Behaviour of Solutions of Linear Systems

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Abstract

This paper deals with the behaviour of solutions of linear systems. The notions of stability, boundedness and asymptotic behaviour of solutions of a general linear system are studied.

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1.INTRODUCTION

We consider the behaviour of the solutions of the linear differential equation

$$\frac{dz}{dt} = [A + B(t)]z \rightarrow (1)$$

where A is a constant matrix and B (t) is small as $t \rightarrow \infty$.

Two particularly important cases are those where $||B(t)|| \rightarrow 0$ or where $\int_0^\infty ||B(t)||dt < \infty$

The solutions of (1) share many properties with the solutions of

$$\frac{dy}{dt} = Ay \rightarrow (2)$$

so far as their behaviours are concerned.

In this paper, section 2 deals with the concept of stability of linear equations. In section 3, the boundedness property of solutions and the sufficient conditions for boundedness of solutions are studied in detail. Section 4 illustrates the asymptotic behaviour of solutions of linear systems.

2.STABILITY OF LINEAR EQUATIONS [1]

2.1Defn: The solutions of $\frac{dy}{dt} = A(t)y \rightarrow (3)$ arestable with respect to a property P and perturbationsB(t) of type T if the solutions of $\frac{dz}{dt} = [A(t) + B(t)]z \rightarrow (4)$ also possess property P. If this is not true, the solutions of (3) are said to be unstable with respect to property P under perturbations of type T.

To illustrate the above definition, we consider two simple differential equations:

$$\frac{du}{dt} = -au \rightarrow (5)$$

and

$$\frac{dv}{dt} = [-a + b(t)]v \rightarrow (6)$$

where $a > 0$ and $b(t) \rightarrow 0$ as $t \rightarrow \infty$.

Considering the solutions of (5) and (6)

$\frac{du}{dt} = -au \Rightarrow \frac{du}{u} = -adt$

$\Rightarrow \log u = -at + c$

$\Rightarrow u = ce^{-at} \Rightarrow \lim_{t\to\infty} u = 0$

$$\frac{dv}{dt} = [-a + b(t)]v$$

$$\Rightarrow \frac{dv}{v} = [-a + b(t)]dt$$

$$\Rightarrow \log v = -at + \int b(t)dt$$

$$\Rightarrow v = e^{-at+\int b(t)dt}$$

$$\Rightarrow \lim_{t\to\infty} v = 0$$

Thus,both solutions u and v tend to zero as $t \rightarrow \infty$

Also, $\lim_{t\to\infty} \frac{\log u}{t} = \lim_{t\to\infty} \frac{\log v}{t} = -a$

So, both solutions u and v have the following properties:

i) $\lim_{t\to\infty} u = \lim_{t\to\infty} v = 0$

and

ii) $\lim_{t\to\infty} \frac{\log u}{t} = \lim_{t\to\infty} \frac{\log v}{t} = -a$

Now suppose a=0 and b(t) = $\frac{1}{t}$

Then $\frac{du}{dt} = -au => u = ce^{-at}$

$=> u = c \; when \; a = 0$

and

$\frac{dv}{dt} = [-a + b(t)]v$

$\Rightarrow \frac{dv}{v} = [-a + b(t)]dt = \left[-a + \frac{1}{t}\right]dt$

$\Rightarrow \log v = -at + \log t$

$\Rightarrow v = te^{-at}$

$\therefore v$ is unbounded as $t \rightarrow \infty$

With respect to property (ii), since a=0, $\frac{\log u}{t} = 0$

$$\therefore \lim_{t\to\infty} \frac{\log u}{t}$$

Also $\frac{\log v}{t} = 0 \Rightarrow \lim_{t\to\infty} \frac{\log v}{t} = 0$

Thus, there is stability with respect to property (ii) but instability with respect to property (i) – the property of boundedness.

2.2 Note:

If we replace $\frac{1}{t}$ by a function which is integrable over (t_0, ∞), then boundedness will be preserved.

2.3 Note:
The most important property of solutions is that of boundedness. If a solution is bounded, we are interested in knowing whether or not it approaches zero at $t \to \infty$

3.1 BOUNDEDNESS OF SOLUTIONS([1],[2])

3.1.1 Definition:

We call the coefficient matrix A(t) of the differential equation $\frac{dz}{dt} = A(t)z$ almost constant if $\lim_{t\to\infty} A(t) = A$, a constant matrix.

3.1.2 Lemma(Fundamental lemma):

If u,v≥ 0, if C_1 is a positive constant, and if

$$u \le C_1 + \int_0^t uv dt_1 \to (7)$$

then, $u \le C_1 \exp(\int_0^t v dt_1) \to (8)$

Proof: From (7), we have

$$\frac{uv}{C_1 + \int_0^t uv dt_1} \le v \to (9)$$

Integrating both sides of (9) between 0 and t, we get

$$\log[C_1 + \int_0^t uv\, dt_1] - \log C_1 \le \int_0^t v\, dt_1$$

i.e., $\log\left(\frac{C_1+\int_0^t uv\, dt_1}{C_1}\right) \le \int_0^t v\, dt_1$

$$C_1 + \int_0^t uv\, dt_1 \le C_1 \exp\left(\int_0^t v\, dt_1\right) \to (10)$$

$$\therefore u \le C_1 + \int_0^t uv\, dt_1 \le C_1 \exp\left(\int_0^t v\, dt_1\right) \to (11)$$

which is the fundamental lemma.

3.1.3Theorem:

If all solutions of

$$dy/dt = Ay \to (12)$$

where A is a constant matrix, are bounded as $t \to \infty$, then the same is true of the solutions of

$$\frac{dz}{dt} = [A + B(t)]z \to (13)$$

provided $\int_0^\infty ||B(t)|| dt < \infty$

Proof:

Equation (13) can be written as

$$\frac{dz}{dt} = Az + B(t)z \to (14)$$

Every solution of (14) satisfies a linear integral equation

$$z = y + \int_0^t Y(t - t_1) B(t_1)\, z(t_1)\, dt_1 \to (15)$$

where y is the solution of (12) for which y(0) = z(0) and Y is the matrix solution of

$$\frac{dY}{dt} = AY, Y(0) = I \to (16)$$

$\therefore$ we have $y = Yy(0) = Yz(0)$

Let $C_1 = \max(\sup_{t\ge0}||y||, \sup_{t\ge0}||Y||)$

Then, from (15), we get

$$||z|| \le ||y|| + \int_0^t ||Y(t-t_1)||\, ||B(t_1)||\, ||z(t_1)||\, dt_1$$

$$\le C_1 + C_1 \int_0^t ||B(t_1)||\, ||z(t_1)||\, dt_1 \to (17)$$

Applying the fundamental lemma in (17), we get

$$||z|| \le C_1 \exp\left(C_1 \int_0^t ||B(t_1)||\, dt_1\right) \to (18)$$

Since $\int_0^\infty ||B(t)||\, dt < \infty$, from (18), it follows that $||z||$ is bounded.

i.e., the solutions of equation (13) are bounded.

Hence, the theorem.

3.1.4Theorem

If all the solutions of the equation

$$\frac{dy}{dt} = Ay \to (19)$$

approach zero as $t \to \infty$, the same holds for the solutions of

$$\frac{dz}{dt} = [A + B(t)]z \to (20)$$

provided that $||B(t)|| \le C_1$ for $t \ge t_0$, where C_1 is a constant which depends upon A.

Proof

Every solution of (20) satisfies a linear integral equation

$$z = y + \int_0^t Y(t - t_1)\, B(t_1)\, z(t_1)\, dt_1 \to (21)$$

where Y is the matrix solution of $\frac{dY}{dt} = AY, Y(0) = I$

Now $\frac{dy}{dt} = AY \Rightarrow \frac{dY}{Y} = A\, dt$

$\Rightarrow \log Y = At + B$

$$\Rightarrow Y = e^{At+B} = Ce^{At}$$

Since $||Y|| \to 0$ as $t \to \infty$, $\exists$ a positive constant $\propto$ such that $||Y|| \le C_2 e^{-\propto t}$ for $t \ge 0 \to$ (22) By theorem 3.1.3, we have $y = Yy(0) \therefore ||y|| \le C_2 e^{-\propto t}$ for $t \ge 0 \to$ (23)

Hence, from (21),

$$||z|| \le C_2 e^{-\propto t} + C_2 \int_0^t e^{-\propto(t-t_1)} ||B(t_1)||\, ||z(t_1)||\, dt_1 \to ($$

Since $||B(t)|| \le C_1$ for $t \ge t_0$, we get

$$||z||e^{\propto t} \le C_2 + C_1 C_2 \int_0^t e^{\propto t_1} ||z(t_1)||\, dt_1 \to (25)$$

Applying fundamental lemma in equation (25), we get

$$||z||e^{\propto t} \le C_2 \exp\left(\int_0^t C_1 C_2\, dt_1\right) \to (26)$$

i.e., $||z||e^{\propto t} \le C_2 e^{C_1 C_2 t} \to (27)$

If $C_1C_2 < \propto$, then the above equation gives $||z|| \to 0$ as $t \to \infty$. But the constants C_2 and $\propto$ depend upon the characteristic roots of A. Hence, it follows that C_1 depends upon A. Hence the theorem.

3.2 SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF SOLUTIONS ([1])

In this section, the sufficient conditions required for the solutions of a linear system to be bounded, are being dealt with

The boundedness of the solutions of

$$\frac{dy}{dt} = A(t)y \to (28)$$

together with the condition $||B(t)|| \to 0$ as $t \to \infty$ is not sufficient to ensure the boundedness of all solutions of

$$\frac{dz}{dt} = [A(t) + B(t)]z \to (29)$$

Even if we amend the condition $||B(t)|| \to 0$ as $t \to \infty$, by the condition $\int_0^\infty ||B(t)|| \, dt < \infty$, the sufficiency remains unjustified. This fact is illustrated in the following theorem:

3.2.1 Theorem

There is an equation of the type

$$\frac{dy}{dt} = A(t)y$$

with the property that all solutions approach zero as $t \to \infty$, and a matrix $B(t)$ for which $\int_0^\infty ||B(t)|| \, dt < \infty$, such that all solutions of the equation $\frac{dz}{dt} = [A(t) + B(t)]z$ are not bounded.

Proof: Consider the equations

$$\frac{dy_1}{dt} = -ay_1 \to (30)$$

and

$$\frac{dy_2}{dt} = [\sin(\log t) + \cos(\log t) - 2a]y_2 \to (31)$$

We solve for y_1 and y_2

Equation (30) $\to$ $\frac{dy_1}{y_1} = -a\, dt$

Integrating, $\log y_1 = -at + \log C_1$

i.e., $y_1 = C_1 e^{-at}$

Equation (31)

$$\to \frac{dy_2}{y_2} = [\sin(\log t) + \cos(\log t) - 2a]dt$$

Integrating,

$$\log y_2 = [t \sin(\log t) - 2at] + \log C_2$$

$\therefore y_2 = C_2 e^{t\sin(\log t) - 2at}$

Thus, the general solutions of (30) and (31) are respectively

$$y_1 = C_1 e^{-at} \to (32)$$

and

$$y_2 = C_2 e^{t\sin(\log t) - 2at} \to (33)$$

If 'a' is any non-negative constant, then every solution of (32) and (33) approach zero as $t \to \infty$

Let

$$B(t) = \begin{bmatrix} 0 & 0 \\ e^{-at} & 0 \end{bmatrix} \text{ be the perturbing matrix}$$

The perturbed equation has the form

$$\frac{dz_1}{dt} = -az_1 \to (34)$$

and

$$\frac{dz_2}{dt} = [\sin(\log t) + \cos(\log t) - 2a]z_2 + z_1 e^{-at} \to (35)$$

Equation (34) gives $z_1 = C_1 e^{-at}$ and

Equation (35) gives $z_2 = e^{[t\sin(\log t) - 2at]} \left\{ C_2 + C_1 \int_0^t e^{-t_1 \sin(\log t_1)} dt_1 \right\}$

Let $t = e^{\left(2n + \frac{1}{2}\right)\pi}$

If $1 < 2a < 1 + e^{\frac{-\pi}{2}}$, then

$$\int_0^t e^{-t_1 \sin(\log t_1)} dt_1 > \int_{te^{-\pi}}^{te^{\frac{-2\pi}{3}}} e^{-t_1 \sin(\log t_1)} dt_1 > t\left(e^{\frac{-2\pi}{3}} - e^{-\pi}\right) \exp\left(-\frac{e^{-\pi} t}{2}\right),$$

which implies the solutions of (34) and (35) will be bounded only if $C_1 = 0$

Now, $C_1 = 0 \Leftrightarrow z_1(0) = 0$

Thus, if $z_1(0) \neq 0$, the solutions of (35) are not bounded

3.2.2 Theorem:

If all the solutions of the equation $\frac{dy}{dt} = A(t)y$ are bounded, then all the solutions of the equation $\frac{dz}{dt} = [A(t) + B(t)]z$ are bounded, provided

(a) $\int_0^\infty ||B(t)|| dt < \infty$

&

(b) $\lim_{t \to \infty} \int_0^t tr(A) dt > -\infty$

Proof: Expressing z in terms of y, we have

$$z = y + \int_0^t Y(t)\, Y^{-1}(t_1) B(t_1)\, z(t_1) dt_1$$

$$\therefore ||z|| \le ||y|| + \int_0^t ||Y(t)||\,||Y^{-1}(t_1)||\ \ ||B(t_1)||\ \ ||z(t_1)|| dt_1$$

Since $\det Y = \exp\left[\int_0^t tr(A) dt\right]$, if condition (b) is satisfied, then $||Y^{-1}(t)||$ is bounded as $t \to \infty$

$$\therefore ||z|| \le C_1 + C_1 \int_0^t ||B(t_1)||\,||z(t_1)||\, dt_1$$

Applying the fundamental lemma in the above equation, we get

$$||z|| \le C_1 \exp\left(C_1 \int_0^t ||B(t_1)|| dt_1\right)$$

Since $\int_0^\infty ||B(t)|| dt < \infty$, $||z||$ is bounded

Hence, the theorem.

4. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS [3]

In the previous section, we have dealt with the necessary and sufficient conditions for the boundedness of a solution. If a solution is bounded, we are interested in knowing whether or not it approaches zero as $t \rightarrow +\infty$, which depicts the asymptotic behaviour of the solutions.

Consider the linear system

$$\frac{dy}{dt} = A(t)y, \; t \geq 0 \rightarrow (36)$$

where A(t) is a real-valued continuous nxn matrix on $0 \leq t < \infty$

We want to find the behaviour of solutions of (36) as $t \rightarrow \infty$

If the eigen values of the matrix A are known, all solutions of (36) are completely determined. Hence, the eigen values determine the behaviour of solutions as $t \rightarrow \infty$

4.1Theorem:

Let A(t) be a real- valued, continuous, nxn matrix on $[0, \infty)$.

Let M(t) be the largest eigen value of $(t) + A^T(t)$, where $A^T(t)$ is the transpose of the matrix $A(t)$.

If $\lim_{t\to\infty} \int_{t_0}^{t} M(s)ds = -\infty \quad (t_0 > 0 \; is \; fixed) \rightarrow$ (37) , then every solution of (36) tends to zero as $t \rightarrow +\infty$.

Proof:

Let ϕ be a solution of (36)

Then $|\phi(t)|^2 = \phi^T(t)\phi(t)$

$$\therefore \frac{d}{dt}|\phi(t)|^2 = \phi^T(t)\phi^{\prime}(t) + \phi^{T^{\prime}}(t)\phi(t)$$

$$= \phi^T(t)A(t)\,\phi(t) + A^T(t)\,\phi^T(t)\,\phi(t)$$

$$= \phi^T(t)[A(t) + A^T(t)]\phi(t)$$

Since M(t) is the largest eigen value of the symmetric matrix $A(t) + A^T(t)$, we get

$$|\phi^T(t)[A(t) + A^T(t)]\phi(t)| \leq M(t)|\phi(t)|^2$$

Thus,

$$0 \leq |\phi(t)|^2 \leq |\phi(t_0)|^2 \left(\exp\left[\int_{t_0}^{t} M(s)ds\right]\right) \rightarrow (38)$$

By condition (37), the right side of (38) tends to zero

Hence, $\lim_{t\rightarrow\infty}\phi(t) = 0$

Hence, the theorem

4.2Theorem:

Let m(t) be the smallest eigen value of $A(t) + A^T(t)$. If

$$\limsup_{t\to\infty} \int_{t_0}^{t} m(s)ds = +\infty \; (t_0 > 0 \; is \; fixed) \rightarrow (39)$$

then every non-zero solution of (36) is unbounded as $t \rightarrow +\infty$.

Proof: Let ϕ be a solution of (36)

As in the previous theorem, we have

$$\frac{d}{dt}|\phi(t)|^2 = \phi^T(t)\,[A(t) + A^T(t)]\phi(t) \rightarrow (40)$$

Since m(t) is the smallest eigen value of $(t) + A^T(t)$, we get

$$\frac{d}{dt}|\phi(t)|^2 \geq m(t)|\phi(t)|^2$$

Thus, $\frac{d}{dt}\left\{e^{-\int_{t_0}^{t} m(s)ds}|\phi(t)|^2\right\}$

$$= e^{-\int_{t_0}^{t} m(s)ds}\left\{\frac{d}{dt}|\phi(t)|^2 - m(t)|\phi(t)|^2\right\} \geq 0$$

$$\therefore |\phi(t)|^2 \geq |\phi(t_0)|^2\, e^{\int_{t_0}^{t} m(s)ds} \rightarrow (41)$$

By condition (39) the right side of (41) tends to $+\infty$ as $t \rightarrow +\infty$.

$$\therefore \lim_{t\to\infty}|\phi(t)| = +\infty$$

i.e, the solution ϕ is unbounded

Hence, the theorem

5.CONCLUSION

This paper is a work on the behaviour of solutions of linear systems, when the time is increased indefinitely; which is a kind of stability property. This provides an insight into the necessary steps to be taken to avoid unwanted phenomena or criteria in a system.

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