Distance Closed Domatic Number of Graphs

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Abstract: *In a graph G = (V, E), a set $S \subset V(G)$ is said to be a distance closed set if for each vertex $u \in S$ and for each $w \in V - S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. A dominating set S is said to be a Distance Closed Dominating (D.C.D) set if S is distance closed.The cardinality of a minimum distance closed dominating set of G is called the distance closed domination number of G and is denoted by $\gamma_{dcl}(G)$. The definition and the extensive study of the distance closed dominating sets in graphs are studied in [6].In this paper,the distance closed domatic number of some special classes of graphs are studied.Also, a general algorithm to find the structure of graphs with a given domatic number is proposed.*

Keywords:*Distance, eccentricity, radius, diameter, degree, paths, cycles, trees, self-centred graphs, complete graphs, complete bipartite graphs, regular graphs, distance closed dominating set, distance closed domination number, distance closed domatic number.*

1. INTRODUCTION

For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is the number of edges incident with v and is denoted by $deg_G(v)$. A graph with every vertex of degree k is called a k-regular graph. The length of the shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$.For a connected graph G, the *eccentricity* $e_G(v) = \max\{d_G(u, v) : \forall u \in V(G)\}$. The minimum and maximum eccentricities are called the *radius* and *diameter* of G, denoted by r(G) and d(G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered* graph. A graph G is called *complete* if every pair of vertices of G are adjacent. We denote a complete graph of order n byK_n.The *complement* $\overline{G}$ of a graph G has $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. In particular, $\overline{K_n}$ has n vertices and no edges. A *bipartite graph* is a graph G for which V can be partitioned as $V = V_1 \cup V_2$ with no two adjacent vertices in the same V_i and if G contains every edge joining V_1 and V_2, then G is a *complete bipartite graph*. In this case, if V_1 and V_2 have m and n vertices, we write $G = K_{m, n}$. The concept of distance and related properties are studied in [1] and [2]. One important aspect of the concept of distance and eccentricity is the existence of polynomial time algorithm to analyze them.

A *cut vertex* of a connected graph is a vertex whose removal increases the number of components. Thus if v is a cut vertex of a connected graph G, then G – v is disconnected. A *non-separable graph* is connected, non-trivial and has no cut vertices. A *block* of a graph is a maximal non-separable subgraph.An *Openpath* of G is a finite, alternating sequence $v_0, e_1, v_1, e_2 \ldots v_{n-1}, e_n, v_n$ beginning with vertex v_0 and ending with vertex v_n, such that $e_i = v_{i-1}v_i$ and $v_i \neq v_j$ for i, j = 1, 2 … n. The number n (the number of occurrences of edges) is called the *length* of the path. A path of order n is denoted by P_n. Therefore, $P_n = v_1v_2 \ldots v_n$ indicates a path of order n on the vertices $v_1, v_2 \ldots v_n$. A *cycle* of G is a path $v_1v_2 \ldots v_n$ ($n \geq 3$) with the additional edge v_nv_1. A cycle of order n is denoted by C_n, represented as $C_n = v_1v_2 \ldots v_nv_1$. A *tree* is a connected graph, which has no cycles.

The concept of domination in graphs was introduced by Ore [7] in 1962. It is originated from the chess game theory which paved the way to the development of the study of various domination parameters and then relation to various other graph parameters. A set $D \subseteq V(G)$ is called a *dominating set* of G if every vertex in V(G) – D is adjacent to some vertex in D and the *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. Different types of dominating sets have been studied by imposing conditions on the dominating sets. The list of survey of domination theory papers are in [3] and [4]. At present, domination is considered to be one of the fundamental concepts in graph theory and its various applications to ad-hoc networks, biological networks, distributed computing, social networks and web graphs partly explain the increased interest. Such applications usually aim at selecting a subset of nodes that will provide some definite services such that every node in the network is 'close' to some node in the subset.

An interesting variant of domination problem is to ask how many dominating sets one can pack into a given graph G and the main question is how to partition the vertex set of a graph into maximum number of disjoint dominating sets. The word “domatic” arises from two words “dominating” and “chromatic”, since the definition is related to both domination and coloring concepts. The domaticnumber of a graph was defined by Cockayne and Hedetniemi (1977) and the concepts were defined for undirected graphs in [3]. Zelinka (1984) transferred the concept of domatic number to directed graphs in [8]. The concept of domatic partition deals with the partition of a given graph into vertex disjoint distance closed dominating sets. Thus, it will be useful to partition the given communication network into parallel fault tolerant central location models, each of which behaves as a sub-model/network which could perform independently without affecting the remaining system. Thus, identification of such disjoint sub-networks is helpful in the parallel architecture design of the given network.

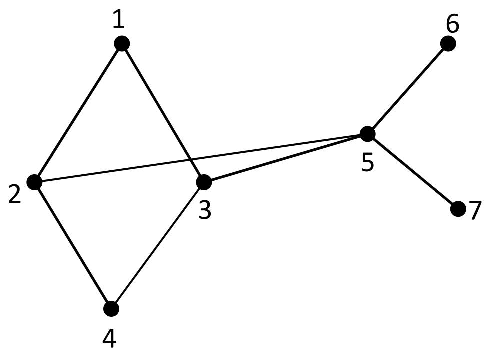

Figure 2.1 - ***An example of D.C.D set of a graph***

Clearly from the definition, $1 \leq \gamma_{dcl} \leq p$ and the graph with $\gamma_{dcl} = p$ is called a 0-distance closed dominating graph. The definition and the extensive study of the above said distance closeddominating sets in graphs are studied in[6]. The following are some important results proved in [6] has used here.

Proposition 2.1: If T is a tree with number of vertices $p \geq 2$, then $\gamma_{dcl}(T) = p - k + 2$, where k is the number of pendant vertices in T.

Proposition2.2: If G is a 2 self-centered graph with a dominating edge, then $\gamma_{dcl}(G) = 4$.

Theorem 2.1: Let G be a graph of order p. Then $\gamma_{dcl}(G) = 2$ if and only if G has at least two vertices of degree $p - 1$.

Theorem 2.2: Let G be a graph of order p. Then $\gamma_{dcl}(G) = 3$ if and only if G has exactly one vertex of degree $p - 1$.

2. PRIOR RESULTS

The concept of distance closed set is defined and studied in the doctoral thesis of Janakiraman [5] and the concept of distance closed sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the distance closed dominating set is defined with respect to the distance closed property and the dominating set of the graph. Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a **D**istance **C**losed **D**ominating (D.C.D) set if

(i) S is distance closed and

(ii) S is a dominating set.

The cardinality of a minimum distance closed dominating set of G is called the distance closed domination number of G and is denoted by $\gamma_{dcl}(G)$. For example, in the graph G given in Figure 2.1, the set S = {1, 3, 5, 6} forms a D.C.D set and $\gamma_{dcl}(G) = 4$.

Theorem 2.3: If a graph G is connected and $d(G) \geq 3$, then $\gamma_{dcl}(\overline{G}) = 4$.

3. MAIN RESULTS

In this paper, the distance closed domatic number of some special classes of graphs are discussed. Also, the structure of graphs with a given distance closed domatic number are analyzed.

Definition 3.1:

The *distance closed domatic number* $d_{dcl}(G) = n$ of a graph G is the maximum partition $\{V_1, V_2 \ldots V_n\}$ of V(G) such that each V_i, $1 \leq i \leq n$ is a D.C.D set of G.

3.1 The bounds on distance closed domatic number

The following theorems give the bounds of distance closed domatic number in terms of the number of vertices and minimum degree of a graph.

Proposition 3.1.1: For any graph G, $d_{dcl}(G) \leq \delta$.

Proof: If a graph G has domatic number k, then clearly every vertex must be adjacent to at least k vertices, one in each dominating subset of a distance closed domatic partition of order k. Hence, $d_{dcl}(G) \leq \delta$.

Corollary 3.1.1: If G is a graph with $\delta(G) = 1$, then G must have $d_{dcl}(G) = 1$.

Corollary 3.1.2: For any tree T with $p \geq 1$, $d_{dcl}(T) = 1$.

Proposition 3.1.2: If G is a graph with $d_{dcl}(G) \geq 2$, then G is a block.

Proof: If G has a cut vertex v, then every D.C.D set of G must contain the vertex v, which is not possible as $d_{dcl}(G) \geq 2$. Hence, G must be a block.

Proposition 3.1.3: For any cycle C_n, $d_{dcl}(C_n) = 1$, $n \geq 3$.

Proposition 3.1.4: If G is a complete graph on p vertices, then $d_{dcl}(G) = \left\lfloor \frac{p}{2} \right\rfloor$.

Proof: If $G = K_p$, then clearly $\gamma_{dcl}(K_p) = 2$. Also for any graph G,

$$\gamma_{dcl}(G)*d_{dcl}(G) = p.$$

$$\Rightarrow d_{dcl}(G) = \left\lfloor \frac{p}{2} \right\rfloor.$$

Proposition 3.1.5: If G is a complete bipartite graph $K_{m,\,n}$ on p vertices, $p = m + n$, then

$$d_{dcl}(G) = \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor, & \text{if } m < n \\ \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n < m \end{cases}$$

Proof: Let G be a complete bipartite graph $K_{m,\,n}$ on p vertices and let $\{V_1, V_2\}$, where $|V_1| = m$ and $|V_2| = n$ be the bipartition of V(G). Then G is 2 self-centered and $\gamma_{dcl}(G) = 4$. Also, any D.C.D set of G must have exactly two vertices from each of V_1 and V_2.

$$\text{Hence, } d_{dcl}(G) = \begin{cases} \left\lfloor \frac{m}{2} \right\rfloor, & \text{if } m < n \\ \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n < m \end{cases}$$

Proposition 3.1.6: If G is a (p – 2) regular graph, then $d_{dcl}(G) = \left\lfloor \frac{p}{4} \right\rfloor$.

Proof: If G is a (p – 2) regular graph, then G is 2 self-centered and $\gamma_{dcl}(G) = 4$.

Hence, $d_{dcl}(G) = \left\lfloor \frac{p}{4} \right\rfloor$.

Proposition 3.1.7: For the Petersen graph G (2 self-centered), $d_{dcl}(G) = 2$.

Proposition3.1.8: If G is a 2 self-centered graph such that for every $v \in V(G)$, both $\langle N_1(v) \rangle$ and $\langle N_2(v) \rangle$ are independent, then $d_{dcl}(G) \leq \left\lfloor \frac{p}{4} \right\rfloor$.

Proof: Let G be a 2 self-centered graph such that for every $v \in V(G)$, both $\langle N_1(v) \rangle$ and $\langle N_2(v) \rangle$ are independent. Then every vertex in $N_1(v)$ is adjacent to all the vertices of $N_2(v)$ and vice versa. Thus, G can be partitioned into at most $\left\lfloor \frac{p}{4} \right\rfloor$ disjoint cycles of length 4 and also each cycle represents a D.C.D set of G. Hence, $d_{dcl}(G) \leq \left\lfloor \frac{p}{4} \right\rfloor$.

Remark 3.1.1: The upper bound for the above result $d_{dcl} = \left\lfloor \frac{p}{4} \right\rfloor$ is attainable for any complete bipartite graph $K_{n,\,n}$ and the lower bound $d_{dcl} = 1$ is attainable for any $K_{2,\,m}$.

Proposition 3.1.9: Let G be a graph with p vertices and radius r, then $d_{dcl}(G) \leq \left\lfloor \frac{p}{2r} \right\rfloor$.

Proof: For any graph G with radius r, the radius of any D.C.D set is at least r. Thus, any D.C.D set of G must have cardinality at least 2r. Hence, $\gamma_{dcl}(G) \geq 2r$ and hence $d_{dcl}(G) \leq \left\lfloor \frac{p}{2r} \right\rfloor$.

Proposition 3.1.10: If G is a graph with p vertices and it contains m_1 number of vertices of degree p – 1, then

$$d_{dcl}(G) \leq \left\lfloor \frac{p + m_1}{4} \right\rfloor.$$

Proof: Since G has a vertex of degree p – 1, G is of radius 1 and diameter 2. Let $S = \{v \in V(G) \mid \deg(v) = p - 1\}$ and $|S| = m_1$. Then, we have the following two cases.

Case (i): $\frac{p - m_1}{2} < m_1$

If $\frac{p-m_1}{2} < m_1$, then G contains $\left\lfloor \frac{m_1}{2} \right\rfloor$ number of disjoint D.C.D sets for <S> and for the remaining graph the maximum partition is $\left\lfloor \frac{p-m_1}{4} \right\rfloor$.

Therefore, $d_{dcl}(G) \leq \left\lfloor \frac{p-m_1}{4} \right\rfloor + \left\lfloor \frac{m_1}{2} \right\rfloor$

$\Rightarrow d_{dcl}(G) \leq \left\lfloor \frac{p+m_1}{4} \right\rfloor$

Case (ii): $\frac{p-m_1}{2} > m_1$

If $\frac{p-m_1}{2} > m_1$, then G contains at most m_1 number of disjoint D.C.D sets, each has cardinality equal to 3 and for the remaining $(p - 3m_1)$ vertices, the maximum partition is $\left\lfloor \frac{p-3m_1}{4} \right\rfloor$.

Therefore, $d_{dcl}(G) \leq m_1 + \left\lfloor \frac{p-3m_1}{4} \right\rfloor$

$$\leq \left\lfloor \frac{4m_1 + p - 3m_1}{4} \right\rfloor$$

$\Rightarrow d_{dcl}(G) \leq \left\lfloor \frac{p+m_1}{4} \right\rfloor$

Hence, from cases (i) and (ii) we have the result.

Theorem 3.1.4: Let G be a graph with diameter ≥ 3. If $d_{dcl}(G) = k$, then $d_{dcl}(\overline{G}) \geq k$.

Proof: If G is a graph with diameter ≥ 3, then $\gamma_{dcl}(\overline{G}) = 4$. Also, if $d_{dcl}(G) = k$, then G has k disjoint D.C.D sets D_i, i = 1 to k such that each $\langle D_i \rangle$, i = 1 to k is either a tree or a cycle. In both the cases, $\langle D_i \rangle$, i = 1 to k has at least one induced P_4 on it. Clearly the 4 vertices in P_4 of each $\langle D_i \rangle$, i = 1 to k forms a D.C.D set for $\overline{G}$. Hence, each D_i, i = 1 to k has at least one subset which is a D.C.D set of $\overline{G}$ and hence $d_{dcl}(\overline{G}) \geq k$.

3.2 Nordhaus-Gaddum results for distance closed domatic number

Proposition 3.2.1: For any graph G with p vertices,

$$d_{dcl}(G) + d_{dcl}(\overline{G}) \leq p - 1.$$

Proof: We know that for any graph G, $d_{dcl}(G) \leq \delta(G)$.

Hence, $d_{dcl}(\overline{G}) \leq \delta(\overline{G}) = p - \Delta(G) - 1$

Therefore,

$$d_{dcl}(G) + d_{dcl}(\overline{G}) \leq \delta(G) + p - \Delta(G) - 1$$
$$\leq p + \delta(G) - \Delta(G) - 1$$

Hence, $d_{dcl}(G) + d_{dcl}(\overline{G}) \leq p - 1$.

Proposition 3.2.2: For any cycle C_n with $n \geq 6$,

$d_{dcl}(C_n) + d_{dcl}(\overline{C_n}) \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$.

Proof: For any cycle C_n with $n \geq 6$, $d_{dcl}(C_n) = 1$. Also, any two vertices at distance 3 in C_n form a dominating edge for $\overline{C_n}$ and therefore $\overline{C_n}$ is 2 self-centred. Hence, $\gamma_{dcl}(\overline{C_n}) = 4$ and hence $d_{dcl}(\overline{C_n}) \leq \left\lfloor \frac{n}{4} \right\rfloor$

.Therefore, $d_{dcl}(C_n) + d_{dcl}(\overline{C_n}) \leq \left\lfloor \frac{n}{4} \right\rfloor + 1$.

3.3 Structure of graphs with a given distance closed domatic number

We can construct graphs with a given distance closed domatic number. For example, the structure of graphs with $d_{dcl}(G) = \frac{(p+m)}{4}$, where m is the number of (p – 1) degree vertices in G is given below.

If $d_{dcl}(G) = \frac{(p+m)}{4} = k$, then G has k disjoint D.C.D sets $V_1, V_2 \ldots V_k$ such that each V_i, i =1 to k is an D.C.D set of G and the structure of G is obtained as follows:

Construction procedure:

1. Consider m cycles $C_1, C_2 \ldots$ Cm, each of length 7 (that is $|C_i| = 7$, i = 1, 2 …m) and label the vertices of each C_i (i = 1, 2,…m) by $v_1, v_2, \ldots v_7$.
2. Now, join the vertex v_1 of C_1 to the vertices in $C_1 - \{v_2, v_3\}$ and all the vertices of $C_2, C_3 \ldots C_m$. Also, join the vertex v_4 of C_1 to the vertices v_2 and v_4 of $C_2, C_3 \ldots C_m$. Similarly, join the vertex v_5 of C_1 to the vertices v_3 and v_5 of $C_2, C_3 \ldots C_m$.
3. Finally, join the vertex v_6 of C_1 to the vertex v_6 of $C_2, C_3 \ldots C_m$ and join the vertex v_7 of C_1

to the vertex v_7 of C_2, $C_3 \ldots C_m$. Repeat this procedure for the remaining cycles C_2, $C_3 \ldots C_m$.

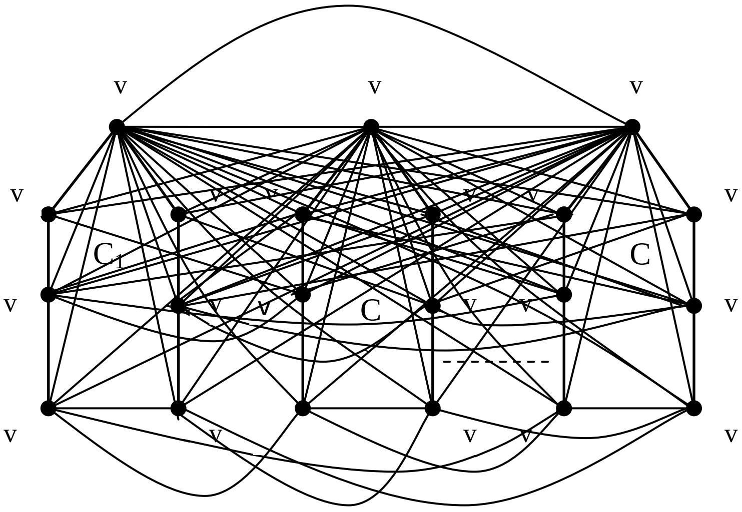

Figure 3.1 - ***Structure of graphs with*** $d_{dcl} = \dfrac{(p+m)}{4}$

Now the resultant graph G is of radius 1 and diameter 2. Also G has exactly mvertices of degree p – 1. In each C_i, i = 1, 2 …m the set of vertices $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6, v_7\}$ form two disjoint D.C.D sets for G. That is G has exactly 2m = $\dfrac{(p+m)}{4}$ number of disjoint D.C.D sets.

3.4 Algorithm to construct a graph with a given domatic number

In this section, an algorithm to construct an r self-centered graph G with a given domatic number $d_{dcl}(G) = \dfrac{p}{2r}$ is given. The validity and complexity of the algorithm is also given and it can be checked in polynomial time.

In general, the distance closed property in a graph can be checked in polynomial time. However, finding the connected domination number of a graph G is NP-complete. Hence, finding a minimum distance closed dominating set in a general graph is NP-complete and so attempts are made to develop polynomial time algorithms for finding D.C.D sets in some special classes of graphs. In all the algorithms, the distance matrix of G is computed as a pre-processing step, whose time complexity is $O(p^3)$. Further it is assumed that any graph G given as input is in the form of adjacency matrix and it is also assumed that the degrees of all vertices are part of the input. Hence, the complexity of the algorithm discussed in this section is given excluding the pre-processing time.

Algorithm to construct a 2k regular graph G with $d_{rdcl}(G) = k$, where G is r self-centered and $k = \dfrac{p}{2r}$

Input : Positive integers r, k
Output : A 2k regular graph G with $d_{rdcl}(G) = k$

Pseudocode:

Step 1:

Consider k cycles C_i (i = 1 to k) each of length 2r;

Set$V(C_i) = \{v_1^{(i)}, v_2^{(i)}, \ldots v_{2r}^{(i)}\}$ (i=1 to k),

$V = \bigcup_{i=1}^{k} V(C_i)$;

Step 2:

For eachC_i(i = 1 to k) **do** {

For each $v_j^{(i)} \in V(C_i)$ (j = 1 to 2r) **do**

$u v_j^{(i)} \in E$, $\forall\, u \in N(v_j^{(m)})$, m = 1 to k and $m \neq i$;

}

Step 3:

Output G = (V, E); *//G is a 2k regular graph with $d_{rdcl}(G) = k$*

Step 4:

Exit

Validity of the algorithm:

Validity follows directly from the construction of a 2k regular graph G, which is r self-centered with $d_{rdcl}(G) = k$ and $k = \dfrac{p}{2r}$.

Complexity of the algorithm:

Step 1 takes constant time to set V and forming the edge set E using the **for** loops given in step 2 takes $O(2kr(k-1)) = O(p^2)$ as $k = \dfrac{p}{2r}$. Thus the complexity of the algorithm is $O(p^2)$.

4 CONCLUSION

In this paper, the distance closed domatic number of some special classes of graphs are studied.

Also the Nordhaus-Gaddum results for the above domatic number of a graph and its complement are given and the structure of a graph with a given distance closed domatic number is analyzed. Also, a general algorithm to find the structure of graphs with a given domatic number is proposed. The validity and complexity of the algorithm is also given and it can be checked in polynomial time. Since the distance closed dominating set is distance preserving, in most of the cases, it is useful to find a sub-network in a given communication network, which is fault tolerant. It is not true that all graphs have at least one distance preserving subset which is a dominating set. Also, every graph has at least one distance closed dominating set even if it does not have a distance preserving dominating set. In those cases, we can cover at most all the vertices using this distance closed domination. Hence, this concept is very useful to analyze the worst case complexity in fault tolerance. Also, the above obtained results are used to analyze the behavior of different communication networks in different situations. In particular, in fault tolerance analysis of networks, parallel architecture designing and signal processing.

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