Common Fixed Point Theorems for Weakly Compatible Maps in Intuitionistic Fuzzy Metric Space

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Abstract — *The aim of this paper is to prove a common fixed point theorem in an intuitionistic fuzzy metric space by using the notion of property E.A.*

Keywords — *Intuitionistic fuzzy metric space, property E.A., fixed point, weakly compatible maps.*
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I. INTRODUCTION

In 1986 Jungck [6] introduced the notion of compatible maps for a pair of self mappings. In 1975 Kramosil and Michalek [7], in 1994 George and Veeramani [5] gave papers involving compatible maps proved the existence of common fixed points in the classical and fuzzy metric spaces. In 2002 Amari and Moutawakil [1] generalized the concept of non compatibility by defining the notion of property E.A. and proved common fixed point under strict contractive conditions. In 1986 Atanassove [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets and later there has been much progress in the study of intuitionistic fuzzy sets by many author (Coker,1997 [4]; Park,2004 [10]; Alaca, Park et al.,2006 [2]; Manro et al.,2010 [8],2012 [9]. In 2004 Park [10] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani (1994) [5]. In 2006 Turkoglu et al. [14] gave a generalization of Jungck's common fixed point theorems [6] to intuitionistic fuzzy metric spaces. In this paper we prove a common fixed point theorem in an intuitionistic fuzzy metric space by using the notion of property E.A.

II. PRELIMINARIES

The concept of triangular norms (t-norms) and triangular connorms (t-conorm) were originally introduced by Schweizer and Sklar (1960) [13] in the study of statistical metric spaces.

Definition 2.1.[13] A binary operation $*$: $[0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative
(ii) $*$ is continuous
(iii) $a * 1 = a$ for all $a \in [0, 1]$
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2.[13] A binary operation $\Diamond$: $[0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-conorm if $\Diamond$ satisfies the following conditions:

(i) $\Diamond$ is commutative and associative
(ii) $\Diamond$ is continuous
(iii) $a \Diamond 0 = a$ for all $a \in [0, 1]$
(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Alaca et al. [2] using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to Kramosil and Michalek [7] as :

Definition 2.3. [2] A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$
(ii) $M(x, y, 0) = 0$ for all $x, y \in X$
(iii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ if and only if $x = y$
(iv) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$
(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$
(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is left continuous
(vii) $\lim_{n\rightarrow\infty} M(x, y, t) = 1$; for all $x, y \in X$ and $t > 0$
(viii) $N(x, y, 0) = 1$; for all $x, y \in X$
(ix) $N(x, y, t) = 0$; for all $x, y \in X$ and $t > 0$ if and only if $x = y$
(x) $N(x, y, t) = N(y, x, t)$; for all $x, y \in X$ and $t > 0$

(xi) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$; for all x, y, z ∈X and s, t > 0

(xii) for all x, y ∈ X, $N(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is right continuous

(xiii) $\lim_{n\to\infty} N(x,y,t) = 0$; for all x, y ∈ X.

Then (M, N) is called an intuitionistic fuzzy metric space on X. The functions M(x, y, t) and N(x, y, t) denote the degree of nearness and the degree of non-nearness between x and y with respect to t respectively.

Remark 2.1.[2] Every fuzzy metric space (X, M, *) is an intuitionistic fuzzy metric space of the form (X, M, 1-M, *, ◊) such that t-norm * and t-conorm ◊ are associated as $x \Diamond y = ((1-x)(1-y))$; ∀ x , y ∈ X

Remark 2.2.[2] In intuitionistic fuzzy metric space (X, M, N, *, ◊),M(x, y, .) is non-decreasing and N(x, y, .) is non-increasing for all x,y∈ X.
Alaca et al. [2] introduced the following notions:

Definition 2.4.[2] Let (X, M, N, *, ◊) be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all t >0 and p >0, $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ and $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point x ∈ X if, for all t > 0, $\lim_{n\to\infty} M(x_n, x, t) = 1$ and $\lim_{n\to\infty} N(x_n, x, t) = 0$

Definition 2.5. [2] An intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.6. A pair of self mappings (f, g) of an intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be commuting, if
M(fgx , gfx, t)=1 and N(fgx , gfx, t) = 0;for all x∈X

Definition 2.7.[1]A pair of self mappings (f , g) of an intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to satisfy the property E.A. if there exist a sequence $\{x_n\}$ in X such that
$\lim_{n\to\infty} M(fx_n, gx_n, t) = 1$ and $\lim_{n\to\infty} N(fx_n, gx_n, t) = 0$

Definition 2.8. [1] A pair of self mappings (f, g) of an intuitionistic fuzzy metric space (X, M, N, *, ◊) is said to be weakly compatible if they commute at coincidence points i.e. if fu = gu for some u ∈ X, then fgu = gfu.

It is easy to see that two compatible maps are weakly compatible.

Lemma 2.1. [2] Let (X, M, N, *, ◊) be intuitionistic fuzzy metric space and for all x, y ∈ X,
t > 0 and if for a number k ∈ (0 , 1),
M(x , y, kt) ≥ M(x, y, t) and
N(x , y, kt) ≤ N(x, y, t), then x = y.

III. MAIN RESULT

We prove the following theorem.

Theorem

Let (X, M, N, *, ◊) be a complete intuitionistic fuzzy metric space with continuous t-norm and continuous t-conorm defined by a * a ≥ a and (1-a) ◊ (1-a) ≤ (1-a) where a∈[0,1]. Let g and h be two weakly compatible mappings of X satisfying the following conditions:
(a) g and h satisfy the property E.A.
(b) for x, y ∈ X , t > 0 there exist 0 < k < 1 such that

$$M(hx, hy, kt) \geq \min[M(hx, gx, t) * M(gx, hy, t), M(gy, hx, t) * M(gy, hy, t)]$$

(3.1)

$$N(hx, hy, kt) \leq \max[N(hx, gx, t) \Diamond N(gx, hy, t), N(gy, hx, t) \Diamond N(gy, hy, t)]$$

(3.2)

(c) g(X) and h(X) are complete subspaces of X.
Then g and h have a unique common fixed point.

Proof – Let $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} hx_n = \lim_{n\to\infty} gx_n = u$ for some u ∈ X, by property E.A.
Since g(X) is a complete subspace of X.
Therefore, every convergent sequence of points of g(X) has a limit point in g(X).
=> $\lim_{n\to\infty} gx_n = ga = u = \lim_{n\to\infty} hx_n$, for some a ∈ X
=> u = ga ∈ g(X)
Now we prove that ha = ga
Take $x = x_n$ and y = a in (3.1) and (3.2)
$M(hx_n, ha, kt) \geq \min[M(hx_n, gx_n, t) * M(gx_n, ha, t), M(ga, hx_n, t) * M(gx_n, hx_n, t)]$
and
$N(hx_n, ha, kt) \leq \max[N(hx_n, gx_n, t) \Diamond N(gx_n, ha, t), N(ga, hx_n, t) \Diamond N(gx_n, hx_n, t)]$
Taking limit $n \to \infty$ on both the sides, we get
$M(ga, ha, kt) \geq \min[M(ga, ga, t) * M(ga, ha, t), M(ga, ga, t) * M(ga, ha, t)]$ and
$N(ga, ha, kt) \leq \max[N(ga, ga, t) \Diamond M(ga, ha, t), N(ga, ga, t) \Diamond N(ga, ha, t)]$
=> $M(ga, ha, kt) \geq \min[1 * M(ga, ha, t), 1 * M(ga, ha, t)]$ and
$N(ga, ha, kt) \leq \max[0 \Diamond N(ga, ha, t), 0 \Diamond N(ga, ha, t)]$
=> M(ga, ha, kt) ≥ M(ga, ha, t) and

N(ga,ha,kt) ≤ N(ga,ha,t),[by Definition (2.1)and (2.2)]

=>ga = ha [by Lemma (2.1)]

Therefore u = ga = ha (3.3)

This shows that a coincident point of h and g.

Since h and g are weakly compatible.

Therefore hga = gha = gga = hha (3.4)

Now we show that ha is the common fixed point of h and g.

Now we take x = a and y = ha

M(ha, hha, kt)≥min[M(ha, ga, t) * M(ga, hha, t), M(gha, ha, t) * M(gha, hha, t)]

and

N(ha, hha, kt) ≤ max[N(ha, ga, t) ◊ N(ga, hha, t), N(gha, ha, t) ◊ N(gha, hha, t)]

=>M(ha, hha, kt) ≥ min[M(ha, ha, t) * M(ha, hha, t), M(hha, ha, t)*M(hha, hha, t)]

and

N(ha, hha, kt) ≤ max[N(ha, ha, t) ◊ N(ha, hha, t), N(hha, ha, t) ◊ N(hha, hha, t)]

=>M(ha, hha, kt) ≥ min[M(ha, ha, t) * M(ha, hha, t), M(ha, hha, t)*M(hha, hha, t)]

and

N(ha, hha, kt) ≤ max[N(ha, ha, t) ◊ N(ha, hha, t), N(ha, hha, t) ◊ N(hha, hha, t)]

=> M(ha, hha, kt) ≥ min[1*M(ha, hha, t), M(ha, hha, t) * 1] and

N(ha, hha, kt) ≤ max[0 ◊ N(ha, hha, t), N(ha, hha, t) ◊ 0] by definition (2.3)

=> M(ha, hha, kt) ≥ [M(ha, hha, t)] and

N(ha, hha, kt) ≤ [N(ha, hha, t)]

=> ha=hha by Lemma (2.1)

=> ha = hha = gha by using (3.4)

This proves that ha is the common fixed point of h and g.

Uniqueness of common fixed point of h and g

If possible suppose that u and v are two common fixed points of h and g.

Then by (3.1)

M(u,v,kt)=M(hu,hv,kt)≥min[M(hu,gu,t)*M(gu,hv,t), M(gv,hu,t)*M(gv,hv,t)]

=min[M(hu,gu,t)*M(gu,hv,t), M(hu,gv,t)*M(gv,hv,t)]

= min[M(u, u, t) * M(u, v, t), M(u, v, t) * M(v, v, t)]

= min[1* M(u, v, t), M(u, v, t) * 1]

= M(u, v, t)

=> M(u, v, kt) ≥ M(u, v, t)

and by (3.2)

N(u, v, kt)=N(hu, hv, kt) ≤ max[N(hu ,gu, t) ◊ N(gu,hv, t),N(gv, hu, t) ◊ N(gv, hv, t)]

=max[N(hu, gu, t) ◊ N(gu, hv, t),N(hu, gv, t) ◊ N(gv, hv, t)]

= max[N(u, u, t) ◊ N(u, v, t), N(u, v, t) ◊ N(v, v, t)]

= max[0 ◊ N(u, v, t), N(u, v, t) ◊ 0]

= N(u, v, t)

=> N(u, v, kt) ≤ N(u, v, t)

By using Lemma (2.1), we have u = v

Hence g and h have a unique common fixed point.

This completes the proof.

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