(III a) Shape function for Node 5

Corner node 5

z,y

At Node5(1,1)

$\therefore z=1, y=1$ at node 5

$$Node4\left(1,\frac{1}{2}\right)^{z_1,y_1} \text{ and } Node6\left(\frac{1}{2},1\right)^{z_2,y_2}$$

Two point formula for Node4 and Node6

$$\frac{z-z_1}{z_2-z_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{z-1}{\frac{1}{2}-1}=\frac{y-\frac{1}{2}}{1-\frac{1}{2}}\Rightarrow\frac{z-1}{-\frac{1}{2}}=\frac{y-\frac{1}{2}}{\frac{1}{2}}$$

$$\Rightarrow z-1=-\frac{1}{2}\left(\frac{y-\frac{1}{2}}{\frac{1}{2}}\right)$$

$$\Rightarrow z-1=-y+\frac{1}{2}\Rightarrow z+y-1-\frac{1}{2}=0$$

$$z+y+\left(\frac{-2-1}{2}\right)=0\Rightarrow z+y-\frac{3}{2}=0$$

$N_5=1$ at node 5 and 0 at other nodes.

Thus N_5 has to vanish along the lines

y=0,z=0 and z+y-$\frac{3}{2}=0$

$\left(\begin{array}{l}\text{For node 5 we should not}\\ \text{take y=1 \& z=1}\end{array}\right)$

Consequently, N_5 is of the form

$$N_5=C(y-0)(z-0)\left(z+y-\frac{3}{2}\right)\quad(5)$$

Where C is some constant. The constant is determined from the condition $N_5=1$ at node 5.

we should find constant c value

*Node*5(1,1)

$z=1, y=1$ at node 5

$$N_5=C(1-0)(1-0)\left(1+1-\frac{3}{2}\right)$$

$$=C(1)(1)\left(2-\frac{3}{2}\right)=C\left(\frac{4-3}{2}\right)\quad(\because N_5=1)$$

$$1=C\left(\frac{1}{2}\right)\Rightarrow C=2$$

$$(5)\Rightarrow N_5=2yz\left(z+y-\frac{3}{2}\right)\quad(6)$$

(IV a) Shape function for Node 7

Corner node 7

z,y

At Node5(0,1)

$\therefore z=0, y=1$ at node 7

$$Node6\left(\frac{1}{2},1\right)^{z_1,y_1} \text{ and } Node8\left(0,\frac{1}{2}\right)^{z_2,y_2}$$

Two point formula for Node6 and Node8

$$\frac{z-z_1}{z_2-z_1}=\frac{y-y_1}{y_2-y_1}$$

$$\frac{z-\frac{1}{2}}{0-\frac{1}{2}}=\frac{y-1}{\frac{1}{2}-1}\Rightarrow\frac{z-\frac{1}{2}}{-\frac{1}{2}}=\frac{y-1}{-\frac{1}{2}}$$

$$\Rightarrow z-\frac{1}{2}=-\frac{1}{2}\left(\frac{y-1}{-\frac{1}{2}}\right)$$

$$\Rightarrow z-\frac{1}{2}=y-1\Rightarrow z-y-\frac{1}{2}+1=0$$

$$z-y+\left(\frac{-1+2}{2}\right)=0\Rightarrow z-y+\frac{1}{2}=0$$

$N_7=1$ at node 7 and 0 at other nodes.

Thus N_7 has to vanish along the

lines y=0,z=1 and z-y+$\frac{1}{2}=0$

$\left(\begin{array}{l}\text{For node 7 we should not}\\ \text{take z=0 \& y=1}\end{array}\right)$

Consequently, N_7 is of the form

$$N_7=C(y-0)(z-1)\left(z-y+\frac{1}{2}\right)\quad(7)$$

Where C is some constant. The constant is determined from the condition $N_7=1$ at node 7.

we should find constant c value

*Node*7(0,1)

$z=0, y=1$ at node 7

$$N_7 = C(1-0)(0-1)\left(0-1+\frac{1}{2}\right)$$

$$=C(1)(-1)\left(-1+\frac{1}{2}\right) = C(-1)\left(\frac{-2+1}{2}\right)$$

$(\because N_7 = 1)$

$$1 = C\left(\frac{1}{2}\right) \Rightarrow C = 2$$

$$(7) \Rightarrow N_7 = 2y(z-1)\left(z-y+\frac{1}{2}\right) \quad (8)$$

V a) Shape function for Mid Node 2

Shape function for Node $2\left(\frac{1}{2},0\right)$

$z = \frac{1}{2}, y = 0$

$N_2 = 1$ at node 2 and 0 at other nodes.
Thus N_2 has to vanish along the lines z=0,y=1 and z=1
(For Node 2 we should not take y=0)
Consequently, N_2 is of the form

$$N_2 = C(z-0)(y-1)(z-1) \quad (9)$$

We should find constant C value

Node $2\left(\frac{1}{2},0\right) \Rightarrow z = \frac{1}{2}, y = 0$

$$N_2 = C\left(\frac{1}{2}-0\right)(0-1)(z-1)$$

$$= C\left(\frac{1}{2}\right)(-1)\left(\frac{1}{2}-1\right) = C\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$= C\left(\frac{1}{4}\right) \quad (\because N_1 = 1)$$

$$1 = C\left(\frac{1}{4}\right) \Rightarrow C = 4$$

$$(9) \Rightarrow N_2 = 4z(y-1)(z-1) \quad (10)$$

VI a) Shape function for Mid Node 4

Shape function for Node $4\left(1,\frac{1}{2}\right)$

$z = 1, y = \frac{1}{2}$ at node 4

$N_4 = 1$ at node 4 and 0 at other nodes.
Thus N_4 has to vanish along the lines y=0,z=0 and y=1
(For Node 4 we should not take z=1)
Consequently, N_4 is of the form

$$N_4 = C(y-0)(z-0)(y-1) \quad (11)$$

We should find constant C value

Node $4\left(1,\frac{1}{2}\right) \Rightarrow z = 1, y = \frac{1}{2}$

$$N_4 = C\left(\frac{1}{2}-0\right)(1-0)\left(\frac{1}{2}-1\right)$$

$$= C\left(\frac{1}{2}\right)(1)\left(\frac{1-2}{2}\right) = C\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$= C\left(-\frac{1}{4}\right) \quad (\because N_4 = 1)$$

$$1 = C\left(-\frac{1}{4}\right) \Rightarrow C = -4$$

$$(11) \Rightarrow N_4 = -4yz(y-1) \quad (12)$$

VII a) Shape function for Mid node 6

Shape function for Node $6\left(\frac{1}{2},1\right)$

$z = \frac{1}{2}, y = 1$ at node 6

$N_6 = 1$ at node 6 and 0 at other nodes.
Thus N_6 has to vanish along the lines z=0,y=0 and z=1
(For Node 6 we should not take y=1)
Consequently, N_6 is of the form

$$N_6 = C(z-0)(y-0)(z-1) \quad (13)$$

We should find constant C value

Node $6\left(\frac{1}{2},1\right) \Rightarrow z = \frac{1}{2}, y = 1$

$$N_6 = C\left(\frac{1}{2}-0\right)(1-0)\left(\frac{1}{2}-1\right)$$

$$= C\left(\frac{1}{2}\right)(1)\left(\frac{1-2}{2}\right) \quad (\because N_6 = 1)$$

$$1 = C\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \Rightarrow 1 = C\left(-\frac{1}{4}\right)$$

$$\Rightarrow C = \frac{4}{-1} \Rightarrow C = -4$$

$(13) \Rightarrow N_6 = -4zy(z-1) \qquad (14)$

($VIII$ a) Shape function for *Mid* Node 8

Shape function for Node 6 $\left(0, \frac{1}{2}\right)$

$z = 0, y = \frac{1}{2}$ at node 6

$N_8 = 1$ at node 8 and 0 at other nodes.

Thus N_8 has to vanish along the lines y=0,z=1 and y=1

(For Node 8 we should not take z=0)

Consequently, N_8 is of the form

$N_8 = C(y-0)(z-1)(y-1) \qquad (15)$

We should find constant C value

Node 8 $\left(0, \frac{1}{2}\right)$

$z = 0, y = \frac{1}{2}$ at node 8

$$N_8 = C\left(\frac{1}{2} - 0\right)(0-1)\left(\frac{1}{2} - 1\right)$$

$$= C\left(\frac{1}{2}\right)(-1)\left(\frac{1-2}{2}\right) = C\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$$

$$N_8 = C\left(\frac{1}{4}\right) \Rightarrow 1 = C\left(\frac{1}{4}\right)$$

$\Rightarrow C = 4 \quad (\because N_8 = 1)$

$(15) \Rightarrow N_8 = 4y(z-1)(y-1) \qquad (16)$

IV. VERIFICATION

(I b) 1[st] Condition

Sum of all the shape functions is equal to one

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8 = (2) + (4) + (6) + (8) + (10) + (12) + (14) + (16)$$

Output

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7 + N_8 = 1$$

(II b) 2[nd] Condition

Each shape function has a value of one at its own node and zero at the other nodes.

(i) *At* Node 1 (0,0) z=0 , y=0

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
Output	1	0	0	0	0	0	0	0

(ii) *At* Node 2 $(\frac{1}{2},0)$ z=$\frac{1}{2}$ y=0

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
Output	0	1	0	0	0	0	0	0

(iii) *At* Node 3 (1,0) z=1,y=0

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
Output	0	0	1	0	0	0	0	0

(iv) *At* Node 4$(1,\frac{1}{2})$ z=1,y=$\frac{1}{2}$

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
Output	0	0	0	1	0	0	0	0

(v) *At* Node 5(1,1) z=1,y=1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
Output	0	0	0	0	1	0	0	0

(vi) *At* Node 6$(\frac{1}{2},1)$ z=$\frac{1}{2}$, $y=1$

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
Output	0	0	0	0	0	1	0	0

(vii) *At* Node 7(0,1) z=0,y=1

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
Output	0	0	0	0	0	0	1	0

($viii$) *At* Node 8$(0,\frac{1}{2})$ z=0,y=$\frac{1}{2}$

	N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8
Output	0	0	0	0	0	0	0	1

V. AUTHOR'S CONTRIBUTION

1. Derived Shape functions for 8-noded Rectangular serendipity element in

Horizontal Channel Geometry.

2. Verified sum of all the shape functions is equal to one.
3. Verified each shape function has a value of one at its own node and zero at the other nodes.

References

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