Transmuted Exponentiated Inverse Weibull Distribution with Applications in Medical Sciences

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Abstract: *This manuscript considers the transmuted model of the Exponentiated Inverse Weibull distribution. A comprehensive description of the mathematical properties of the proposed model is given in this article. The various properties which include reliability analysis, moments, quantile function, median, moment generating function, characteristic function and order statistics have been discussed in the paper. The method of maximum likelihood estimation has been used for estimating the parameters of the newly proposed distribution. The usefulness of the newly developed model over its sub models for better fitting is illustrated both by the simulated as well as real life data sets.*

Keywords: *Transmuted Exponentiated Inverse, Weibull Distribution, Reliability analysis, Moments, Quantile function, Order Statistics, AIC, HQIC.*

1. INTRODUCTION

The Weibull distribution was introduced by Waladdi Weibull, a Swedish physicist in 1951. This is one of the most widely used probability model for analyzing lifetime data and reliability of components. The basic model of Inverse Weibull distribution was studied by Keller et. al. (1982). Further, Drapella (1993) worked on Inverse Weibull model and suggested the name as complementary Weibull distribution. This model is applicable in reliability analysis, medical sciences and modeling infant mortality rate, wear out periods, degradation of mechanical components etc. Aleem and Pasha (2003) studied some additional distributional properties of Inverse Weibull distribution.

Mudholkar et. al. (1995) studied the Exponentiated Weibull distribution and applied the model for the analysis bus motor failure data. Nassar and Eissa (2003) gave the detailed account of the exponentiated Weibull model. Gupta and Kundu (2001) introduced the Exponentiated exponential family of distributions as an alternate to gamma and Weibull distributions. Al-Hussaini (2010) discussed the exponentiated family of distributions in detail. A generalization of Inverted Weibull distribiton called Exponentiated Inverted Weibull distribution was introduced by Flaih et.al (2012) by adding a new shape parameter θ through exponentiation to the distribution function.

A random variable X is said to follow standard exponentiated inverted Weibull distribution if its distribution function is given as:

$$F(x)=\left(e^{-x^{-\beta}}\right)^{\theta},\quad x,\beta \text{ and } \theta>0. \qquad (1)$$

where β and θ are shape parameters.

Then, the probability density function is of the following form:

$$f(x)=\theta\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta},\quad x,\beta \text{ and } \theta>0. \qquad (2)$$

2. TRANSMUTED EXPONENTIATED INVERTED WEIBULL DISTRIBUTION

In the recent past, a significant progress has been made towards the generalization of some well known distributions. These extended distributions find their application in many lifetime problems like engineering, finance, economics and biomedical sciences. Shaw and Buckley (2009) devised the new quadratic rank transmutation map (QRTM) technique for generalizing the different classical models and provide more flexible extension of these models for life testing and best fit. According quadratic rank transmutation map technique (QRTM) approach, a random variable X is said to have transmuted distribution if its cumulative distribution function is given by:

$$G(x)=(1+\lambda)F(x)-\lambda[F(x)]^2.\quad |\lambda|\le 1. \qquad (3)$$

where $G(x)$ is the cdf of the base distribution which on differentiation yields:

$$g(x)=f(x)[1+\lambda-2\lambda F(x)]. \qquad (4)$$

Here $f(x)$ and $g(x)$ are the corresponding *pdfs* associated with *cdfs* $F(x)$and $G(x)$ respectively. It might be noted at $\lambda = 0$; the model reduces to parent distribution.

The main aim of this paper is to study a more flexible extension of exponentiated inverted Weibull distribution using the transmutation map technique called transmuted exponentiated inverted Weibull model. Different probability models have been discussed in the statistical literature based on transmutation map approach. Ashour and Eltehiwy (2013) introduced the transmuted Lomax distribution. Afaq et al. (2014) formulated the transmuted inverse Rayleigh distribution and presented a comprehensive account of its various structural properties. In addition to this, Ahmad et al. (2015) derived the structural properties of transmuted Weibull distribution.

Now using the equations (1) and (3), the cumulative distribution function (cdf) of the random variable X following Transmuted Exponentiated Inverse Weibull (TEIW) distribution is given as:

$$G(x) = \left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]. \quad (5)$$

Also, the probability density function of the Transmuted Exponentiated Inverse Weibull (TEIW) distribution using the equations (2) and (4) with parameters θ, β and λ is given as:

$$g(x) = \theta\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right], \quad (6)$$

where θ and β are the shape parameters and λ is the transmuted parameter.

Figure 1 and 2 represents the possible shapes of the density function and distribution function of the proposed transmuted exponentiated inverse Weibull distribution for various possible values of the parameters.

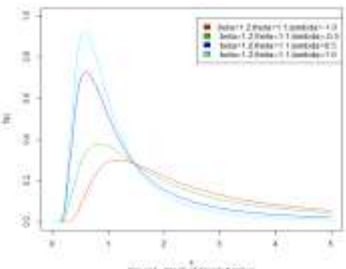


Figure1: Graph of density function

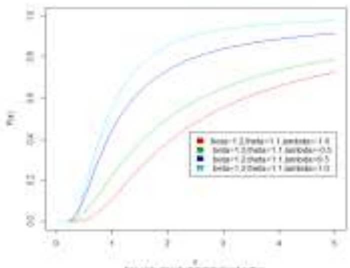


Figure2: Graph of distribution function

3. RELATIONSHIP WITH OTHER DISTRIBUTIONS

The different possible theoretical distributions which can be derived from the proposed TEIW distribution are given as follows:

1. When $\lambda = 0$, the equation (6) reduces to two parameter Exponentiated Inverse Weibull distribution with probability density function as:
$$g(x) = \theta\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta},$$
$$x, \theta \text{ and } \beta > 0.$$
2. When $\theta = 1$, the equation(6) reduces to two parameter Transmuted Inverse Weibull distribution with probability density function as:

$$g(x) = \beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)\right],$$

$$x, \beta > 0 \text{ and } |\lambda| \geq 1$$

3. When $\theta = 1$ and $\lambda = 0$,the equation (6) reduces to one parameter Inverse Weibull distribution with probability density function as:
$$g(x) = \beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right),$$
$$x, \beta > 0.$$
4. When $\beta = 2$, the equation (6) reduces to two parameter transmuted inverse Rayleigh distribution with probability density function as:

$$g(x) = \frac{2\theta}{x^3} e^{-\frac{\theta}{x^2}}\left[1+\lambda-2\lambda e^{\frac{-\theta}{x^2}}\right],$$

$$x,\ \theta > 0 \text{ and } |\lambda| \geq 1.$$

5. When $\beta = 2$ and $\lambda = 0$, the equation (6) reduces to one parameter Inverse Rayleigh distribution with probability density function as:
$$g(x) = \frac{2\theta}{x^3} e^{\frac{-\theta}{x^3}}, \qquad x, \theta > 0.$$
6. When $\beta = 2$ and $\theta = 1$, the equation (6) reduces to one parameter Transmuted

Standard Inverse Rayleigh distribution with probability density function as:

$$g(x)=\frac{2}{x^3}e^{-\frac{1}{x^2}}\left[1+\lambda-2\lambda e^{\frac{-1}{x^2}}\right],$$

$x>0,\ |\lambda|\ge 1.$

7. When $\beta=2,\ \theta=1$ *and* $\lambda=0,$ the equation (6) reduces to Standard Inverse Rayleigh distribution with probability density function as:

$$g(x)=\frac{2}{x^3}e^{-\frac{1}{x^2}},\qquad x>0.$$

8. When $\beta=1,$ the equation (6) reduces to two parameter transmuted Inverse Exponential distribution with probability density function as:

$$g(x)=\frac{\theta}{x^2}e^{-\frac{\theta}{x}}\left[1+\lambda-2\lambda e^{\frac{-\theta}{x}}\right],$$

$x,\ \theta>0$ and $|\lambda|\ge 1.$

9. When $\beta=\theta=1,$ the equation (6) reduces to one parameter transmuted Standard Inverse Exponential distribution with probability density function as:

$$g(x)=\frac{1}{x^2}e^{-\frac{1}{x}}\left[1+\lambda-2\lambda e^{\frac{-1}{x}}\right],$$

$x>0$ and $|\lambda|\ge 1$.

10. When $\beta=1$ and $\lambda=0,$ the equation (6) reduces to one parameter Inverse Exponential distribution with probability density function as:

$$g(x)=\frac{\theta}{x^2}e^{-\frac{\theta}{x}},\qquad x,\ \theta>0.$$

11. When $\beta=1,\ \theta=1$ *and* $\lambda=0,$ the equation (6) reduces to Standard Inverse Exponential distribution with probability density function as:

$$g(x)=\frac{1}{x^2}e^{-\frac{1}{x}},\qquad x>0.$$

4. RELIABILITY ANALYSIS

This section gives a comprehensive account of the survival function, hazard rate and reverse hazard rate of the transmuted exponentiated inverted Weibull distribution.

4.1 Reliability function: The reliability function of the model is defined as the probability that an item does not fail prior to sometime t. The other names given to reliability function are the survival or survivor function of the model. Denoted by $R(x)$, the survivor function can be mathematically computed as:

$$R(x)=1-G(x)$$

$$R(x)=1-\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]. \quad (7)$$

4.2 Hazard function: The hazard rate of the model can be derived as the ratio of the probability density function and the reliability function. Denoted by $h(x)$, it is also termed as the hazard rate, failure rate or force of mortality and is given as:

$$h(x)=\frac{g(x)}{R(x)}$$

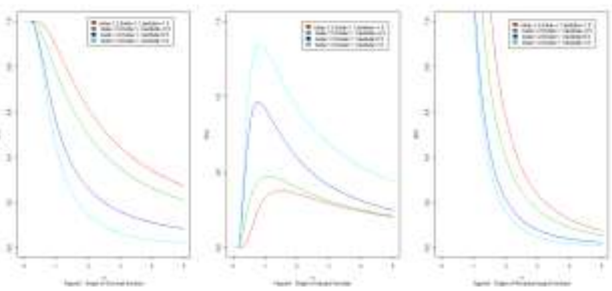
$$h(x)=\frac{\theta\beta\ x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]}{1-\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]}. \quad (8)$$

4.3 Reverse Hazard function: The reverse hazard rate is derived as the ratio of the probability density function and the cumulative distribution function. It is also an essential criterion for characterizing lifetime data and is given as:

$$\phi(x)=\frac{g(x)}{G(x)}$$

$$\phi(x)=\frac{\theta\beta\ x^{-(\beta+1)}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]}{\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]}. \quad (9)$$

The graphical representation of survival function, hazard rate and reverse hazard rate for different values of parameters are given in figure 3, 4 and 5 respectively.



5. STATISTICAL PROPERTIES OF THE TEIW DISTRIBUTION

In this section, certain fundamental statistical properties of the transmuted exponentiated Inverse Weibull distribution have been obtained:

5.1 Moments: The rth moments of the transmuted exponentiated inverted Weibull distribution are computed as follows:

$$\mu_r' = E\left[x^r\right] = \int_0^{\infty} x^r \theta\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]dx$$

$$\mu_r' = \theta\beta\int_0^{\infty} x^{r-\beta-1}\left(e^{-x^{-\beta}}\right)^{\theta}dx + \lambda\theta\beta\int_0^{\infty} x^{r-\beta-1}\left(e^{-x^{-\beta}}\right)^{\theta}dx + 2\lambda\theta\beta\int_0^{\infty} x^{r-\beta-1}\left(e^{-x^{-\beta}}\right)^{2\theta}dx.$$

Setting $\dfrac{\theta}{x^{\beta}} = u$

$$\mu_r' = \theta^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right) + \lambda\theta^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right) - \lambda(2\theta)^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right).$$

This can be finally obtained as:

$$\mu_r' = \theta^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right)\left[1+\lambda-2^{\frac{r}{\beta}}\lambda\right], \quad \beta > r.$$

(10)

Put $r = 1$ in equation (10), we get the expected value of transmuted exponentiated inverse Weibull distribution as:

$$\mu_1' = \theta^{\frac{1}{\beta}}\Gamma\left(1-\frac{1}{\beta}\right)\left[1+\lambda-2^{\frac{1}{\beta}}\lambda\right], \quad \beta > 1. \quad (11)$$

For different values of r in equation (10), we can find the other moments of the proposed model.

5.2 Harmonic mean: The harmonic mean of the proposed transmuted model can be calculated as:

$$H.M = E\left[\frac{1}{X}\right] = \int_0^{\infty}\frac{1}{x}f(x)dx$$

$$H.M = \theta^{\frac{-1}{\beta}}\Gamma\left(1+\frac{1}{\beta}\right)\left[1+\lambda-2^{-\frac{1}{\beta}}\lambda\right]. \quad (12)$$

5.3 Moment Generating Function: The moment generating function of the proposed distribution can be derived as follows:

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} g(x)dx$$

$$= \int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}+....\right)g(x)dx$$

$$= \int_0^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r g(x)dx =$$

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}E(x)^r$$

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\theta^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right)\left[1+\lambda-2^{\frac{r}{\beta}}\lambda\right], \quad \beta > r.$$

(13)

5.4 Characteristic function: The characteristic function of the transmuted exponentiated Weibull distribution is given as follows:

$$\phi_X(t) = E\left(e^{itx}\right) = \int_0^{\infty} e^{itx} g(x)dx$$

$$= \int_0^{\infty}\left(1+itx+\frac{(itx)^2}{2!}+....\right)g(x)dx$$

$$= \int_0^{\infty}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}x^r g(x)dx =$$

$$\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(x)^r$$

$$\phi_X(t) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\theta^{\frac{r}{\beta}}\Gamma\left(1-\frac{r}{\beta}\right)\left[1+\lambda-2^{\frac{r}{\beta}}\lambda\right], \quad \beta > r.$$

(14)

6. QUANTILE FUNCTION AND MEDIAN

Quantile function and median Denoted by $Q(u)$, the quantile function can be mathematically computed as:

$Q(u) = G^{-1}(x)$, where $u \sim U(0,1)$.

Since $G(x) = \left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]$, the quantile function for TEIW distribution is given as:

$$x = \left[\frac{-1}{\log A}\right]^{\frac{1}{\beta}}, \text{where } A = \left[\frac{(\lambda+1)-\sqrt{(\lambda+1)^2-4\lambda u}}{2\lambda}\right]^{\frac{1}{\theta}}. \tag{15}$$

Putting u=0.5 in equation (23), we get the median of TEIW distribution:

$$x = \left[\frac{-1}{\log A_1}\right]^{\frac{1}{\beta}}, \text{where } A_1 = \left[\frac{(\lambda+1)-\sqrt{\lambda^2+1}}{2\lambda}\right]^{\frac{1}{\theta}}. \tag{16}$$

Once the quantile function is computed, we can generate the random numbers for the distribution under discussion using the quantile function.

7. RENYI ENTROPY

The entropy of a random variable X with probability density TEIW $(x;\theta,\beta,\lambda)$ is a measure of the variation of the uncertainty. The larger the entropy indicates the greater uncertainty in the data. The Renyi entropy (1960) denoted by $I_R(\rho)$ for X is a measure of variation of uncertainty and is defined as:

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_{-\infty}^{\infty} f(x)^{\rho}\,dx\right\} \tag{17}$$

If X has TEIWD $(x;\theta,\beta,\lambda)$, then by substituting equation (6) in (17) we have:

$$I_R(\rho) = \frac{1}{1-\rho}\log\left[\int_0^{\infty}(\theta\beta)^{\rho}x^{-\rho(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\rho\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]^{\rho}dx\right]$$

For the convenience, let

$$m(x) = \int_0^{\infty}(\theta\beta)^{\rho}x^{-\rho(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\rho\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]^{\rho}dx$$

.

Setting $\frac{\theta}{x^{\beta}} = u$ then $x = \left(\frac{u}{\theta}\right)^{\frac{-1}{\beta}}$, therefore

$$m(x) = (\theta\beta)^{\rho-1}\int_0^{\infty}\left(\frac{u}{\theta}\right)^{\frac{(\rho-1)(\beta+1)}{\beta}}e^{-\rho u}\left[1+\lambda-2\lambda e^{-u}\right]^{\rho}du \tag{18}$$

.

Using $\left[1-2\lambda e^{-u}+\lambda\right]^{\rho} = \sum_{j=0}^{\infty}{}^{\rho}c_j\left(1-2\lambda e^{-u}\right)^{j}\lambda^{\rho-j}$ in (18), we have

$$m(x) = \sum_{j=0}^{\infty}v_j\theta^{\frac{1-\rho}{\beta}}\beta^{\rho-1}\sum_{k=0}^{\infty}a_k\int_0^{\infty}u^{\frac{(\rho-1)(\beta+1)}{\beta}}e^{-u(\rho+k)}du \tag{19}$$

Where $v_j = {}^{\rho}c_j\lambda^{\rho-j}$ and $a_k = \frac{(-1)^k\Gamma(j+1)}{\Gamma(j+1-k)k!}$.

$$m(x) = \sum_{j=0}^{\infty}v_j\theta^{\frac{1-\rho}{\beta}}\beta^{\rho-1}\sum_{k=0}^{\infty}a_k\frac{\Gamma\left(\frac{(\rho-1)(\beta+1)}{\beta}+1\right)}{(\rho+k)^{\frac{(\rho-1)(\beta+1)}{\beta}+1}}.$$

Now,

$$I_R(\rho) = \frac{1}{\beta}\log\theta - \log\beta + \frac{1}{1-\rho}\log\sum_{j=0}^{\infty}v_j\sum_{k=0}^{\infty}a_k\frac{\Gamma\left(\frac{(\rho-1)(\beta+1)}{\beta}+1\right)}{(\rho+k)^{\frac{(\rho-1)(\beta+1)}{\beta}+1}}. \tag{20}$$

The β or q-entropy introduced by Havrda and Charvat (1967) is denoted by $I_H(q)$ and can be computed as:

$$I_H(q) = \frac{1}{q-1}\left\{1-\int_{-\infty}^{\infty}f(x)^q\,dx\right\}, \text{where}$$

$q > 0$ *and* $q \neq 1$ (21) Suppose X has TEIWD $(x;\theta,\beta,\lambda)$, then by substituting (6) in (21), we get the β entropy as follows:

$$I_H(q) = \frac{1}{q-1}\left[1-\int_0^{\infty}(\theta\beta)^q x^{-q(\beta+1)}\left(e^{-x^{-\beta}}\right)^{q\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]^q dx\right]$$

$$I_H(q) = \frac{1}{q-1}\left[1-\sum_{j=0}^{\infty}v_j\theta^{\frac{1-q}{\beta}}\beta^{q-1}\sum_{k=0}^{\infty}a_k\frac{\Gamma\left(\frac{(q-1)(\beta+1)}{\beta}+1\right)}{(q+k)^{\frac{(q-1)(\beta+1)}{\beta}+1}}\right]. \tag{22}$$

8. ORDER STATISTICS

If $X_1, X_2, X_3, \ldots X_n$ be a random sample of size n taken from TEIW distribution with ordered values as $X_{(1)}, X_{(2)}, X_{(3)}, \ldots X_{(n)}$, then the probability density function of the ordered statistic is given as:

$$g_r(x) = \frac{n!}{(n-r)!(r-1)!}g(x)[G(x)]^{r-1}[1-G(x)]^{n-r}. \tag{23}$$

The density function of the rth order statistics following transmuted exponentiated inverse Weibull distribution is obtained by using the equation (5) and (6) in equation (23):

$$g_r(x)=\frac{n!}{(n-r)!(r-1)!}\theta\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]$$

$$\left[\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]\right]^{r-1}\left[1-\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]\right]^{n-r}.$$

The density function of the smallest order statistics is obtained by putting $r=1$ in equation (20) as:

$$g_1(x)=n\theta\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]$$

$$\times\left[1-\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]\right]^{n-1}.$$

Similarly, by substituting $r=n$ in equation (20) the density function of largest order statistic as follows:

$$g_n(x)=n\theta\beta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]$$

$$\left[\left(e^{-x^{-\beta}}\right)^{\theta}\left[1+\lambda-\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]\right]^{n-1}.$$

9. PARAMETER ESTIMATION

In this section, the unknown parameters β,θ and λ of the transmuted exponentiated Inverted Weibull distribution are estimated by the maximum likelihood estimation procedure. The sample consisting of n observations $x_1,x_2,x_3,\ldots.x_n$ is considered. The likelihood function of the proposed distribution is as follows:

$$L(x\mid\theta,\beta)=(\theta\beta)^n\prod_{i=1}^{n}\left\{\begin{matrix}x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta}\\ \left[1+\lambda-2\lambda\left(e^{-x^{-\beta}}\right)^{\theta}\right]\end{matrix}\right\}.$$

(24)

The corresponding log likelihood function of the equation (24) is given as under:

$$\log L=n\log\theta+n\log\beta-(\beta+1)\sum_{i=1}^{n}\log x_i-\theta\sum_{i=1}^{n}x_i^{\beta}$$

$$+\sum_{i=1}^{n}\log\left(1+\lambda-2\lambda\exp\left(\frac{-\theta}{x^{\beta}}\right)\right).$$

(25)

Hence, on differentiating the equation (25) with respect to the unknown parameters β,θ and λ of the TEIW model and equating these to zero yield the following three normal equations respectively:

$$\frac{d}{d\beta}\log L=\frac{n}{\beta}-\sum_{i=1}^{n}\log x_i+\theta\sum_{i=1}^{n}x^{-\beta}\log x$$

$$-2\lambda\theta\sum_{i=1}^{n}\frac{x^{-\beta}\exp\left(-\theta x^{-\beta}\right)\log x}{\left(1+\lambda-2\lambda\exp\left(-\theta x^{-\beta}\right)\right)}.$$

(26)

$$\frac{d}{d\theta}\log L=\frac{n}{\theta}-\sum_{i=1}^{n}x^{-\beta}+2\lambda\sum_{i=1}^{n}\frac{x^{-\beta}\exp\left(-\theta x^{-\beta}\right)}{\left(1+\lambda-2\lambda\exp\left(-\theta x^{-\beta}\right)\right)}.$$

(27)

$$\frac{d}{d\lambda}\log L=\sum_{i=1}^{n}\frac{\left(1-2\exp\left(-\theta x^{-\beta}\right)\right)}{\left(1+\lambda-2\lambda\exp\left(-\theta x^{-\beta}\right)\right)}.$$

(28)

It can be clearly seen that the equations are not in explicit form as such the estimates of the unknown parameters are obtained by solving the normal equations simultaneously using the Newton Raphson algorithm.

10. APPLICATION

In this section, both the simulated as well as real life data sets are considered for the comparison of the flexibility of the proposed transmuted model of exponentiated Inverse Weibull distribution over its different sub models. In order to compare the different models, criteria like AIC (Akaike information criterion) and HQIC (Hannan–Quinn information criterion) are used. The distribution which provides us lesser values of AIC and HQIC is rendered as best. The values of AIC and HQIC can be computed as follows:

AIC=2k-2logL and HQIC=2klog (logn)-2logL,

where k is the number of parameters in the probability model, n is the sample size and -2log*L* is the maximized value of the log-likelihood function of the model under discussion. The analysis of both the data sets is performed through R software. The MLEs of the parameters are obtained with standard errors shown in parentheses. Moreover, the corresponding log-likelihood values, AIC and HQIC are displayed in Table 1and 2.

10.1 Simulated Data Set: In the simulation study, a sample of size 100 has been generated from the R software to evaluate the performance of the proposed model over its sub models. Choosing the values of the parameters as $\beta=0.7,\theta=1.3$ and transmutation parameter $\lambda = 1.0$, the data set is obtained by using the inverse cdf method as discussed in section (8). The summary of the analysis is displayed in the table 1as under:

Table 1: MLEs of the model parameters using Generated data set, the resulting SEs in parentheses and Criteria for Comparison

N	Distribution	β	λ	θ	Log-likelihood	AIC	HQIC
100	TEIWD	0.79934 (0.12732)	1.000000 (0.31917)	1.36156 (0.22408)	**-163.9756**	**333.9511**	**337.1143**
	EIWD	1.12250 (0.08409)	_	0.75853 (0.08664)	-165.8498	335.6996	337.8083
	TIWD	0.99074 (0.07132)	0.45494 (0.18799)	_	-166.1402	336.2804	338.3891
	IWD	1.02665 (0.06755)	_	_	-169.0442	340.0883	341.1428
	TIRD	_	-0.71781 (0.11434)	0.26125 (0.03266)	-195.282	394.564	396.6727
	IRD	_	_	0.34507 (0.03451)	-207.4305	416.861	417.9154
	TSIRD	_	0.10924 (0.13486)	_	-290.4998	582.9996	584.054
	TIED	_	0.47911 (0.29826)	1.01965 (0.15169)	-166.1403	336.2807	338.3893

10.2 Real Life Data Sets: In order to confirm our simulated results, we use two real data sets to show that the transmuted Exponentiated Inverse Weibull distribution can be a better model than the sub models.

Data Set I: Consider a data set corresponding to remission times (in months) of a random sample of 124 bladder cancer patients given in Lee and Wang (2003). The data set is given as follows : 0.08, 2.09, 2.73, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.22, 3.52, 4.98, 6.99, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23,
5.41, 7.62, 10.75, 15.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.93, 8.65, 12.63 and 22.69.

Data Set II: This data represents the survival times of 121 patients with breast cancer obtained from a large hospital which is widely reported in some literatures like Ramos et al. (2013).
0.3,0.3,4.0,5.0,5.6,6.2,6.3,6.6,6.8,7.4,7.5,8.4,8.4,10.3, 11.0,11.8,12.2,12.3,13.5,14.4,14.4,14.8,15.5,15.7,16. 2,16.3,16.5,16.8,17.2,17.3,17.5,17.9,19.8,20.4,20.9,2 1.0,21.0,21.1,23.0,23.4,23.6,24.0,24.0,27.9,28.2,29.1, 30.0,31.0,31.0,32.0,35.0,35.0,37.0,37.0,37.0,38.0,38. 0,38.0,39.0,39.0,40.0,40.0,40.0,41.0,41.0,41.0,42.0,4 3.0,43.0,43.0,44.0,45.0,45.0,46.0,46.0,47.0,48.0,49.0, 51.0,51.0,51.0,52.0,54.0,55.0,56.0,57.0,58.0,59.0,60. 0,60.0,60.0,61.0,62.0,65.0,65.0,67.0,67.0,68.0,69.0,7 8.0,80.0,83.0,88.0,89.0,90.0,93.0,96.0,103.0,105.0,10 9.0,109.0,111.0,115.0,117.0,125.0,126.0,127.0,129.0, 129.0, 139.0, 154.0. These data sets are used here only for illustrative purposes. The required numerical evaluations are carried out using R software.

Table 2: MLEs of the model parameters using real data sets, the resulting SEs parentheses and Criteria for Comparison

Data	Distribution	β	λ	θ	Log-likelihood	AIC	HQIC
Data Set I	TEIWD	0.83317 (0.04759)	-0.85352 (0.09540)	1.54257 (0.17873)	**-422.0538**	**850.1076**	**853.5036**
	EIWD	0.74968 (2.39809)	_	2.39809 (0.21942)	-429.1313	862.2625	864.5266
	TIWD	0.76989 (0.04883)	-0.95757 (0.04022)	_	-428.9819	861.9638	864.2278
	IWD	0.00237 (0.67417)	_	_	-464.1575	930.3151	931.447
	TIRD	_	-0.95345 (0.03121)	0.54227 (0.04981)	-687.5907	1379.181	1381.445

	IRD	–	–	0.59867 (0.05376)	-749.5542	1501.108	1502.24
	TSIRD	–	-0.91974 (0.04027)	–	-714.3615	1430.723	1431.855
	TIED	–	-0.85782 (0.07868)	1.65136 (0.17241)	-428.1652	860.3303	862.5944
Data set II	TEIWD	0.75163 (0.03998)	-0.91633 (0.05973)	4.91329 (0.62503)	**-626.2907**	**1258.581**	**1261.988**
	EIWD	0.66602 (0.03644)	–	6.81583 (0.78861)	- 636.6115	1277.223	1279.494
	TIWD	0.45686 (0.03033)	-1.00000 (0.20954)	–	-679.7679	1363.536	1365.807
	IWD	0.38132 (0.02878)	–	–	-731.6091	1465.218	1466.354
	TIRD	–	-0.96623 (0.02366)	5.23753 (0.47615)	-1021.173	2046.346	2048.617
	IRD	–	–	5.33823 (0.48529)	-1092.548	2187.096	2188.231
	TSIRD	–	-0.96684 (0.02326)	–	-1123.63	2249.259	2250.395
	TIED	–	-0.94016 (0.04189)	7.35186 (0.68099)	-645.7979	1295.596	1297.867

CONCLUSION

In this paper, we have studied the new distribution that is transmuted exponentiated inverse Weibull distribution, a generalization of inverse Weibull distribution to improve the flexibility of the parent model by adding an additional transmuted parameter. Different mathematical properties like reliability analysis, moments, moment generating function and characteristic function are derived. The parameters have been obtained using the maximum likelihood technique. In order to analyze the flexibility and applicability of the proposed distribution both the simulated as well as real life data sets are considered. Due to the lesser values of AIC and HQIC in data analysis, it can be concluded that the newly developed model has superiority over its sub models.

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