Barycentric Map and Layer Topology on Simplices

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Abstract—*Simplicial maps mapssimplicial complexes to equal or lower dimensional complexes. In this paper, we wish to introduce a new topology on simplices which will help us to define mapping from a complex to higher dimensional complexes.*

Keywords—*Simplices, Barycentric Coordinates, Simplicial complexes, Simplicial Maps, Barycentric Maps, Layer Toplogy etc..*

I. INTRODUCTION

Many of the Topological properties of a space can be explored through its triangulations. We are comparing topological properties of two spaces with the help of continuous functions or homeomorphisms from one space to other. But it is difficult to check the existence of such a homeomorphism in many cases. Simplicial maps are continuous functions from one complex to another complex.But it is a simplicial homeomorphism or an isomorphism if and only if the induced vertex map is a bijection[1].We can't define a simplicial map from a complex to another if the later has higher dimension.

II. PRELIMINARIES

We use standard definitions related to simplicial complex and simplicial maps which can be found in textbooks such as Munkers[1]

Simplices and Simplicial Complexes.

Let $\{a_0, a_1, \dots . a_n\}$be geometrically independent set in $\boldsymbol{R^N}$.We define the **n-simplex** σ spanned by $a_0, a_1, \dots . a_n$ be the set of all points x of $\boldsymbol{R^N}$such that

$$x = \sum_{i=0}^{n} t_i a_i \text{ , where } \sum_{i=0}^{n} t_i = 1$$

and $t_i \geq 0$ for all i. The numbers t_i are uniquely determined by x ; they are called the **barycentriccoordinates** of the point x of σ with respect to $a_0, a_1, \dots . a_n$.

The points $a_0, a_1, \dots . a_n$ that spansσ are called **vertices** of σ; the number n is called the **dimension** of σ.Any simplex spanned by a subset of $\{a_0, a_1, \dots . a_n\}$ is called a **face** of σ. In particular, the face of σ spanned by $a_1, \dots . a_n$ is called the face **opposite**a_0. The faces of σ different from σ itself are called the **proper faces** of σ; their union is called the **boundary** of σ and is denoted by Bdσ. The **interior** of σ is defined by the equation

$Int\,\sigma = \sigma - Bd\,\sigma$; the set $Int\sigma$ is sometimes called **open simplex.**

A **simplicial complex** K in $\boldsymbol{R^N}$is a collection of simplices in $\boldsymbol{R^N}$such that

(1) Every face of a simplex of K is in K
(2) The intersection of any two simplexes of K is a face of each of them.

If L is a sub collectionof K that contains all faces of its elements, then L is a simplicial complex called **subcomplex** of K. Asubcomplex of K is the collection of all simplices of K of dimension at most p, called **p-skeleton** of K and is denoted by $K^{(p)}$. The points of the collection $K^{(0)}$ are called **vertices** of K. Let $|K|$be the subset of $\boldsymbol{R^N}$ that is the union of the simplices of K. Giving each simplex, its natural topology as a subspace of $\boldsymbol{R^N}$,we toplogize$|K|$by declaring a subset A of $|K|$ to be closed in $|K|$ if and only if $A \cap \sigma$ is closed in σ, for each σ in K. The space $|K|$ is called the **underlying space** of K, or the **polytop**of K.

BARYCENTRIC COORDINATES AND BARYCENTRIC MAPS

According to Munkers[1], if x is a point of the polyhedron $|K|$, then x is a point of one simplex K whose vertices are $a_0, a_1, \dots . a_n$.Then

$$x = \sum_{i=0}^{n} t_i a_i \text{ , where } \sum_{i=0}^{n} t_i = 1$$

and$t_i \geq 0$ for all i.If v is an arbitrary vertex of K, then the barycentric coordinates $t_v(x)$ of x with respect to v is $t_v(x) = 0\ if\ v \neq a_i$and

$$t_v(x) = t_i, if\ v = a_i$$

Lemma 1: Let $v = a_i$ be any vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$, then $t_v\left(a_j\right) = 1\ for\ i = j$ and $t_v\left(a_j\right) = 0\ for\ i \neq j$.

Proof:

If $a_0, a_1, \dots . a_n$are the vertices of the simplex σ,then$a_0, a_1, \dots . a_n$ are geometrically independent. We have $x = \sum_{i=0}^{n} t_i a_i$, where $\sum_{i=0}^{n} t_i = 1$ for all$x \in$ σ.

Therefore, $a_i = \sum_{i=0}^{n} t_i a_i$ ie, $t_1 a_1 + t_2 a_2 + \cdots + (t_i - 1) a_i + \cdots + t_n a_n = 0$.

By independence of $a_0, a_1, \dots . a_n$,we obtain

$$t_1 = 0, t_2 = 0, \dots, t_i = 1, \quad ..t_n = 0$$

$$t_v(a_j) = 1 \; for \; i = j \text{ and}$$
$$t_v(a_j) = 0 \; for \; i \neq j$$
■

For fixed v on a simplex σ, the function $t_v(x)$ is continuous. If the dimension ofσ ≥ 1, then
$$0 \leq t_v(x) \leq 1.$$
Without loss of generality hereafter we are taking $v = a_0$.If σ is a simplex of dimension$n \geq 1$, each 1-face of the form $a_0 a_i$are mapped to [0, 1] by the map t_v in such a way that $t_v(a_0) = 1$ and
$t_v(a_i) = 0$. Moreover, since t_v is continuous, by intermediate value theorem the map
$t_v: a_0 a_i \to [0,1]$issurjective and hence
$t_v: \sigma \to [0,1]$is also surjective.

Lemma 2: Let $v = a_0$ be a vertex of a simplex σ $= a_0 a_1 \dots a_n$and let x be a point on $a_0 a_i$ for some i, then $t_{a_j}(x) = 0$ for $j \neq 0, i$.

Proof:

Since x being a point on $a_0 a_i$, we have
$x = t_0 a_0 + t_i a_i$where$0 \leq t_0, t_i \leq 1$ and
$t_0 + t_i = 1$. And at the same time since x being a point on the simplex σ $= a_0 a_1 \dots a_n$,
$x = s_0 a_0 + s_1 a_1 + \cdots + s_i a_i + \cdots + s_n a_n$.
Then$t_{a_0}(x) = s_0, t_{a_1}(x) = s_1, \dots,$
$t_{a_i}(x) = s_i, \dots, t_{a_n}(x) = s_n$.
Therefore, $t_0 a_0 + t_i a_i = s_0 a_0 + s_1 a_1 + \cdots + s_i a_i + \cdots + s_n a_n$.
ie, $(s_0 - t_0) a_0 + s_1 a_1 + \cdots + (s_i - t_i) a_i + \cdots + s_n a_n = 0$.
By independenceof $a_0, a_1, \dots, a_n$, we have
$$s_0 = t_0, s_1 = 0, s_2 = 0, \dots, s_i = t_i, \dots, s_n = 0$$
$$t_{a_0}(x) = t_0, t_{a_1}(x) = 0, t_{a_2}(x) = 0 \dots,$$
$t_{a_i}(x) = s_i, \dots, t_{a_n}(x) = 0$.
Hence
$t_{a_j}(x) = 0$ for $j \neq 0, i$. ■

Lemma 3:Let$v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$then the map$t_v: a_0 a_i \to [0,1]$is injective.

Proof:

Let x, y be points on the simplex $a_0 a_i$.
Then $x = t_0 a_0 + t_1 a_1$where$t_1 = 1 - t_0$ and
$y = s_0 a_0 + s_1 a_1$where$s_1 = 1 - s_0$.
Let $t_v(x) = t_v(y)$. Then $t_0 = s_0$ and $t_1 = s_1$which imply
$x = y$ ■

Theorem 1: Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$ and let $t_v(x_i) = t \; for \; i = 1,2,3, \dots, n$, where x_i is a point on the face $a_0 a_i$. Then $t_v(z) = t$ for all z belongs to the convex hull determined by $x_1, x_2, \dots, x_n$.

Proof:

Let
$$x_1 = t'_1 a_0 + t'_2 a_1 + \cdots + t''_n a_n$$
$$x_2 = t''_1 a_0 + t''_2 a_1 + \cdots + t''_n a_n$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x_n = t_1^{(n)} a_0 + t_2^{(n)} a_1 + \cdots + t_n^{(n)} a_n$$
Since $t_v(x_i) = t \;\; for \; i = 1,2,3, \dots, n$, we have
$$t'_1 = t''_1 = \cdots = t_1^{(n)} = t$$
Let z belongs to the convex hull determined by $x_1, x_2, \dots, x_n$. Then
$z = \sum_{i=1}^{n} s_i x_i$where $\sum_{i=1}^{n} s_i = 1$.
Therefore
$z = (s_1 t'_1 + s_2 t''_1 + \cdots + s_n t_1^{(n)}) a_0 + (s_1 t'_2 + s_2 t''_2 + \cdots + s_n t_2^{(n)}) a_1 + \cdots + (s_1 t'_n + s_2 t''_n + \cdots + s_n t_n^{(n)}) a_0$.
Then $t_v(z) = \left(s_1 t'_1 + s_2 t''_1 + \cdots + s_n t_1^{(n)}\right)$
$= (s_1 t + s_2 t + \cdots + s_n t) = t \sum_{i=1}^{n} s_i = t$ ■

Definition: Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices$a_0, a_1, \dots . a_n$, then the mapt_v: σ $\to [0,1]$is called **barycentric map** with respect to v.

From the above discussions, we have barycentric map is continuous and surjctive.

Definition: Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$and let $t \in [0,1]$, then the set $l_t(\sigma) = \{x \in \sigma : t_v(x) = t\}$ is called the ***t*^th^ layer** of σ.

By lemma1, we have 1st layer of σ is $a_0$and by using lemma 1 and theorem 1 we have 0th layer of σ is face opposite to a_0.Also we have
$\sigma = \cup_{t \in I} l_t(\sigma), where \; I = [0,1]$.
Here we can define some subsets of σ as follows.
Let U be a subset of I = [0, 1], then $l_U = \{l_t : t \in U\}$.
Whenever U is open we can call l_U as **open layer.**

Proposition 1: The collection$\mathcal{L}$ of open layers $l_U$form a basis for a topology on σ

Proof:

We note the following

a) Suppose$x \in \sigma$and $t_v(x) = t$. Then there exist an open set U in [0,1] such that $t \in U$ under the usual topology on [0,1]. Thus $x \in l_U \in \mathcal{L}$

b) Let $x \in l_{U_1} \cap l_{U_2}$ for some $l_{U_1}, l_{U_2} \in \mathcal{L}$.Then $t_v(x) \in U_1$ and $t_v(x) \in U_2$. That is $t_v(x) \in U$ where $U = U_1 \cap U_2$. Since U_1andU_2 are open in [0,1], U is also open in [0,1]. Therefore $l_U \in \mathcal{L}$and $x \in l_U$. Since $U \subset U_1$and$U \subset U_2$, we have $l_U \subset l_{U_1} \cap l_{U_2}$.

This completes the proof. ■

Definition: Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots . a_n$. Then the topology on σ in which open layers form a basis is called **layer topology.**

In view of this layer topology, we can easily verify that the inverse of the barycentric map is the multivalued function

$$F: [0,1] \to \sigma$$

whichis continuous. This multivalued function can be defined as

$$F(t) = l_t(\sigma), t \in [0,1]$$

III. CONCLUSIONS

The intention to define layer topology is to generate a relationship among simplices and the closed set [0,1]. As in the introduction,simplicial maps are restricted in the sense that it can be defined from one simplex to its lower dimensional simplices. But in the light of layer topology the inverse of barycentric map may be treated as a map from 1-simplex to higher dimensional simplices. Such a map will open the ways of understanding of simplicial complexes and homology in a broader sense.

ACKNOWLEDGMENT

I wish to acknowledge my sincere thanks to the organisers of UAE math day 2017 and MESMAC International conference 2017 for giving me an opportunity to present papers on "Multivalued functions on simplices" and "Topology on simplices" respectively, the presentation and the related discussions lead me to new thoughts and encouraged me to prepare this paper.

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