Minimum Total Irregularity of Totally Segregated Trees

$JorryT.F.^1, DrParvathyK.S.^2$

1.Assistant Professor, Mercy College, Palakkad, Kerala, India. Pin:678006.
2.Associate Professor, Post Graduate and Research Department of Mathematics, St.Mary's College, Thrissur, Kerala, India, 680020.

Abstract:A tree, in which any two adjacent vertices have distinct degrees, is a Totally Segregated Tree (TST). Total Irregularity of a graph is defined as: $irr_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)|$. In this paper, total irregularity of TST is discussed and TST with $\triangle = 3$ is characterized. We determine the minimum total irregularity of TST on n vertices. Degree sequence of TST with minimum total irregularity is also found.

Keywords:Totally Segregated Tree; Total Irregularity; Minimum Total Irregularity; Maximum degree; Degree sequence.

1 Introduction

For a graph G, let $\triangle(G)$ be the maximum degree of the vertices of G, and V(G), E(G) be the set of all vertices and edges respectively in G. Number of edges incident to a vertex is the degree of that vertex. Degree sequence of a graph is a monotonic non-increasing sequence of the vertex degrees of the graph. A graph is regular if all its vertices have the same degree. Y. Alavi and G. Chartrand [7] defined a connected graph G to be highly Irregular if each of its vertices is adjacent only to vertices with distinct degrees. Jackson and Entringer [1] extended this concept by considering those graphs in which any two adjacent vertices have distinct degrees, and these graphs are named as totally segregated. Jorry T. F. and Parvathy K. S. [4] studied a special case, by considering those graphs in which degrees of any two adjacent vertices are differed by a constant $k \neq 0$, and these graphs are named as k-segregated. A graph G is k-segregated, if $|d_G(u) - d_G(v)| = k \neq 0$ for all edges $uv \in E(G)$ [4]. However, in many applications and problems, it is of great importance to know how irregular a given graph is. Several graph topological indices have been proposed for that purpose. Among them, the most investigated one is the total irregularity of a graph introduced by H. Abdo and D. Dimitrov in [3], which is defined as:

$$irr_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)| \tag{1}$$

and the upper bound of the set $\{irr_t(G) : |V(G)| = n\}$ is obtained. If T is a tree on n vertices, $irr_t(T) \leq (n-1)(n-2)$, and they showed the star graph is

the tree with maximum total irregularity among all trees on n vertices. Star on n vertices($K_{1,n-1}, n \geq 3$), is TST, particularly it is $(n-2)-$segregated tree. Hence, maximum total irregularity of totally segregated trees on n vertices is
$(n-1)(n-2)$, and the maximum is attained for $(n-2)-$ segregated tree ie $K_{1,n-1}$.

Yingxue Zhu, Lihua You, Jieshan Yang [6] determined minimum total irregularity of trees by the following theorem:
Theorem: Let $G = (V, E)$ be a tree on n vertices. Then $irr_t(G) \geq 2n-4$, and the equality holds if and only if $G \cong P_n$, P_n be the path on n vertices[6].

In this paper, minimum total irregularity of totally segregated trees is investigated. Throughout this paper let n_i denote the number of vertices with degree i. Here, equation (1) is rewritten as:

$$irr_t(G) = \sum_{d_i > d_j} (d_i - d_j) n_{d_i} n_{d_j} \tag{2}$$

where d_i, d_j are distinct degrees of vertices in G . Note that total irregularity of a given graph is completely determined by its degree sequence. Graphs with the same degree sequences have the same total irregularity. For convenience sake, we denote TST on n vertices with minimum total irregularity by minT_n, and degree sequence of TST is denoted by T, with two rows of numbers; the first row representing the degree of vertices and the second row representing number of vertices of corresponding degree. $T \begin{matrix} 3 & 2 & 1 \\ n_3 & n_2 & n_1 \end{matrix}$ represents degree sequence of tree T with $\triangle(T) = 3$.

2 Totally Segregated Tree on n vertices with minimum Total Irregularity ($minT_n$)

Theorem 2.1. *If T be a tree and $\triangle(T) = k$,*

$$n_1(T) = n_3(T) + 2n_4(T) + \cdots + (k-2)n_k(T) + 2$$

Proof. Let T be a tree with n vertices.Then $n = n_1 + n_2 + \cdots + n_k$.
By first theorem of graph theory, $2(n-1) = n_1 + 2n_2 + \ + kn_k$.
Eliminating n and solving for n_1, we get $n_1 = n_3 + 2n_4 + 3n_5 + (k-2)n_k + 2$
That is $n_1(T) = n_3(T) + 2n_4(T) + 3n_5(T) + (k-2)n_k(T) + 2$ □

Theorem 2.2. *In a TST with $\triangle = k.forj \neq 1, n_j\,(T) \leq \sum_{i=1,i\neq j}^{k} n_i - 1$*

Proof. Assume the result is not true. Then, for $j \neq 1, n_j\,(T) > \sum_{i=1,i\neq j}^{k} n_i - 1$. That is number of vertices of degree j exceeds one less than the total number of other vertices.Then, it is not TST since, in this case, at least two vertices of degree j are adjacent. Hence the result. □

Corollary 2.1. *In a TST, T with $\triangle = k, n_2\,(T) \leq (k-1)n_k\,(T) + (k-2)\,n_{k-1}\,(T) + \cdots + 2n_3\,(T) + 1$*

Theorem 2.3. *Let T be a TST on n vertices and* $\triangle(T) = 3$. *Then* $n_3(T)-1 \leq n_2(T) \leq 2n_3(T)+1$.

Proof. By crollory 2.1 $n_2(T) \leq 2n_3(T)+1$, We prove $n_3(T) \leq n_2(T)+1$ by mathematical induction on n_2. If $n_2(T) = 1$ possible values of $n_3 = 0, 1, 2$. Hence the result is true.
Let $n_2 \geq 2$. Assume the result is true for TST with $n_2 = k-1$. Let $n_2(T) = k$, and v be a vertex of maximum eccentricity among vertices of degree 2. Let u be a vertex at maximum distance from v. T has two branches at v.
Break T into two branches at v. One of these branches say T_1 contain $u-v$ path and

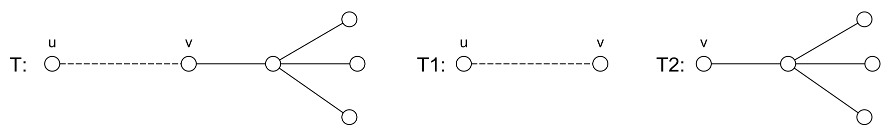

Figure 1:

$n_2(T_1) = k-1$.Then, by our assumption, we have $n_3(T_1) \leq (k-1)+1$ Other branch T_2 has no vertex of degree 2. If such vertex exists , eccentricity of that vertex is more than the eccentricity of v. Hence $n_3(T_2) \leq 1$.
Then $n_3(T) = n_3(T_1) + n_3(T_2) \leq k+1$.
That is, $n_3(T) \leq n_2(T)+1$. ie $n_2(T) \geq n_3(T)-1$. Hence the result. □

Remark 2.1. *In a tree T on n vertices with* $\triangle = 3$,
$n_2 \leq 2n_3+1 \Leftrightarrow n_3 \geq \frac{n-3}{4}$, $n_2 \leq 2n_3+1 \Leftrightarrow n_2 \leq \frac{n-1}{2}$
$n_2 \geq n_3-1 \Leftrightarrow n_3 \leq \frac{n-1}{3}$, $n_2 \geq n_3-1 \Leftrightarrow n_2 \geq \frac{n-4}{3}$

Subdivision of a graph G: A graph H is a subdivision of a graph G, if H can be obtained from G by inserting vertices of degree 2 into the edges of G [2].

Theorem 2.4. *Let k be any positive integer , then for any integer x such that* $k-1 \leq x \leq 2k+1$, *there exists TST with* $\triangle = 3$, $n_2 = x, n_3 = k$, *and* $n_1 = k+2$.

Proof. Consider a path P joining k vertices $v_1, v_2, \cdots, v_k$
Add new vertices $u_1, u_2, \cdots, u_{k+2}$.
join u_i to v_i for $i = 1, 2, \cdots, k$, Join u_{k+1} to v_1 and u_{k+2} to v_k. Let the resulting graph be T. Subdivide each edge in the path P in T to obtain a TST, say T' with $n_2\left(T'\right) = x = k-1$. If $k-1 < x \leq 2k+1$, subdivide $x-(k-1)$ edges in the remaining $k+2$ edges incident at $u_1, u_2, \cdots, u_{k+2}$. When $n_2(T') = x = 2k+1$, TST is 1-segregated, and is shown in Figure 2

□

Branch-transformation: [6]
Let G be a simple graph with at least two pendent vertices, and let u be a vertex of G with $d_G(u) \geq 3$, T be a hanging tree of G connecting to u with $|V(T)| \geq 1$, and v be a pendent vertex of G with $v \notin T$. Let G' be the graph obtained from G by deleting T

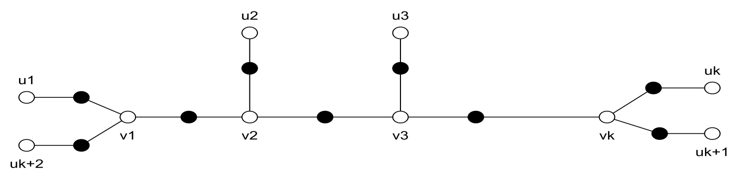

Figure 2:

from vertex u and attaching it to vertex v. We call the transformation from G to $G^{'}$ is a branch transformation on G from vertex u to vertex v.
In the following theorem, we determine minimum total irregularity of TST on n vertices.

Lemma 2.1. *Let T be a TST on $n(n \geq 4)$ vertices and $\triangle(T) = k(k \geq 3)$. Then there exists tree , T^{*} on n vertices with $\triangle(T^{*}) = 3$ satisfying following conditions.*
(i) $n_3(T^{*}) \geq \frac{n_2(T^{*})-1}{2}$
(ii) $irr_t(T^{*}) \leq irr_t(T)$

Proof. **(i)** Let T be a TST on $n(n \geq 4)$ vertices and $\triangle(T) = k(k \geq 3)$. Let v_k be any vertex of degree k in T, and degree sequence of T is given as follows.

$$T \begin{matrix} k & k-1 & \dots & 3 & 2 & 1 \\ n_k & n_{k-1} & \dots & n_3 & n_2 & n_1 \end{matrix}$$

By Corollary 2.1

$$n_2(T) \leq (k-1)n_k(T) + (k-2)n_{k-1}(T) + 3n_4(T) + 2n_3(T) + 1 \tag{3}$$

By Theorem 2.1

$$n_1(T) = (k-2)n_k(T) + (k-3)n_(k-1)(T) + 2n_4(T) + n_3(T) + 2 \tag{4}$$

If $n_2(T) \neq 0$, do the branch-transformation on T from v_k to v_2. If $n_2(T) = 0$, do the branch-transformation on T from v_k to v_1. Repeat the branch-transformation on T till maximum degree of resulting graph T^{*} is 3.
If $N_{Br}(T)$ denotes the number of branch transformations that are done in T,

$$N_{Br}(T) = (k-3)n_k(T) + (k-4)n_{k-1}(T) + \ + 2n_5(T) + n_4(T) \tag{5}$$

We prove this lemma by the following two cases:
Case 1
If $n_2(T) \leq N_{Br}(T)$. That is,

$$n_2(T) \leq (k-3)n_k(T) + (k-4)n_{k-1}(T) + \ + 2n_5(T) + n_4(T) \tag{6}$$

By equation (6), in the process of branch transformation from v_k to v_2, at some stage, number of vertices of degree two in the resulting tree is zero. Then $n_2(T^{*}) = 0$ or 1.

ie $n_2(T^*) - 1 \leq 2n_3(T^*)$, That is $n_3(T^*) \geq \frac{n_2(T^*)-1}{2}$
Case 2
If $n_2(T) > N_{Br}(T)$ .That is, if

$$n_2(T) > (k-3)n_k(T) + (k-4)n_{k-1}(T) + \ + 2n_5(T) + n_4(T) \tag{7}$$

When the process of branch transformation ends, equation (3)-(7) gives upper bound for $n_2(T^*)$

$$n_2(T^*) \leq 2n_k(T) + 2n_{k-1}(T) + \cdots + 2n_5(T) + 2n_4(T) + 2n_3(T) + 1 \tag{8}$$

Number of branch transformations that is done in T and number of vertices of degree greater than or equal to three in T gives $n_3(T^*)$. That is,

$$\begin{aligned} n_3(T^*) &= (k-3)n_k(T) + (k-4)n_{k-1}(T) + \cdots + 2n_5(T) + n_4(T) \\ &\quad + n_k(T) + n_{k-1}(T) + \cdots + n_5(T) + n_4(T) + n_3(T) \\ n_3(T^*) &= (k-2)n_k(T) + (k-3)n_{k-1}(T) + \cdots + 3n_5(T) \\ &\qquad + 2n_4(T) + n_3(T). \end{aligned} \tag{9}$$

From equations (8) and (9), $n_2(T^*) - 1 \leq 2n_3(T^*)$
Hence, in both cases , $n_3(T^*) \geq \frac{n_2(T^*)-1}{2}$.

(ii) We also prove this result by the following two cases.
Case 1
$T^{''}$ is formed from T by branch transformation on T from v_k to v_2.
Degree sequence of T and $T^{''}$ is given by

$T \begin{array}{cccccc} k & k-1 & \ldots & 3 & 2 & 1 \\ n_k & n_{k-1} & \ldots & n_3 & n_2 & n_1 \end{array}$ and

$T^{''} \begin{array}{cccccc} k & k-1 & \ldots & 3 & 2 & 1 \\ n_k - 1 & n_{k-1}+1 & \ldots & n_3+1 & n_2-1 & n_1 \end{array}$

By equation (2)

$$\begin{aligned} &irr_t(T) = \\ &\quad n_k n_{k-1} + 2n_k n_{k-2} + \cdots + (k-3)n_k n_3 + (k-2)n_k n_2 + (k-1)n_k n_1 \\ &\quad n_{k-1}n_{k-2} + 2n_{k-1}n_{k-3} + \cdots + (k-4)n_{k-1}n_3 + (k-3)n_{k-1}n_2 + (k-2)n_{k-1}n_1 \\ &\quad + n_{k-2}n_{k-3} + 2n_{k-2}n_{k-4} + \cdots + (k-5)n_{k-2}n_3 + (k-4)n_{k-2}n_2 + (k-3)n_{k-2}n_1 \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + n_5 n_4 + 2n_5 n_3 + 3n_5 n_2 + 4n_5 n_1 \\ &\quad + n_4 n_3 + 2n_4 n_2 + 3n_4 n_1 \\ &\quad + n_3 n_2 + 2n_3 n_1 \\ &\quad + n_2 n_1 \end{aligned}$$

$irr_t(T^{''}) =$

$$(n_k-1)(n_{k-1}+1)+2(n_k-1)n_{k-2}+\cdots+(k-3)(n_k-1)(n_3+1)$$
$$+(k-2)(n_k-1)(n_2-1)+(k-1)(n_k-1)n_1$$
$$+(n_{k-1}+1)n_{k-2}+2(n_{k-1}+1)n_{k-3}+\cdots+(k-4)(n_{k-1}+1)(n_3+1)$$
$$+(k-3)(n_{k-1}+1)(n_2-1)+(k-2)(n_{k-1}+1)n_1$$
$$+n_{k-2}n_{k-3}+2n_{k-2}n_{k-4}+\cdots+(k-5)n_{k-2}(n_3+1)+(k-4)n_{k-2}(n_2-1)$$
$$+(k-3)n_{k-2}n_1$$
$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+$$
$$n_5n_4+2n_5(n_3+1)+3n_5(n_2-1)+4n_5n_1+$$
$$n_4(n_3+1)+2n_4(n_2-1)+3n_4n_1+$$
$$(n_3+1)(n_2-1)+2(n_3+1)n_1+$$
$$(n_2-1)n_1$$

$$irr_t(T^{''}) = irr_t(T)-2n_{k-1}-2n_{k-2}-2n_{k-3}-\cdots-2n_5-2n_4-2n_3-2$$

Hence $irr_t(T^{''}) < irr_t(T)$
Case 2
$T^{'}$ is formed from T by branch transformation on T from v_k to v_1
Degree sequence of $T^{'}$ is given by

$T^{'}$	k	$k-1$	$\dots$	3	2	1
	n_k-1	$n_{k-1}+1$	$\dots$	n_3	n_2+1	n_1-1

$irr_t(T^{'}) =$

$$(n_k-1)(n_{k-1}+1)+2(n_k-1)n_{k-2}+\cdots+(k-2)(n_k-1)(n_2+1)$$
$$+(k-1)(n_k-1)(n_1-1)$$
$$+(n_{k-1}+1)n_{k-2}+2(n_{k-1}+1)n_{k-3}+\cdots+(k-3)(n_{k-1}+1)(n_2+1)$$
$$+(k-2)(n_{k-1}+1)(n_1-1)$$
$$+n_{k-2}n_{k-3}+2n_{k-2}n_{k-4}+\cdots+(k-5)n_{k-2}n_3+(k-4)n_{k-2}(n_2+1)$$
$$+(k-3)n_{k-2}(n_1-1)$$
$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+n_5n_4+2n_5n_3+3n_5(n_2+1)+4n_5(n_1-1)$$
$$+n_4n_3+2n_4(n_2+1)+3n_4(n_1-1)$$
$$+n_3(n_2+1)+2n_3(n_1-1)$$
$$+(n_2+1)(n_1-1)$$

$$irr_t(T^{'}) = irr_t(T)-2n_{k-1}-2n_{k-2}-2n_{k-3}-\cdots-2n_5-2n_4-2n_3-2n_2-2$$

Hence $irr_t(T^{'}) < irr_t(T)$
We keep on repeating this process and obtain a tree T^* with $\triangle = 3$.
Then $irr_t(T^*) \leq irr_t(T)$ $\square$

Lemma 2.2. *If T^* is any tree on n vertices with $\triangle(T^*) = 3$ and $n_3(T^*) \geq \frac{n_2(T^*)-1}{2}$, there exists TST T_1, on n vertices with $\triangle(T_1) = 3$, $n_3(T_1) = \lceil\frac{n-3}{4}\rceil$, and $irr_t(T_1) \leq irr_t(T^*)$*

Proof. Let T^* be a tree on n vertices with $\triangle(T^*) = 3$ and $n_3(T^*) \geq \frac{n_2(T^*)-1}{2}$. ie $n_3(T^*) \geq \frac{n-3}{4}$ by remark 2.1.ie $n_3(T^*) \geq \lceil\frac{n-3}{4}\rceil$. Let degree sequence of T^* be $T^* \begin{array}{ccc} 3 & 2 & 1 \\ n_3 & n_2 & n_1 \end{array}$ and v_3 a vertex of degree three in T^* . Let T' be the tree obtained from T^* by branch transformation on T^* from v_3 to v_1. Degree sequence of T' is given by $T' \begin{array}{ccc} 3 & 2 & 1 \\ n_3-1 & n_2+2 & n_1-1 \end{array}$

$$\begin{aligned} irr_t(T') &= (n_3(T^*)-1)(n_2(T^*)+2) + 2(n_3(T^*)-1)(n_1(T^*)-1) \\ &\quad +(n_2(T^*)+2)(n_1(T^*)-1) \\ irr_t(T') &= irr_t(T^*) - 2n_2(T^*) - 2 \end{aligned}$$

Hence $irr_t(T') < irr_t(T^*)$. Repeat the above branch transformation on T^* till number of vertices of degree 3 in the resulting graph T_1 is $\lceil\frac{n-3}{4}\rceil$.ie $n_3(T_1) \geq \frac{n-3}{4}$. ie $n_2(T_1) \leq 2n_3 + 1$,by remark 2.1. By theorem 2.3 there exists TST with the same degree sequence. Then we obtain required TST, T_1 on n vertices such that $irr_t(T_1) \leq irr_t(T^*)$ □

Theorem 2.5. *Let T be a TST on $n(n > 3)$ vertices. Then $irr_t(T) \geq 2n + 2n\lceil\frac{n-3}{4}\rceil - 2\lceil\frac{n-3}{4}\rceil^2 - 4\lceil\frac{n-3}{4}\rceil - 4$, and the equality holds if and only if $\triangle(T) = 3, n_3(T) = \lceil\frac{n-3}{4}\rceil$.*

Proof. By lemma 2.1 and lemma 2.2, if T be a TST on $n(n > 3)$ vertices with $\triangle(T) = k(k \geq 3)$. Then there exists TST T_1 on n vertices with $\triangle(T_1) = 3$, $n_3(T_1) = \lceil\frac{n-3}{4}\rceil$ and $irr_t(T_1) \leq irr_t(T)$. Then $n_2(T_1) = n - 2\lceil\frac{n-3}{4}\rceil - 2$, and $n_1(T_1) = \lceil\frac{n-3}{4}\rceil + 2$.
By equation(2), $irr_t(T_1) =$
$\left\lceil\frac{n-3}{4}\right\rceil \left(n - 2\left\lceil\frac{n-3}{4}\right\rceil - 2\right) + 2\left\lceil\frac{n-3}{4}\right\rceil \left(\left\lceil\frac{n-3}{4}\right\rceil + 2\right) + \left(n - 2\left\lceil\frac{n-3}{4}\right\rceil - 2\right)\left(\left\lceil\frac{n-3}{4}\right\rceil + 2\right)$
$irr_t(T) \geq irr_t(T_1) = 2n + 2n\lceil\frac{n-3}{4}\rceil - 2\lceil\frac{n-3}{4}\rceil^2 - 4\lceil\frac{n-3}{4}\rceil - 4$ □

Corollary 2.2. *Among TST on n vertices with $\triangle = 3$,*
When $n = 4k$, $irr_t(T) \geq \frac{3n^2}{8} + n - 4$, equality holds when $n_3(T) = k, n_2(T) = 2k-2, n_1(T) = k+2$
When $n = 4k+1$, $irr_t(T) \geq \frac{3n^2}{8} + \frac{3n}{4} - \frac{25}{8}$, equality holds when $n_3(T) = k, n_2(T) = 2k-1, n_1(T) = k+2$
When $n = 4k+2$, $irr_t(T) \geq \frac{3n^2}{8} + \frac{n}{2} - \frac{5}{2}$, equality holds when $n_3(T) = k, n_2(T) = 2k, n_1(T) = k+2$
When $n = 4k+3$, $irr_t(T) \geq \frac{3n^2}{8} + \frac{n}{4} - \frac{17}{8}$, equality holds when $n_3(T) = k, n_2(T) = 2k+1, n_1(T) = k+2$

Total irregularity is completely determined by its degree sequence. Degree sequence of TST with minimum Total Irregularity (min T_n) for each n is given below:

$No.of\ vertices$	$Degree\ sequence\ of\ minT_n$
$1, 2$	$does\ not\ exist$
3	$T\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}$
$4k, k = 1, 2, 3\cdots$	$T\begin{matrix} 3 & 2 & 1 \\ k & 2k-2 & k+2 \end{matrix}$
$4k+1, k = 1, 2, 3\cdots$	$T\begin{matrix} 3 & 2 & 1 \\ k & 2k-1 & k+2 \end{matrix}$
$n = 4k+2, k = 1, 2, 3\cdots$	$T\begin{matrix} 3 & 2 & 1 \\ k & 2k & k+2 \end{matrix}$
$n = 4k+3, k = 1, 2, 3\cdots$	$T\begin{matrix} 3 & 2 & 1 \\ k & 2k+1 & k+2 \end{matrix}$

Acknowledgement

University of Calicut and St. Mary's College, Thrissur, for providing necessary facilities for pursuing research.

References

[1] D. E. Jackson and R. Entringer, (1986) Totally Segregated Graphs, *Congress. Numer.*, *Vol. 55*, 159-165

[2] Gary Chartrand and Ping Zhang, (2008) *Chromatic Graph Theory*, Boca Raton, London, New York: A Chapman and Hall Book,

[3] H. Abdo and D. Dimitrov, (2015). The Total irregularity of a Graph. *Discrete Mathematics and Theoretical Computer Science*, *Vol.16* (No.1), 201-206.

[4] Jorry T. F. and Parvathy K. S. (2015),Uniformly Segregated Tree, *Bulletin of Kerala Mathematical Association*, *Vol 12* (No.2), 135-143.

[5] M. O. Albertson ,(1997) The Irregularity of a graph, *Ars.Combin. Vol.46*, 219-225.

[6] Yingxue Zhu, Lihua You and Jieshan Yang, (2016). The Minimal Total Irregularity of Some Classes of Graphs. *Filomat*, *Vol.30* (No.5),1203-1211.

[7] Y. Alavi, G. Chartrand, F. R. K. Chung, P. Erdos, R. L. Graham and O. R. Ollermann,(1987) Highly Irregular Graphs. *Journal of Graph Theory*, *Vol. 11* (No.2), 235-249