Uniform Continuity and its Example

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ABSTRACT

The aim of this paper is to study about uniform continuity and its some examples.

1. INTRODUCTION:-

The first published definition of uniform continuity was by Heine in 1870, and in 1872 he published a proof that a continuous function on an open interval need not be uniformly continuous. The proofs are almost verbatim given by Dirichlet in his lectures on definite integrals in 1854. The definition of uniform continuity appears earlier in the work of Bolzano where he also proved that continuous functions on an open interval do not need to be uniformly continuous. In addition he also states that a continuous function on a closed interval is uniformly continuous, but he does not give a complete proof.

Uniform continuity is a much stronger condition than continuity and it is used in lots of places. One very fundamental usage of uniform continuity is in the proof that every continuous function of a closed interval is Riemann integrable. Another one of the reasons it is useful is that if we are working with a continuous function on a closed interval, we get uniform continuity for free.

2. UNIFORM CONTINUITY:-

Continuity itself is a *local* property of a function—that is, a function *f* is continuous, or not, at a particular point, and this can be determined by looking only at the values of the function in an (arbitrarily small) neighbourhood of that point. When we speak of a function being continuous on an interval, we mean only that it is continuous at each point of the interval. In contrast, uniform continuity is a *global* property of *f*, in the sense that the standard definition refers to *pairs* of points rather than individual.

Let $A \subset IR$ and $f : A \to IR$ be continuous. Then for each $a \in A$ and for given $\varepsilon > 0$, there exists a $\delta(\varepsilon, a) > 0$ such that $x \in A$ and $| x - a | < \delta$ imply $| f(x) - f(a) | < \varepsilon$. We emphasize that δ depends, in general, on ϵ as well as the point a. Intuitively this is clear because the function f may change its values rapidly near certain points and slowly near other points.

Now we will give definition of uniform continuity.

A function $f : A \to IR$, where $A \subset IR$ is said to be uniformly continous on A if given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in A$ and $| x - y | < \delta$, we have $| f(x) - f(y) | < \varepsilon$.

Next,we will give sequential definition of uniform continuity .

SEQUENTIAL DEFINITION OF UNIFORM CONTINUITY:- Let $f:A\subset IR\to IR$ be a real valued function, then the following are equivalent:

1.f is uniformly continous on A

2.If $<x_n>$ and $<y_n>$ be any two sequence in A such that $|x_n-y_n|\to 0$ implies $|f(x_n)-f(y_n)|\to 0$ when $n\to\infty$

Hence, if there exist two sequences $<x_n>$ and $<y_n>$ in A such that $|x_n-y_n|\to 0$ but $|f(x_n)-f(y_n)|$ -/→ 0 as $n\to\infty$,then f is not uniform continous on A.

Next, we will give some examples which are not uniform continous.

EXAMPLES:-

1. f(x)=1/x , where x€ (0,1)
   Let $x_n=1/n$ and $y_n=1/(n+1)$
   Now,$|x_n-y_n|=|(1/n)-(1/(n+1))|\to 0$ as $n\to\infty$, but $|f(x_n)-f(y_n)|=$ $|n-(n+1)|=1$-/→0 as $n\to\infty$.Hence,f(x) is not uniform continous.

Now,we will use some results to check the uniform continuity of functions.

Theorem1:-If f:A→IR be continous function on a compact set A,then f is uniformly continous on A.

Ex:-1 f(x)=sin(x) on [a,b]

2 f(x)=x^n on [a,b]

Theorem2:-If f(a,b)→IR where a,b€IR be a continous function,then f is uniformly continous on (a,b) iff lim f(x) and

x→a+

lim f(x) both exist finitely.

x→b-

Ex:-1. By theorem 2,

f(x)=1/x is not uniformly continous on (0,1) as limit of f(x) does not exist at x=0.

2.f(x)=sin(1/x) on (0,1)is not uniform continous as limit of f(x) doesn't exist at x=0.

THEOREM3:- If f: A ⊂ IR→IR be a differentiable function with bounded derivative,then f is uniformly continous on A,but not conversely.

Ex:-1.f(x)=√x on (0,1) is uniformly continous however f'(x)=1/2√x is not bounded on (0,1)

2.f(x)=$\sin^2$(x) on IR is uniformly continous as it is differentiable on IR and its derivative f'(x)=sin2x is bounded on IR

THEOREM4:-If f is a real valued continous function which is periodic,then f is uniform continous.

Ex:-1.f(x)=sin(x)

2.f(x)=cos(x)

3.f(x)=cos(sinx)

4.f(x)=$e^{\sin(\cos(x))}$; where x€IR

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