ON COMMUTATIVITY OF $*$-PRIME NEAR-RINGS WITH DERIVATIONS

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Abstract: The primary purpose of this paper is to introduce the notion of $*$-prime near-rings, which is a special class of distributive near-rings and to investigate their commutativity. Let N be a left near-ring. N is called distributive near-ring if $(x+y)z = xz + yz$ for all $x, y, z \in N$. Further, an additive mapping $x \mapsto x^*$ on N is said to be an involution on N if (i) $(x^*)^* = x$ and (ii) $(xy)^* = y^*x^*$ hold for all $x, y \in N$. A near-ring equipped with an involution '$*$' is called a $*$-near-ring. A $*$-near-ring N is called $*$-prime near-ring if $xNy = xNy^* = \{0\}$ implies that either $x = 0$ or $y = 0$. Analogues of some ring theoretic results, regarding commutativity have been obtained in the setting of $*$-prime near-rings satisfying some properties and identities involving derivations.

1. INTRODUCTION

Throughout the present paper, unless otherwise mentioned, N will denote a left near-ring. N is called a prime near-ring if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semiprime if $xNx = \{0\}$ implies $x = 0$. Given an integer $n > 1$, near-ring N is said to be n-torsion free, if for $x \in N$, $nx = 0$ implies $x = 0$. If K is a nonempty subset of N, then a normal subgroup $(K, +)$ of $(N, +)$ is called a right ideal (resp. a left ideal) of N if $(x + k)y - xy \in K$ (resp. $xk \in K$) holds for all $x, y \in N$ and for all $k \in K$. K is called an ideal of N if it is both a left ideal as well as a right ideal of N. The symbol Z will denote the multiplicative center of N, that is, $Z = \{x \in N \mid xy = yx \text{ for all } y \in N\}$. For any $x, y \in N$ the symbol $[x, y] = xy - yx$ stands for multiplicative commutator of x and y, while the symbol xoy will represent $xy + yx$. For terminologies concerning near-rings, we refer to G.Pilz $[1, 2]$. Following [3], an additive mapping $d : N \longrightarrow N$ satisfying $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ is called a derivation on N. A $*$-near ring N is called $*$-prime near-ring if $xNy = xNy^* = \{0\}$ implies that either $x = 0$ or $y = 0$. Let N be a $*$-near-ring. An ideal I of N is called $*$-ideal if $I^* = I$. An element $x \in N$ is called a symmetric element if $x^* = x$ and an element $x \in N$ is called a skewsymmetric element if $x^* = -x$. We denote the collection of all symmetric and skewsymmetric elements

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of N by $Sa_*(N)$ i.e.; $Sa_*(N) = \{x \in N \mid x^* = \pm x\}$. There has been a lot of work on commutativity of $*$-prime rings costrained with derivations (see $4-7$, where further references can be found). Motivated by these works, we have investigated commutativity of $*$-prime near-rings constrained with derivations.

2. PRELIMINARY RESULTS

We begin with the following lemmas which are essential for developing the proofs of our main results.

Lemma 2.1. Let N be a $*$-near-ring. Then

- (i) N is a distributive near-ring.
- (ii) $xy + zt = zt + xy$ for all $x, y, z, t \in N$.
- (iii) $n(xy) = (nx)y = x(ny)$ for all $x, y \in N$ and $n \in Z$, where Z stands for the set of integers.
- (iv) $[x, y + z] = [x, y] + [x, z]$ and $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in N$.
- (v) $[x, yz] = y[x, z] + [x, y]z$ and $[xy, z] = x[y, z] + [x, z]y$ for all $x, y, z \in N$.
- (vi) If I is an ideal of N then $NI \subseteq I$ and $IN \subseteq I$.

Proof. (i) For all $x, y, z \in N$ we have $\{(y + z)x\}^* = x^*y^* + x^*z^*$, now taking the image of both the sides under $*$ we get $(y + z)x = yx + zx$. This means that N is a distributive near-ring.

(ii) Since N has both distributive properties, expanding $(x+z)(t+y)$ for all $x, y, z, t \in N$, we have $xt + xy + zt + zy = xt + zt + xy + zy$. This implies our required result.

(iii) Since $(N, +)$ is a group and N has both distributive properties, the result is obvious.

(iv) Using both distributive properties of N and (ii), we get the result.

(v) Same trick as used in (iv).

(vi) Under hypothesis it is a trivial fact.

Lemma 2.2. Let N be a $*$-near-ring.

(i) If N is a prime near-ring then it is a $*$-prime near-ring.

(ii) If N is $*$-prime near-ring then it is a semiprime near-ring.

(iii) N is $*$-prime near-ring if and only if $xNy = x^*Ny = \{0\}$ yields $x = 0$ or $y = 0$.

Proof. (i) Suppose that $xNy = xNy^* = \{0\}$. If first case holds then primeness of N insures that either $x = 0$ or $y = 0$. On the other hand if second case holds then primeness of N again provides us either $x = 0$ or $y^* = 0$. Including both the cases we arrive at either $x = 0$ or $y = 0$. Hence N is $*$-prime near-ring .

(ii) Assume that $xNx = \{0\}$ then $xNxNx^* = \{0\}$. By $*$-primeness of N we get that either $x = 0$ or $xNx^* = \{0\}$. But $xNx^* = \{0\}$ together with $xNx = \{0\}$ implies that $x = 0$.

(iii) Let N be a $*$-prime near-ring. Further suppose that $xNy = x^*Ny = \{0\}$. This provides us $y^*Nx^* = y^*Nx = \{0\}$. Using $*$-primeness of N we obtain that either $y^* = 0$ or $x = 0$. This implies that either $x = 0$ or $y = 0$. Converse can be proved in a similar way.

Lemma 2.3. Let N be a $*$-prime near-ring.

(i) If $Z \neq \{0\}$ then N is a ring.

(ii) If $z \in Z\backslash\{0\}$ and x is an element of N such that $xz, xz^* \in Z$ (resp. $xz, x^*z \in Z$) then $x \in Z$.

Proof. (i) Since $Z \neq \{0\}$, there exists $0 \neq z \in Z$. By Lemma 2.1 we obtain that $zx + zy = zy + zx$ for all $x, y \in N$. Now we infer that $z(x + y - x - y) = 0$ for all $x, y \in N$. This implies that $zN(x+y-x-y) = \{0\}$ and $zN(x+y-x-y)^* = \{0\}$. Now $*$-primeness of N provides us $x + y = y + x$ for all $x, y \in N$. Hence $(N, +)$ is abelian. Using Lemma 2.1 again we conclude that N is a ring.

(ii) If $xz, xz^* \in Z$, we have $xzr = rxz$ and $xz^*r = rxz^*$ for all $r \in N$. It is obvious that $z^* \in Z$. These facts provide us $zN[x, r] = \{0\}$ and $z^*N[x, r] = \{0\}$ for all $r \in N$. Using Lemma 2.2 we obtain that $x \in Z$. On the other hand if $xz, x^*z \in Z$, we have $xzr = rxz$ and $x^*zr = rx^*z$ for all $r \in N$. It follows that $zN[x, r] = \{0\}$ and $zN[x^*, r] = \{0\}$ for all $r \in N$. Replacing r by r^* in the relation $zN[x^*, r] = \{0\}$ we obtain that $zN[x^*, r^*] = \{0\}$ i.e.; $zN[x^*, r^*] = \{0\}$. Now we arrive at $zN[x, r] = \{0\}$ and $zN[x, r]^* = \{0\}$ for all $r \in N$.

Finally $*$-primeness of N finishes the proof.

In the year 2006, L.Oukhtite and S.Salhi [4, Lemma 3.1] proved that if R is a $*$-prime ring possessing a nonzero $*$-ideal I and $x, y \in R$ such that $xIy = \{0\} = xIy^*$, then $x = 0$ or $y = 0$. We have obtained its analogue in the setting of $*$-prime near-rings.

Lemma 2.4. Let N be a $*$-prime near-ring and I be a nonzero $*$-ideal of N. If $x, y \in N$ satisfy $xIy = xIy^* = \{0\}$ (resp. $xIy = x^*Iy = \{0\}$), then $x = 0$ or $y = 0$.

Proof. Assume $x \neq 0$, there exists some $z \in I$ such that $xz \neq 0$. For otherwise $xNy = \{0\}$ and $xNy^* = \{0\}$ for all $y \in I$ and thus $*$-primeness of N gives us $x = 0$. Since $xINy = \{0\}$ and $xINy^* = \{0\}$, we then obtain $xzNy = xzNy^* = \{0\}$. Now $*$-primeness of N provides us $y = 0$. Using similar arguments with necessary variations one can easily prove that $xIy = x^*Iy = \{0\}$) implies that $x = 0$ or $y = 0$.

Recently, L.Oukhtite and S.Salhi [6, Lemma $2-5$] studied derivations in $*$-prime rings and proved the following: Let R be a $*$-prime ring having nonzero $*$-ideal I then (i) If d is a nonzero derivation on R which commutes with $*$ and $[x, R]Id(x) = \{0\}$ for all $x \in I$, then R is commutative. (ii) If d is a nonzero derivation on R which commutes with $*$ and $[d(x), x] = 0$ for all $x \in I$, then R is commutative. (iii) Let d be a derivation of R satisfying $d* = \pm * d$. If $d^2(I) = \{0\}$, then $d = 0$. (iv) Let d_1 and d_2 be derivations of R such that $d_1* = \pm * d_1$ and $d_2* = \pm * d_2$. If $d_2(I) \subseteq I$ and $d_1d_2(I) = \{0\}$, then $d_1 = 0$ or $d_2 = 0$. We have obtained the analogues of these results in the setting of $*$-prime near-rings as below.

Lemma 2.5. Let N be a $*$-prime near-ring admitting a nonzero derivation d, which commutes with $*$. If I is a nonzero $*$-ideal of N and $[x, N]Id(x) = \{0\}$ for all $x \in I$, then N is a commutative ring.

Proof. Let $x \in I$. Since $y = x - x^* \in I$, then $[y, z]Id(y) = 0$ for all $z \in N$. As $y \in Sa_*(N)$, then using Lemma 2.1 we arrive at $[y, z]Id(y) = [y, z]^*Id(y) = \{0\}$ for all $z \in N$. By Lemma 2.4 we obtain that $d(y) = 0$ or $[y, z] = 0$ for all $z \in N$. If $d(y) = 0$, then $d(x) = d(x^*) = (d(x))^*$. Therefore $[x, z]Id(x) = [x, z]I(d(x))^* = \{0\}$ and by Lemma 2.4 we infer that either $d(x) = 0$ or $x \in Z$. On the other hand if $[y, z] = 0$ for all $z \in N$, then $y \in Z$ and hence $[x - x^*, z] = 0$ for all $z \in N$. By Lemma 2.1 we have that $[x, z] = [x^*, z]$ for all $z \in N$. Therefore using Lemma 2.1 again we obtain that $[x, z]Id(x) = [x, z]^*Id(x) = \{0\}$. Again using Lemma 2.4 we get either $d(x) = 0$ or $x \in Z$. Now we conclude that for each $x \in I$ either $d(x) = 0$ or $x \in Z$.

Let us consider $H = \{x \in I \mid d(x) = 0\}$ and $K = \{x \in I \mid x \in Z\}$. Using Lemma 2.1 it can be easily shown that H and K are additive subgroups of I such that $I = H \cup K$. But a group can not be a union of two of its proper subgroups and hence $I = H$ or $I = K$. If $I = H$, then $d(x) = 0$ for all $x \in I$. For any $t \in N$, replacing x by xt we get $xd(t) = 0$, for all $x \in I$ i.e.; $Id(t) = \{0\}$ for all $t \in N$. In particular $pId(t) = p^*Id(t) = \{0\}$ for all $t \in N$, where $0 \neq p \in N$. Now Lemma 2.4 gives us $d = 0$, a contradiction. Hence $I = K$ so that $I \subseteq Z$. $I \neq \{0\}$ implies that $Z \neq \{0\}$. Hence by Lemma 2.3, N is a ring. Let $z, t \in N$ and $x \in I$. From $ztx = zxt = tzx$ we conclude that $[z, t]I = \{0\}$ and then $[z, t]Ip = [z, t]Ip^* = \{0\}$, where $0 \neq p \in N$. In view of Lemma 2.4, we conclude that $[z, t] = 0$ for all $z, t \in N$. Therefore, N is a commutative ring.

Lemma 2.6. Let N be a $*$-prime near-ring admitting a nonzero derivation d, which commutes with $*$. If I is a nonzero $*$-ideal of N and $[d(x), x] = 0$ for all $x \in I$, then N is a commutative ring.

Proof. Let $x, y \in I$. Linearizing $[d(x), x] = 0$ with the help of Lemma 2.1 and using the hypothesis we get

$$[d(x), y] + [d(y), x] = 0 \tag{2.1}$$

for all $x, y \in I$. Now replacing y by yx and using Lemma 2.1 we obtain that

$$[d(x), y]x + [d(y), x]x + [y, x]d(x) = 0 \tag{2.2}$$

for all $x, y \in I$. Relations 2.1 and 2.2 yield $[x, y]d(x) = 0$, for all $x, y \in I$. Thus, for any $z \in N$, we have $[x, zy]d(x) = [x, z]yd(x) = 0$ by Lemma 2.1 and therefore $[x, N]Id(x) = \{0\}$ for all $x \in I$. Finally by Lemma 2.5 we get the required result.

Lemma 2.7. Let N be a 2-torsion free $*$-prime near-ring admitting a derivation d such that $d* = \pm * d$. If I is a nonzero $*$-ideal of N and $d^2(I) = \{0\}$, then $d = 0$.

Proof. For any $x \in I$, we have $d^2(x) = 0$. Putting xy for x where $y \in I$, and using Lemma 2.1 we arrive at $d^2(x)y + 2d(x)d(y) + xd^2(y) = 0$ for all $x, y \in I$. 2-torsion freeness of N and $d^2(I) = \{0\}$, provide us $d(x)d(y) = 0$. Replacing x by xz where $z \in I$ in the last relation and using Lemma 2.1, we get $d(x)zd(y) = 0$ for all $x, y, z \in N$ i.e.;$d(x)Id(y) = \{0\}$. Since $d* = \pm * d$, by Lemma 2.4 I infer that $d(x) = 0$ for all $x \in I$. Now replacing x by xt where $t \in N$ we have $xd(t) = 0$ and therefore $INd(t) = \{0\}$ for all $t \in N$. Since I is a nonzero $*$-ideal and N is a $*$-prime near-ring, We get $d(z) = 0$ for all $z \in N$ and consequently $d = 0$.

Lemma 2.8. Let N be a 2-torsion free $*$-prime near-ring admitting derivations d_1 and d_2 such that $d_1* = \pm * d_1$ and $d_2* = \pm * d_2$. If I is a nonzero $*$-ideal of N such that $d^2(I) \subseteq I$ and $d_1d_2(I) = 0$, then $d_1 = 0$ or $d_2 = 0$.

Proof. Let $x, y \in I$. Then $d_1d_2(xy) = d_1(x)d_2(y) + d_2(x)d_1(y) = 0$. Replacing x by $d_2(x)$ We get $d_2^2(x)d_1(y) = 0$ for all $x, y \in I$. Now putting yz where $z \in I$ for y we obtain that $d_2^2(x)yd_1(z) = 0$ for all $x, y, z \in I$. It then gives us $d_2^2(x)Id_1(z) = \{0\}$ for all $x, z \in I$. The conditions $I^* = I$ and $d_1* = \pm * d_1$ provide us $d_2^2(x)Id_1(z) = d_2^2(x)I\{d_1(z)\}^* = \{0\}$ and by Lemma 2.4 it follows that either $d_1(z) = 0$ for all $z \in I$ or $d_2^2(x) = 0$ for all $x \in I$. If $d_1(z) = 0$ for all $z \in I$, then replacing z by zt where $t \in N$, we obtain that $zd_1(t) = 0$ i.e.; $Id_1(t) = \{0\}$. As $d_1* = \pm * d_1$, this implies that $pId_1(t) = pI\{d_1(t)\}^* = \{0\}$ for $0 \neq p \in N$. In view of Lemma 2.4 We obtain that $d_1 = 0$. On the other hand if $d_2^2(x) = 0$ for all $x \in I$, We obtain by Lemma 2.7 that $d_2 = 0$.

3. MAIN RESULTS

In the year 2006, L. Oukhtite and S. Salhi [4 ,Theorem 3.2] obtained the following: Let d be a nonzero derivation of a 2-torsion free $*$-prime ring R and I a nonzero $*$-ideal of R. If $r \in Sa_*(R)$ satisfies $[d(x), r] = 0$ for all $x \in I$, then $r \in Z(R)$. Furthermore, if $d(I) \subseteq Z(R)$, then R is commutative. we have obtained its analogue for $*$-prime near-rings with derivation.

Theorem 3.1. Let N be a 2-torsion free $*$-prime near-ring admitting a nonzero derivation d and a nonzero $*$-ideal I. If $t \in Sa_*(N)$ satisfies $[d(x), t] = 0$ for all $x \in I$, then $t \in Z$. Furthermore, if $d(I) \subseteq Z$, then N is a commutative ring.

Proof. Since $[d(xy), t] = 0$ for all $x, y \in I$, using Lemma 2.1 it provides us $d(x)yt + xd(y)t - td(x)y - txd(y) = 0$. Conditions $[d(x), t] = [d(y), t] = 0$ give us

$$d(x)[y, t] + [x, t]d(y) = 0 \tag{3.1}$$

for all $x, y \in I$. Replacing y by yt and using Lemma 2.1, we conclude that $[x, t]Id(t) = \{0\}$. The fact that I is a $*$-ideal together with $t \in Sa_*(N)$ and Lemma 2.1, provide $[x, t]^*Id(t) = [x, t]Id(t) = \{0\}$. Applying Lemma 2.4, either $d(t) = 0$ or $[x, t] = 0$. If $d(t) \neq 0$, then $[x, t] = 0$ for all $x \in I$. Let $s \in N$, from $[sx, t] = 0$ it follows by Lemma 2.1 that $[s, t]x = 0$. Let $0 \neq x_0 \in I$, as $[s, t]Nx_0 = [s, t]Nx_0^*$. Since N is $*$-prime near-ring, which proves that $[s, t] = 0$ i.e.; $t \in Z$. On the other hand if $d(t) = 0$, then $d([x, t]) = [d(x), t] = 0$ and consequently

$$d([I, t]) = \{0\}. \tag{3.2}$$

Replacing y by yz where $z \in I$ in the relation (3.1) and using Lemma 2.1, we see that

$$d(x)y[z,t] + [x,t]yd(z) = 0. \tag{3.3}$$

Taking $[z,t]$ instead of z in relation (3.3) and applying relation (3.2) and Lemma 2.1 we then arrive at $d(x)y[[z,t],t] = 0$ so that $d(x)I[[z,t],t] = \{0\} = d(x)I[[z,t],t]^*$. Hence $d(I) = \{0\}$ or $[[z,t],t] = 0$ for all $z \in I$, by Lemma 2.4. If $d(I) = \{0\}$, then for any $s \in N$ we get $d(sx) = d(s)x = 0$ for all $x \in I$. Therefore $d(s)NI = \{0\} = d(s)NI^*$ and as I is nonzero, then $*$-primeness of N provides us $d(t) = 0$ which implies that $d = 0$, a contradiction. Thus we conclude that $[[z,t],t] = 0$. Now putting zx for z and using Lemma 2.1 we obtain that $0 = [[zx,t],t] = [z,t][x,t] + [z,t][x,t]$. It follows that $[z,t][x,t] = 0$, by 2-torsion freeness of N. Replacing z by sz where $s \in N$ and using Lemma 2.1 again we obtain that $0 = [sz,t][x,t] = [s,t]z[x,t]$ and consequently $[s,t]I[x,t] = \{0\}$ for all $x \in I$. Therefore by Lemma 2.1, $[s,t]I[x,t] = [s,t]I[x,t]^* = \{0\}$. Once again using Lemma 2.4 we arrive at $[s,t] = 0$ or $[x,t] = 0$. If $[s,t] = 0$, then $t \in Z$. If $[x,t] = 0$ for all $x \in I$, then for any $s \in N$ we have $0 = [sx,t] = s[x,t] + [s,t]x = [s,t]x$ by Lemma 2.1. Hence $\{0\} = [s,t]I = [s,t]Ip = [s,t]Ip^*$, where $0 \neq p \in N$. Using Lemma 2.4 once again we conclude that $[s,t] = 0$, which proves that $t \in Z$.

Now suppose that $d(I) \subseteq Z$. Hence from the first part of the theorem we conclude that $Sa_*(N) \subseteq Z$. It is obvious that for each $s \in N$, $s - s^* \in Sa_*(N)$. Now for any given $s \in N$, there are two possibilities either $s - s^* \neq 0$, or $s - s^* = 0$. If first case occurs, then $0 \neq s - s^* \in Z$. Therefore in this case $Z \neq \{0\}$ and by Lemma 2.3, N becomes a ring. If first case does not occur, then $s = s^*$ for all $s \in N$. Which implies that $s_1 s_2 = (s_1 s_2)^* = s_2^* s_1^* = s_2 s_1$ for all $s_1, s_2 \in N$ and we conclude that in this case $N = Z$ i.e.; $Z \neq \{0\}$. Again Lemma 2.3 shows that N is a ring. Finally we get a fact that if $d(I) \subseteq Z$ holds, then N is a ring. As N is a ring, $s + s^*, s - s^* \in Sa_*(N)$. We then obtain $s + s^*, s - s^* \in Z$ and hence $2s \in Z$. Since N is 2-torsion free, then $s \in Z$ proving the commutativity of N. Hence N is a commutative ring.

Recently, L.Oukhtite and S.Salhi [4, Theorem 3.3] proved the following: Let d be a nonzero derivation of a 2-torsion free $*$-prime ring R and let $t \in Sa_*(R)$. If $d([R,t]) = 0$, then $t \in Z(R)$. In particular, if $d[x,y] = 0$, for all $x, y \in R$, then R is commutative. we have obtained its analogue in the setting of $*$-prime near-rings with derivation.

Theorem 3.2. Let N be a 2-torsion free $*$-prime near-ring admitting a nonzero derivation d and let $t \in Sa_*(N)$. If $d([N,t]) = 0$, then $t \in Z$. In particular, if $d[x,y] = 0$, for all $x, y \in N$, then N is commutative ring.

Proof. If $d(t) = 0$, from our hypothesis and using Lemma 2.1, for any $x \in N$ we obtain, $0 = d([x,t]) = d(x)t + xd(t) - d(t)x - td(x) = d(x)t - td(x) = [d(x),t]$. Hence we arrive at

$[d(x), t] = 0$ for all $x \in N$. Applying Theorem 3.2, this gives $t \in Z$ and in this case proof finishes. Now assume that $d(t) \neq 0$. For all $x \in N$, we have $0 = d([tx, t]) = d(t[x, t]) = td[x, t] + d(t)[x, t]$. This implies that

$$d(t)[x, t] = 0. \tag{3.4}$$

Putting xy where $y \in N$ for x, and using Lemma 2.1 we arrive at $0 = d(t)[xy, t] = d(t)x[y, t] + d(t)[x, t]y$. Which reduces to $d(t)x[y, t] = 0$ i.e.; $d(t)R[y, t] = \{0\}$ for all $y \in R$, with the help of relation (3.4). Since $t \in Sa_*(N)$, then by Lemma 2.1 we have $d(t)R[y, t] = d(t)R[y, t]^* = \{0\}$. Now $*$-primeness of N insures that $[y, t] = 0$ i.e.; $t \in Z$. Now suppose that $d[x, y] = 0$, for all $x, y \in N$ and using first part of the theorem, we conclude that $Sa_*(N) \subseteq Z$. Further onward using the same argument as used in the Theorem 3.1, we obtain that N is a commutative ring.

Recently, L.Oukhtite and S.Salhi [6, Theorem 1] proved the following: Let R be a 2-torsion free $*$-prime ring, admitting a nonzero derivation d, which commutes with $*$ and I a nonzero $*$-ideal. If $[d(x), x] \in Z(R)$, for all $x \in I$, then it is commutative. we have proved its analogue for $*$-prime near-rings with derivation.

Theorem 3.3. Let N be a 2-torsion free $*$-prime near-ring, admitting a nonzero derivation d, which commutes with $*$. If $[d(x), x] \in Z$, for all $x \in I$, then N is a commutative ring.

Proof. Linearizing $[d(x), x] \in Z$ with the help of Lemma 2.1, we arrive at $[d(x), y] + [d(y), x] \in Z$ for all $x, y \in I$. Replacing y by x^2 and using Lemma 2.1, we obtain that $4x[d(x), x] \in Z$. Now 2-torsion freeness of N forces $x[d(x), x] \in Z$ for all $x \in I$. Thus for any $t \in N$, we have that $tx[d(x), x] = x[d(x), x]t = xt[d(x), x]$ and so by Lemma 2.1 we arrive at $[t, x][d(x), x] = 0$, for all $x \in I$ and for all $t \in N$. Replacing t by $d(x)$, we obtain $[d(x), x]^2 = 0$ for all $x \in I$. Since $[d(x), x] \in Z$, then $[d(x), x]N[d(x), x][d(x), x]^* = \{0\}$. As $[d(x), x][d(x), x]^* \in Sa_*(N)$. Now $*$-prime ness of N provides us $[d(x), x] = 0$ or $[d(x), x][d(x), x]^* = 0$. Suppose $[d(x), x][d(x), x]^* = 0$ holds then the condition $[d(x), x] \in Z$ gives us $[d(x), x]N[d(x), x]^* = \{0\}$. Including the both cases we infer that $[d(x), x]N[d(x), x]^* = \{0\} = [d(x), x]N[d(x), x]$. $*$-primeness of N yields $[d(x), x] = 0$ for all $x \in I$. Now using Lemma 2.6, we get our required result.

In the year 2007, L.Oukhtite and S.Salhi [7, Theorem 1.2-1.3] obtained the following results: Let R be a 2-torsion free $*$-prime ring admitting a nonzero derivation d, which commutes with $*$ and I a nonzero $*$-ideal. If R satisfies any one of the following conditions: (i) $[d(x), d(y)] = 0$, for all $x, y \in I$, (ii) $d([x, y]) = 0$ for all $x, y \in I$, then it is commutative. we have proved analogues of these results in the setting of $*$-prime near-rings with derivation. Finally it is also shown that the restriction of 2-torsion freeness of R used by authors while proving above (ii) is redundant.

Theorem 3.4. Let N be a 2-torsion free $*$-prime near-ring and I a nonzero $*$-ideal of N. If N admits a nonzero derivation d such that $[d(x), d(y)] = 0$, for all $x, y \in I$ and d commutes with $*$, then N is a commutative ring.

Proof. By hypothesis we have, $[d(x), d(y)] = 0$, for all $x, y \in I$. Now replacing y by xy and using Lemma 2.1, we obtain that $d(x)[d(x), y] + [d(x), x]d(y) = 0$, for all $x, y \in I$. Putting yz where $z \in N$ for y in the last expression and using Lemma 2.1, we get $d(x)y[d(x), z] + [d(x), x]yd(z) = 0$, for all $x, y \in I, z \in N$. Replacing z by $d(t)$ where $t \in I$ and using hypothesis, we arrive at $[d(x), x]yd^2(t) = 0$, for all $x, y, t \in I$. This implies that $[d(x), x]Id^2(t) = \{0\}$. Since $d* = *d$ and d is a $*$-ideal, we get $[d(x), x]Id^2(t) = [d(x), x]^*Id^2(t) = \{0\}$ by Lemma 2.1. Applying Lemma 2.4, either $d^2(t) = 0$ for all $t \in I$ or $[d(x), x] = 0$ for all $x \in I$. If $d^2(t) = 0$ for all $t \in I$, then by Lemma 2.7 we conclude that $d = 0$, a contradiction. Thus we conclude that $[d(x), x] = 0$ for all $x \in I$ and hence Lemma 2.6 finishes the proof.

Theorem 3.5. Let N be a $*$-prime near-ring and I a nonzero $*$-ideal of N. If N admits a nonzero derivation d such that $d([x, y]) = 0$, for all $x, y \in I$ and d commutes with $*$, then N is a commutative ring.

Proof. By hypothesis we have $d([x, y]) = 0$, for all $x, y \in I$. Now replacing y by yx, we obtain that $d([x, yx]) = 0$. Using hypothesis and Lemma 2.1 we obtain that $[x, y]d(x) = 0$ for all $x, y \in I$. This implies that, for any $z \in N$, replacing y by zy and using Lemma 2.1 again we arrive at $[x, z]yd(x) = 0$ for all $x, y \in I$ i.e.;$[x, N]Id(x) = \{0\}$ for all $x \in I$. Now by Lemma 2.5, the result follows.

Theorem 3.6. Let N be a $*$-prime near-ring and I a nonzero $*$-ideal of N. If N admits a nonzero derivation d, which commutes with $*$ and one of the following conditions hold (i) $d([x, y]) = \pm[x, y]$, (ii) $d([x, y]) = \pm(xoy)$, (iii) $d(xoy) = 0$, (iv) $d(xoy) = \pm(xoy)$ and (v) $d(xoy) = \pm[x, y]$ for all $x, y \in I$, then N is a commutative ring.

Proof. It can be proved using the same techniques, as in Theorem 3.5.

The following example justifies the existence of $*$-primeness in the hypotheses of the Theorems 3.5 and 3.6.

Example 3.1. Let S be a left near-ring. Suppose $N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, 0 \in S \right\}$. Define $d, * : N \longrightarrow N$ such that

$$d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to check that N is $*$-near-ring and $I = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid x, 0 \in S \right\}$ is a $*$-ideal of N. If we set $p = \begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $0 \neq s \in S$, then $pNp = \{0\} = pNp^*$ proving that N is not $*$-prime near-ring. Furthermore d is a nonzero derivation, which commutes with $*$ and satisfies the following conditions:
(i) $d([x, y]) = 0$, (ii) $d([x, y]) = \pm[x, y]$, (iii) $d([x, y]) = \pm(xoy)$, (iv) $d(xoy) = 0$, (v) $d(xoy) = \pm(xoy)$ and (vi) $d(xoy) = \pm[x, y]$ for all $x, y \in I$. However N is not a commutative ring.

Recently, L.Oukhtite and S.Salhi [6, Theorem 2] obtained the following result: Let R be $*$-prime ring with characteristic not 2 and I be a nonzero $*$-ideal of R. Suppose there exist derivations d_1 and d_2 which commute with $*$ such that $d_1(x)x - xd_2(x) \in Z(R)$ for all $x \in I$. If $d_2 \neq 0$, then R is commutative. The main purpose of the following theorem is to prove its analogue for $*$-prime near-rings.

Theorem 3.7. Let N be a 2-torsion free $*$-prime near-ring and I a nonzero $*$-ideal of N. Suppose there exist derivations d_1 and d_2 which commute with $*$ such that $d_1(x)x - xd_2(x) \in Z$ for all $x \in I$. If $d_2 \neq 0$, then N is a commutative ring.

Proof. If $I \cap Z = \{0\}$; as $d_1(x)x - xd_2(x) \in I \cap Z$, then $d_1(x)x = xd_2(x)$ for all $x \in I$. Linearizing this relation with the help of Lemma 2.1 we get $d_1(x)y + d_1(y)x = xd_2(y) + yd_2(x)$ for all $x, y \in I$. Replacing y by yx in the last relation and using the same again, combined with the fact that $d_1(x)x = xd_2(x)$ and Lemma 2.1, we arrive at $[x, yd_2(x)] = 0$ for all $x, y \in I$. Now putting ty, where $t \in N$ and using Lemma 2.1, to get $[x, t]yd_2(x) = 0$ i.e.; $[x, N]Id_2(x) = \{0\}$ for all $x \in I$. Since $d_2 \neq 0$, from Lemma 2.5 we conclude that N

is a commutative ring.
Next, assume that $I \cap Z \neq \{0\}$. This implies that $Z \neq \{0\}$. Therefore by Lemma 2.3, N is a ring. Choose $0 \neq z \in I \cap Z$ in such way $z^* = \pm z$. If $z^* = z$, then nothing to do, otherwise we consider $t = z - z^*$, then $t \in I \cap Z$ and $t^* = -t$. Linearizing $d_1(x)x - xd_2(x) \in Z$, we get

$$d_1(x)y + d_1(y)x - xd_2(y) - yd_2(x) \in Z \tag{3.5}$$

for all $x, y \in I$. Replacing y by z and using $d_2(z) \in Z$ in the relation (3.5), we arrive at

$$z(d_1(x) - d_2(x)) + (d_1(z) - d_2(z))x \in Z \tag{3.6}$$

for all $x \in I$. Putting $y = z^2$ in the relation (3.5) and using the relation (3.6), we conclude that $z(d_1(z) - d_2(z))x \in Z$ for all $x \in I$. This implies that $z(d_1(z) - d_2(z))N[x, t] = \{0\} = z(d_1(z) - d_2(z))N[x, t]^*$ for all $x \in I, t \in N$. Which leads us to $I \subseteq Z$ and by Lemma 2.6, N is a commutative ring or $d_1(z) = d_2(z)$. If $d_1(z) = d_2(z)$, then by relation (3.6) we conclude that $z(d_1(x) - d_2(x)) \in Z$ and so $(d_1(x) - d_2(x)) \in Z$ for all $x \in I$. Hence, $d(I) \subseteq Z$ where $d = d_1 - d_2$. Then it follows by Lemma 2.6 that N is a commutative ring. If $d_1 = d_2$ then $d_1(x)x - xd_1(x) = [d_1(x), x] \in Z$ for all $x \in I$ and by Theorem 3.3, we conclude that N is a commutative ring.

Recently, L.Oukhtite and S.Salhi [5, Theorem 3.3] obtained the following result for prime near-rings: Let N be a prime near-ring, which admits a nonzero derivation d. If d acts as a homomorphism on N, then d is the identity map. Motivated by this result we investigated its analogue in the setting of $*$-prime near-rings under some constraints.

Theorem 3.8. Let N be $*$-prime near-ring, admitting a derivation d and a nonzero $*$-ideal I. If d acts as a homomorphism on I and $d* = *d$, then $d = 0$.

Proof. Assume that d acts as a homomorphism on I. Then one obtains that $d(xy) = d(x)d(y) = d(x)y + xd(y)$ for all $x, y \in I$. Replacing y by yz, where $z \in I$, we obtain that $d(x)d(yz) = d(x)yz + xd(yz)$. Since d acts as a homomorphism on I, we deduce that $d(xy)d(z) = d(x)yz + xyd(z) + xd(y)z$. Using Lemma 2.1 we arrive at $xd(y)d(z) + d(x)yd(z) = d(x)yz + xyd(z) + xd(y)z$ i.e.;$xd(yz) + d(x)yd(z) = d(x)yz + xyd(z) + xd(y)z$. This implies that $xyd(z) + xd(y)z + d(x)yd(z) = d(x)yz + xyd(z) + xd(y)z$. Using Lemma 2.1 we arrive at $d(x)yd(z) = d(x)yz$ i.e.; $d(x)y(d(z) - z) = 0$ for all $x, y, z \in I$. Since I is a $*$-ideal and $d* = *d$, we conclude that $d(x)I(d(z) - z) = \{0\} = \{d(x)\}^*I(d(z) - z)$ for all $x, z \in I$. By Lemma 2.4 we infer that either $d(z) = z$ or $d(x) = 0$. If first case holds, then replacing z by zx we obtain that $d(zx) = zx$ i.e.; $zd(x) + d(z)x = zx$. This implies that $zd(x) = 0$ i.e.;$Id(x) = \{0\}$. Finally we get $tId(x) = t^*Id(x) = \{0\}$, where $0 \neq t \in N$.

By Lemma 2.4 we obtain that $d(x) = 0$. Now combining both the cases we conclude that $d(x) = 0$ for all $x \in I$. Putting xt for x where $t \in N$, we obtain that $xd(t) + d(x)t = 0$ i.e.; $xd(t) = 0$. Finally we get that $sId(t) = \{0\} = s^*Id(t)$, where $0 \neq s \in N$. Again Lemma 2.4 insures that $d = 0$.

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